THE NUMBER OF k-COLOURED GRAPHS

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Introduction. In this paper we describe an algorithm for finding the number of non-isomorphic k-coloured graphs with n nodes and e edges. We use Pólya's fundamental enumeration theorem (in a form similar to that given by de Bruijn (see **1**)) which reduces the problem to finding the cycle index for a certain permutation group. Harary (**3**) followed this same program for bi-coloured graphs, but failed to find the cycle index of the relevant group for general k-coloured graphs.

The attentive reader will note that we do not claim to have enumerated k-coloured graphs, only that we have described an algorithm for carrying out such an enumeration. For any fixed n and k there are a finite number of labelled k-coloured graphs (these were enumerated by Read (**4**)), and these can be listed and compared in order to determine the classes of isomorphic graphs. Such a listing and sequence of comparisons constitutes a tremendous, but finite, job even for $(n, k) = (9, 3)$. The algorithm we describe reduces this work so that hand and machine calculations are now feasible for many small n and k. Actually, an explicit formula involving sums over partitions can be given, but this would be too complicated to write down for the general case.

Definitions and notation. Let $0 \leqq n_1 \leqq n_2 \leqq \ldots \leqq n_k$ be a set of k natural numbers such that $n_1 = \ldots = n_{a_1} = \nu_1 < n_{a_1+1} = \ldots = n_{a_1+a_2} = \nu_2 < \ldots < n_{a_1+\ldots+a_{i-1}+1} = \ldots = n_k = \nu_i$, then we shall sometimes write $(n_1, \ldots, n_k) = (\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ to indicate the same *partition of* $n = n_1 + n_2 + \ldots + n_k = a_1\nu_1 + a_2\nu_2 + \ldots + a_i\nu_i$ into exactly $k = \nu_1 + \nu_2 + \ldots + \nu_i$ positive parts. Suppose that $(n_1, \ldots, n_k) = (\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ is a partition of n into k positive parts; a *k-coloured graph* $(N_{n_1}{}^1, \ldots, N_{nk}{}^k : E)$ of *type* $(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ consists of k non-empty sets $N_{n_c}{}^c = \{(c, i) : i = 1, \ldots, n_c\}$ containing *coloured nodes*, and a set of *edges* E, where E is a subset of the *complete edge set* $E(\nu_1^{a_1}, \ldots, \nu_i^{a_i}) = \{\{(a, i), (b, i)\} : (a, i) \in N_{na}{}^a, (b, j) \in N_{nb}{}^b, 1 \leqq a < b \leqq k\}$. The complete k-coloured graph of type $(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ is $G(\nu_1^{a_1}, \ldots, \nu_i^{a_i}) = (N_{a_1}{}^1, \ldots, N_{ak}{}^k : E(\nu_1^{a_1}, \ldots, \nu_i^{a_i}))$; sometimes k-coloured graphs are called *k-partite graphs*.

Now we wish to define certain permutations on the nodes and edges of the complete k-coloured graph of type $(n_1, \ldots, n_k) = (\nu_1^{a_1}, \ldots, \nu_i^{a_i})$. Let S_n denote the symmetric group on $N_n = \{1, \ldots, n\}$. We say that $(\pi : \pi_1, \ldots, \pi_k)$

Received March 8, 1967. This paper was written while the author was a post-doctoral fellow at McMaster University, Hamilton, Ontario, Canada, 1967.

https://doi.org/10.4153/CJM-1968-134-x Published online by Cambridge University Press

is a *node* permutation of $G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ if $\pi \in S_k$ is such that $n_c = n_{\pi c}$, for $c = 1, \ldots, k$, and if $\pi_c \in S_{n_c}$, $c = 1, \ldots, k$, where

$$(1) \qquad (\pi : \pi_1, \ldots, \pi_k) \colon (c, i) \to (\pi c, \pi_c i).$$

The set $\Gamma G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ containing all node permutations of the complete k-coloured graph of type $(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ is a permutation group called the *automorphism group* of $G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$. Each node permutation $(\pi : \pi_1, \ldots, \pi_k)$ induces an *edge permutation* defined by

$$(2) \qquad (\pi : \pi_1, \ldots, \pi_k)^* \colon \{(a, i), (b, j)\} \to \{(\pi a, \pi_a i), (\pi b, \pi_b j)\},$$

and the set $\Gamma^* G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ containing all edge permutations forms the *edge automorphism group* of $G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$.

Two k-coloured graphs X and Y both of type $(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ are said to be *isomorphic* if there is an edge permutation $\pi^* \in \Gamma^* G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ such that π^* maps the edges of X one-to-one onto the edges of Y. Isomorphism is an equivalence relation on the set of all k-coloured graphs of type $(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$. Let $g(\nu_1^{a_1}, \ldots, \nu_i^{a_i} : e)$ denote the number of distinct equivalence classes containing isomorphic k-coloured graphs of type $(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ with e edges, and define the *counting polynomial*

$$(3) \qquad G(\nu_1^{a_1}, \ldots, \nu_i^{a_i} : x) = \sum g(\nu_1^{a_1}, \ldots, \nu_i^{a_i} : e) x^e,$$

where

$$0 \leqq e \leqq \sum a_r a_s \nu_r \nu_s + \sum \binom{a_t}{2} \nu_t^2, \qquad 1 \leqq r < s \leqq i, \ 1 \leqq t \leqq i.$$

An algorithm for computing $G(\nu_1^{a_1}, \ldots, \nu_i^{a_i} : x)$ is given by Pólya's enumeration theorem; for completeness we shall preesnt a brief statement of this theorem due to de Bruijn (see **1**; **2**) and then describe the application of the theorem to the problem we are treating.

Pólya's theorem. We shall find it convenient to have a slightly modified definition for the cycle index which is usually only defined for groups of permutations. Suppose that π is a permutation of degree n expressed as a product of disjoint cycles; the *cycle index* of π is

$$Z(\pi; \bar{x}) = Z(\pi; x_1, x_2, \ldots) = \prod x_k^{i_k}, \qquad 1 \leqq k \leqq \infty,$$

where π has exactly i_k cycles of length k, $k = 1, 2, \ldots$; thus, $i_k = 0$ for almost all k. The *cycle index of a set* S containing permutations of degree n is

$$Z(S; \bar{x}) = Z(S; x_1, x_2, \ldots) = \sum Z(\pi; \bar{x}), \ \pi \in S;$$

finally, the *cycle index of a group* G of permutations of degree n is $|G|^{-1} Z(G; \bar{x})$, where $|G|$ denotes the number of elements in G, and $Z(G; \bar{x})$ is the cycle index of the set G.

Suppose that D and R are finite sets and let R^D denote the set of all mappings of D into R. If G is a permutation group defined on D, two maps $f, g \in R^D$ are

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G-equivalent if there is a permutation $\pi \in G$ such that $f\pi = g$. Let w denote a mapping of R into $C[x]$, the ring of polynomials over the field C of complex numbers; $w(r)$ is called the *weight* of $r \in R$. The *weight of a map* $f \in R^D$ is $W(f) = \Pi\, w[f(d)]$, $d \in D$, and the *weight of an equivalence class* F containing equivalent maps $f, g, \ldots$ is $W(F) = W(f)$, where $f \in F$; it is easy to check that W is a well-defined function. Let $w_i = \sum w^i(r)$, $r \in R$; then Pólya's theorem gives the following relation between the sum of the weights of the equivalence classes induced in R^D by G and the cycle index for G:

$$\sum_F W(F) = |G|^{-1}Z(G : w_1, w_2, \ldots). \tag{4}$$

Suppose that $(n_1, \ldots, n_k) = (\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ is a partition of n. If we put $D = E(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ and $R = \{0, 1\}$, there is an obvious one-to-one correspondence between the maps in R^D and the k-coloured graphs of type $(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$. (The map $f \in R^D$ corresponds to the graph $(N_{n_1}{}^1, \ldots, N_{n_k}{}^k : E)$, where X is in E if, and only if, $f(X) = 1$, $X \in D$.) Furthermore, if we define $w(0) = 1$, $w(1) = x$, then the weight of a map $f \in R^D$ indicates the number of edges possessed by the graph corresponding to f; for example, if $W(f) = x^e$, the graph corresponding to f has exactly e edges. Next, the edge automorphism group of the complete k-coloured graph of type $(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ is a permutation group defined on D such that $f, g \in R^D$ are $\Gamma^*G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$-equivalent if, and only if, the graphs corresponding to f and g are isomorphic. From this it follows that $\sum W(F) = G(\nu_1^{a_1}, \ldots, \nu_i^{a_i} : x)$; hence, from (4) we conclude that

$$G(\nu_1^{a_1}, \ldots, \nu_i^{a_i} : x) = |\Gamma^*G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})|^{-1}Z(\Gamma^*G(\nu_1^{a_1}, \ldots, \nu_i^{a_i}) : 1 + x, 1 + x^2, \ldots). \tag{5}$$

Thus, Pólya's theorem shows that an algorithm for computing $G(\nu_1^{a_1}, \ldots, \nu_i^{a_i} : x)$ can be given if we can describe the cycle index of $\Gamma^*G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$; in the remaining portion of this paper we shall give a solution to this problem.

The cycle index for the group $\Gamma^*G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$. Let $(\pi : \phi_1, \phi_2, \ldots, \phi_k)$ be a given node permutation of $G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$, suppose that $\pi = (1, 2, \ldots, a)(a + 1, \ldots, a + b)\ldots$, and put $\phi_1\phi_2 \ldots \phi_a = \Phi$, $\phi_{a+1}\phi_{a+2} \ldots \phi_{a+b} = \Lambda$. We are going to find $z(a : \Phi)$, (in Lemma A) the cycle index for the restriction of $(\pi : \phi_1, \ldots, \phi_k)^*$ to edges which join nodes in the sets $N_{n_1}{}^1, \ldots, N_{n_a}{}^a$; also, we shall find $z(a, b : \Phi, \Lambda)$ (in Lemma B), the cycle index for the restriction of $(\pi : \phi_1, \phi_2, \ldots, \phi_k)^*$ to edges which join nodes in the sets $N_{n_1}{}^1, \ldots, N_{n_a}{}^a$ to the nodes in the sets $N_{n_{a+1}}^{a+1}, \ldots, N_{n_{a+b}}^{a+b}$. (Of course, $n_1 = n_2 = \ldots = n_a$, $n_{a+1} = \ldots = n_{a+b}$.)

Once we have these results, the cycle index for $(\pi : \phi_1, \ldots, \phi_k)^*$ can be expressed as a product of appropriate cycle indices $z(a : \Phi)$ and $z(a, b : \Phi, \Lambda)$. To see this, write $\pi = \pi_1 \ldots \pi_i$, where π_j is the restriction of π to the a_j sets of nodes having ν_j elements, and suppose that $Z(\pi_j : \bar{x}) = x_1^{p_{j1}}x_2^{p_{j2}} \ldots$, $j = 1, 2, \ldots, i$. Thus, π_j may be written as a product of disjoint cycles of the

https://doi.org/10.4153/CJM-1968-134-x Published online by Cambridge University Press

a_j colours used to colour the node sets containing ν_j nodes; thus, we can let $0 < c_{j1} \leqq c_{j2} \leqq \ldots$ denote the lengths of all the cycles of π_j (so that $c_{j1} + c_{j2} + \ldots = a_j$). Finally, if $\Phi_{j1}, \Phi_{j2}, \ldots \in S_{\nu_j}$ denote the products of the permutations from S_{ν_j} which act on the sets of nodes whose colours are permuted by the colour cycles of length $c_{j1}, c_{j2}, \ldots$, respectively, then

$$(6) \quad Z((\pi : \phi_1, \phi_2, \ldots, \phi_k)^* : \bar{x}) = \prod_{\{u,v\}} \prod_{\{r,s\}} z(c_{uv} : \Phi_{uv}) z(c_{uv}, c_{rs} : \Phi_{uv}, \Phi_{rs}).$$

Thus, for the set $\Gamma^* G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ we have that

$$(7) \quad Z(\Gamma^* G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})) = \sum_{\pi \in S_k} \sum_{\phi_1 \in S_{n_1}} \cdots \sum_{\phi_k \in S_{n_k}} Z((\pi : \phi_1, \ldots, \phi_k)^* : \bar{x}).$$

This formula, along with (6), leads us to think that we must calculate certain products of permutations in order to evaluate (7), but this is not the case. If $\pi, \pi' \in S_k$, and $Z(\pi : \bar{x}) = Z(\pi' : \bar{x})$, and if $\phi_1, \ldots, \phi_k$ and $\phi_1', \ldots, \phi_k'$ give rise to $\Phi_{11}, \ldots$, and $\Phi_{11}', \ldots$, respectively, such that $Z(\Phi_{11} : \bar{x}) = Z(\Phi_{11}' : \bar{x})$, then $Z((\pi : \phi_1, \ldots, \phi_k)^* : \bar{x}) = Z((\pi' : \phi_1, \ldots, \phi_k)^* : \bar{x})$. This follows since the labels of the nodes can be permuted in the cycles of one of the permutations so that it is transformed into the other. Let $[\pi : \Phi_{11}, \Phi_{12}, \ldots]^*$ denote $(\pi : \phi_1, \ldots, \phi_k)^*$, where $\Phi_{j1}, \Phi_{j2}, \ldots$ are the products of permutations defined for (6); also, for $\mu \in S_m$, let $T(\mu)$ denote the number of $\mu' \in S_m$ such that $Z(\mu : \bar{x}) = Z(\mu' : \bar{x})$. We shall write

$$\sum_{\mu}{}', \qquad \mu \in S_m,$$

to indicate a sum whose index ranges over the permutations of S_m having distinct cycle indices. The number of products $\lambda_1 \lambda_2 \ldots \lambda_{c_{j1}} = \Phi_{j1}$ with $\lambda_1, \ldots, \lambda_{c_{j1}} \in S_{\nu_j}$ is $(\nu_j!)^{c_{j1}}/\nu_j! = (\nu_j!)^{c_{j1}-1}$. Using these observations, we can sum in (7) in another way to obtain the following theorem.

THEOREM. *The group $\Gamma^* G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ has the cycle index*

$$(8) \quad [k!(\nu_1!)^{a_1} \ldots (\nu_i!)^{a_i}]^{-1} \times \sum_{\pi \in S_k}{}' \sum_{\Phi_{11} \in S_{\nu_1}}{}' \cdots T(\pi)(\nu_1!)^{c_{11}-1} T(\Phi_{11}) \ldots Z([\pi : \Phi_{11}, \ldots]^* : \bar{x}).$$

Proof. The proof has already been indicated except that we must show that $|\Gamma^* G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})| = k!(\nu_1!)^{a_1} \ldots (\nu_i!)^{a_i}$. There are $k!$ ways to select $\pi \in S_k$ and $(\nu_j!)^{a_j}$ ways to select an a_j-tuple of elements from S_{ν_j}; hence, the group contains $k!(\nu_1!)^{a_1} \ldots (\nu_i!)^{a_i}$ elements altogether. We still have to prove Lemmas A and B before a complete description of the cycle index of $\Gamma^* G(\nu_1^{a_1}, \ldots, \nu_i^{a_i})$ would be given.

LEMMA A. *Suppose that $Z(\Phi : \bar{x}) = x_1^{f_1} x_2^{f_2} \ldots$, then*

$$(9) \qquad z(a : \Phi) = \prod_{d=1}^{\infty} \prod_{e=d+1}^{\infty} y(a : x_d^{f_d}) y(a : x_d^{f_d}, x_e^{f_e}),$$

where

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$$(10)\qquad y(a : x_d{}^f) = \begin{cases} x_{ad/2}^f x_{ad}^{f(fad-fd-1)/2} & \textit{if } a \textit{ is even and } d \textit{ is odd,} \\ x_{ad}^{f^2 d(a-1)/2} & \textit{otherwise,} \end{cases}$$

$$(11)\qquad y(a : x_d{}^f, x_e{}^g) = x_{a[d,e]}^{fg(a-1)(d,e)},$$

where $[d, e]$ *and* (d, e) *denote the least common multiple and greatest common divisor of* d *and* e, *respectively.*

Proof. Suppose that $(\pi : \phi_1, \ldots, \phi_k)$ is a node permutation of $G(\nu_1{}^{a_1}, \ldots, \nu_i{}^{a_i})$ with $\pi = (1, 2, \ldots, a) \ldots$, and $\phi_1 \ldots \phi_a = \Phi$. We wish to show that the restriction of $(\pi : \phi_1, \ldots, \phi_k)^*$ to the edges of $G(\nu_1{}^{a_1}, \ldots, \nu_i{}^{a_i})$ which join nodes in the sets $N_{n_1}{}^1, \ldots, N_{n_a}{}^a$ can be given in terms of $Z(\Phi : \bar{x})$.

If $(1, j) \in N_{n_1}{}'$, and if j is in a t cycle of Φ, then the node cycle of $(\pi : \phi_1, \ldots, \phi_k)$ which contains $(1, j)$ is

$$(12)\quad (1, j) \to (2, \phi_1 j) \to \ldots \to (a, \phi_{a-1} \ldots \phi_1 j) \to (1, \phi j) \to (2, \phi_1 \Phi j) \to \ldots \to (a, \phi_{a-1} \ldots \phi_1 \Phi j) \to \ldots \to (1, \Phi^t j) = (1, j).$$

Thus, every t cycle of Φ corresponds to an at cycle involving the nodes coloured $1, 2, \ldots, a$; hence, if $Z(\Phi : \bar{x}) = x_1{}^{f_1} x_2{}^{f_2} \ldots$, then the cycle index for the restriction of $(\pi : \phi_1, \ldots, \phi_k)$ to $N_{n_1}{}^1 \cup \ldots \cup N_{n_a}{}^a$ is $x_a{}^{f_1} x_{2a}{}^{f_2} \ldots x_{ta}{}^{f_t} \ldots$.

Now we determine the cycle indices for the edge permutations restricted (1) to the edges which join nodes in the same node cycle, and (2) to the edges which join nodes in distinct node cycles involving the nodes coloured $1, 2, \ldots, a$.

Case 1. Every edge cycle contains an edge having one node in $N_{n_1}{}'$, say $(1, j)$, which is joined to (c, p), where $p = \phi_{c-1} \ldots \phi_1 \Phi_j{}^r$ $(c \neq 1)$, since in this case we are assuming that $(1, j)$ and (c, p) are in the same node cycle of length ad. We can assume that $c \leqq (a + 2)/2$; now consider

$$(13)\quad \{(1, j), (c, p)\} \to \ldots \to \{(c, \phi_{c-1} \ldots \phi_1 j), (2c - 1, \phi_{2c-2} \ldots \phi_c p)\} \to \ldots \to \{(1, \Phi^{t-1} j), (c, \phi_{c-1} \ldots \phi_1 \Phi^{r+t-1} j)\} \to \ldots \to \{(c, \phi_{c-1} \ldots \phi_1 \Phi^{t-1} j), (2c - 1, \phi_{2c-2} \ldots \phi_1 \Phi^{r+t-1} j)\} \to \ldots \to \{(1, \Phi^{d-1} j), (c, \phi_{c-1} \ldots \phi_1 \Phi^{r+d-1} j) \to \ldots \to \{(1, j), (c, \phi_{c-1} \ldots \phi_1 \Phi^r j)\}.$$

If this is a single edge cycle of length ad for each choice of $p = \phi_{c-1} \ldots \phi_1 \Phi^r j$ (we are selecting c and r), then the

$$\binom{ad}{2} - a\binom{d}{z}$$

edges which join nodes in the node cycle being considered split into edge cycles of length ad and are

$$\left\{\binom{ad}{2} - a\binom{d}{2}\right\}/ad = d(a - 1)/2$$

in number. It is easy to verify that if this is not the case, then a is even and d is odd; furthermore, there is exactly one choice of p which gives rise to a different situation. Namely, when $p = ((a+2)/2, \phi_{a/2} \ldots \phi_1 \Phi^{(d-1)/2} j)$ (put $r = t - 1 = (d-1)/2$ and $c = (a+2)/2$ above), there are $(ad - d - 1)/2$ edge cycles of length ad and one edge cycle of length $ad/2$.

Case 2. Now suppose that (c, j) and (c', j'), $c \neq c'$, are nodes such that j and j' are contained in distinct cycles of Φ having lengths d and e, respectively. Then

$$\{(c, j),\ (c', j')\} \rightarrow \{(c+1,\ \phi_c j),\ (c'+1,\ \phi_{c'} j')\} \rightarrow \ldots \tag{14}$$

is an edge cycle of length $a[d, e]$, where $[d, e]$ denotes the least common multiple of d and e. Thus, the $a(a-1)de$ edges which join nodes in the node cycle containing (c, j) to nodes in the node cycle containing (c', j') split into $a(a-1)de/a[d, e] = (a-1)(d, e)$ edge cycles of length $a[d, e]$, where (d, e) denotes the greatest common divisor of d and e.

Now let $y(a : x_d^f)$ and $y(a : x_d^f, x_e^g)$ denote the cycle indices of the edge permutation restricted (1) to the edges which join all nodes coloured $1, 2, \ldots, a$, and contained in a node cycle of length ad, and (2) to the edges which join all nodes coloured $1, 2, \ldots, a$ and contained in a node cycle of length ad to all nodes coloured $1, 2, \ldots, a$ and contained in a node cycle of length ac.

Combining Cases 1 and 2 we calculate

$$y(a : x_d^f) = \begin{cases} x_{ad/2}^{f} x_{ad}^{f(ad-d-1)/2} x_{ad}^{\binom{f}{2}(a-1)d} & \text{if } a \text{ is even, } d \text{ odd,} \\ x_{ad}^{fd(a-1)/2} x_{ad}^{\binom{f}{2}(a-1)d} & \text{otherwise,} \end{cases} \tag{15}$$

and

$$y(a : x_d^f, x_e^g) = x_{a[d,e]}^{fg(a-1)(d,e)} \tag{16}$$

which are the same as the definitions given in the statement of the lemma.

To see that $z(a : \Phi)$ is correctly defined, we simply observe that every edge involved is in one of the cycles we have treated.

LEMMA B. *Suppose that* $Z(\Phi : \bar{x}) = x_1^{f_1} x_2^{f_2} \ldots$ *and* $Z(\Lambda : \bar{x}) = x_1^{g_1} x_2^{g_2} \ldots$; *then*

$$z(a, b : \Phi, \Lambda) = \prod_{d=1}^{\infty} \prod_{e=1}^{\infty} y(a, b : x_d^{f_d}, x_e^{g_e}), \tag{17}$$

where

$$y(a, b : x_d^f, x_e^g) = x_{[ad,be]}^{(ad,be)fg}. \tag{18}$$

Proof. Suppose that (c, j), (c', j'), $c \neq c'$, are nodes contained in node cycles of length ad and be involving the nodes coloured $1, 2, \ldots, a$ and $a+1, \ldots, a+b$, respectively. There are $adbe$ edges joining nodes in one of these cycles to nodes in the other; furthermore,

$$\{(c, j)\ (c', j')\} \rightarrow \{(c+1,\ \phi_c j),\ (c'+1,\ \phi_{c'} j')\} \rightarrow \ldots \tag{19}$$

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is an edge cycle of length $[ad, be]$; thus, the $adbe$ edges under discussion split into $adbe/[ad, be] = (ad, be)$ edge cycles of length $[ad, be]$. Since there are fg different cases similar to this one, we have that

$$(20) \qquad y(a, b : x_d^f, x_e^g) = x_{[ad,be]}^{(ad,be)fg}$$

as the cycle index for the edge permutation restricted to this set of edges. The product in the statement of this lemma gives the cycle index for the edge permutation restricted to edges joining nodes coloured $1, 2, \ldots, a$ to nodes coloured $a + 1, a + 2, \ldots, a + b$.

Tri-coloured graphs. We shall give an example of an application of the theorem proved in the last section by calculating the cycle indices for the edge automorphism groups of the complete tri-coloured graphs on n nodes for $n = 3, 4, \ldots, 9$. In this case, we use one of the three relations (21), (22), or (23), depending on whether the node sets $N_{n_1}{}^1$, $N_{n_2}{}^2$, $N_{n_3}{}^3$ are such that $\{n_1, n_2, n_3\} = \{\lambda, \mu, \nu\}, \{\mu, \nu^2\}$, or $\{\nu^3\}$, respectively, where λ, μ, and ν are distinct natural numbers. (We have simplified these expressions by noting that $z(1 : \phi) = 1$, for $\phi \in S_f, f = 1, 2, \ldots$.) The relations are

$$(21) \qquad F(\lambda, \mu, \nu)/\lambda!\, \mu!\, \nu!,$$

$$(22) \qquad \{F(\mu, \nu, \nu) + G(\mu, \nu)\}/2!\, \mu!(\nu!)^2,$$

$$(23) \qquad \{F(\nu, \nu, \nu) + 3G(\nu, \nu) + 2H(\nu)\}/3!(\nu!)^3,$$

where

$$F(\lambda, \mu, \nu) = \sum_{\phi_1 \in S_\lambda}{}' \sum_{\phi_2 \in S_\mu}{}' \sum_{\phi_3 \in S_\nu}{}' T(\phi_1)T(\phi_2)T(\phi_3) \times z(1, 1 : \phi_1, \phi_2)z(1, 1 : \phi_1, \phi_3)z(1, 1 : \phi_2, \phi_3),$$

$$G(\mu, \nu) = \sum_{\phi_1 \in S_\mu}{}' \sum_{\phi_2 \in S_\nu}{}' \nu! T(\phi_1)T(\phi_2)z(2 : \phi_2)z(1, 2 : \phi_1, \phi_2),$$

$$H(\nu) = \sum_{\phi \in S_\nu}{}' (\nu!)^2 T(\phi)z(3 : \phi).$$

In the table below we give the cycle index for the group $\Gamma^*G(n_1, n_2, n_3)$, where (n_1, n_2, n_3) ranges over the partitions of n into three positive parts for $n = 3, 4, \ldots, 9$. If $1 + x^i$ is substituted for x_i in these polynomials, the coefficient of x^e in the resulting polynomial is the number of non-isomorphic tri-coloured graphs with e edges on the nodes $N_{n_1}{}^1$, $N_{n_2}{}^2$, and $N_{n_3}{}^3$.

(1, 1, 1) $\{x_1^3 + 3x_1x_2 + 2x_3\}/3!$,

(1, 1, 2) $\{x_1^5 + 3x_1x_2^2\}/(2!)^2$,

(1, 1, 3) $\{x_1^7 + 3x_1^3x_2^2 + 4x_1x_2^3 + 2x_1x_3^2 + 2x_1x_6\}/2!\cdot 3!$,

(1, 2, 2) $\{x_1^8 + 4x_1^2x_2^3 + x_2^4 + 2x_2^2x_4\}/(2!)^3$,

(1, 1, 4) $\{x_1^9 + 6x_1^5x_2^2 + 8x_1^3x_3^2 + 13x_1x_2^4 + 8x_1x_2x_6 + 12x_1x_4^2\}/2!\cdot 4!$,

$$(1, 2, 3) \quad \{x_1^{11} + 3x_1^5x_2^3 + x_1^3x_2^4 + 2x_1^2x_3^3 + 3x_1x_2^5 + 2x_2x_3x_6\}/2!\cdot 3!,$$

$$(2, 2, 2) \quad \{x_1^{12} + 3x_1^4x_2^4 + 12x_1^2x_2^5 + 4x_2^6 + 8x_3^4 + 12x_4^3 + 8x_6^2\}/3!(2!)^3,$$

$$(1, 1, 5) \quad \{x_1^{11} + 10x_1^7x_2^2 + 20x_1^5x_3^2 + 15x_1^3x_2^4 + 30x_1^3x_4^2 + 26x_1x_2^5 + 20x_1x_2^2x_3^2 + 40x_1x_2^2x_6 + 30x_1x_2x_4^2 + 24x_1x_5^2 + 24x_1x_{10}\}/2!\cdot 5!,$$

$$(1, 2, 4) \quad \{x_1^{14} + 6x_1^8x_2^3 + 8x_1^5x_3^3 + 6x_1^4x_2^5 + 9x_1^2x_2^6 + 6x_1^2x_4^3 + 8x_1x_2^2x_3x_6 + 3x_2^7 + 6x_2x_4^3\}/2!\cdot 4!,$$

$$(1, 3, 3) \quad \{x_1^{15} + 6x_1^7x_2^4 + 15x_1^3x_2^6 + 4x_1^3x_3^4 + 12x_1x_2x_3^2x_6 + 18x_1x_2x_4^3 + 4x_3^5 + 12x_3x_6^2\}/2!(3!)^2,$$

$$(2, 2, 3) \quad \{x_1^{16} + 3x_1^8x_2^4 + 2x_1^6x_2^5 + 2x_1^4x_3^4 + 14x_1^2x_2^7 + 6x_1^2x_2x_6^2 + 4x_2^8 + 4x_2^2x_3^2x_6 + 2x_2^2x_6^2 + 6x_4^4 + 4x_4x_{12}\}/(2!)^3 3!,$$

$$(1, 1, 6) \quad \{x_1^{13} + 15x_1^7x_2^2 + 40x_1^7x_3^2 + 45x_1^5x_2^4 + 90x_1^5x_4^2 + 120x_1^3x_2^2x_3^2 + 144x_1^3x_5^2 + 91x_1x_2^6 + 160x_1x_2^3x_6 + 270x_1x_2^2x_4^2 + 144x_1x_2x_{10} + 40x_1x_3^4 + 280x_1x_6^2\}/2!\cdot 6!,$$

$$(1, 2, 5) \quad \{x_1^{17} + 10x_1^{11}x_2^3 + 20x_1^8x_3^3 + 16x_1^5x_2^6 + 30x_1^5x_4^3 + 10x_1^3x_2^7 + 20x_1^2x_2^3x_3^3 + 20x_1^2x_2^3x_3x_6 + 24x_1^2x_5^3 + 15x_1x_2^8 + 30x_1x_2^2x_4^3 + 20x_2^4x_3x_6 + 24x_2x_5x_{10}\}/2!\cdot 5!,$$

$$(1, 3, 4) \quad \{x_1^{19} + 6x_1^{11}x_2^4 + 3x_1^9x_2^5 + 8x_1^7x_3^4 + 18x_1^5x_2^7 + 2x_1^4x_3^5 + 3x_1^3x_2^8 + 24x_1^3x_2^2x_3^2x_6 + 6x_1^3x_4^4 + 12x_1^2x_2x_3^3x_6 + 9x_1x_2^9 + 18x_1x_2x_4^4 + 16x_1x_3^6 + 6x_2^2x_3x_6^2 + 12x_3x_4x_{12}\}/3!\cdot 4!,$$

$$(2, 2, 4) \quad \{x_1^{20} + 6x_1^{12}x_2^4 + 2x_1^8x_2^6 + 8x_1^8x_3^4 + 15x_1^4x_2^8 + 6x_1^4x_4^4 + 20x_1^2x_2^9 + 16x_1^2x_2^3x_3^2x_6 + 16x_1^2x_2^3x_6^2 + 12x_1^2x_2x_4^4 + 16x_2^{10} + 8x_2^4x_6^2 + 18x_2^2x_4^4 + 32x_4^5 + 16x_4^2x_{12}\}/(2!)^3 4!,$$

$$(2, 3, 3) \quad \{x_1^{21} + 6x_1^{11}x_2^{10} + x_1^9x_2^6 + 4x_1^6x_3^5 + 9x_1^5x_2^8 + 18x_1^3x_2^9 + 12x_1^2x_2^2x_3^3x_6 + 9x_1x_2^{10} + 36x_1x_2^2x_4^4 + 4x_2^3x_3^3x_6 + 12x_2^3x_3x_6^2 + 4x_3^7 + 4x_3^3x_6^2 + 24x_3x_6^3\}/(2!)^2(3!)^2,$$

$$(1, 1, 7) \quad \{x_1^{15} + 21x_1^{11}x_2^2 + 70x_1^9x_3^2 + 105x_1^7x_2^4 + 210x_1^7x_4^2 + 420x_1^5x_2^2x_3^2 + 504x_1^5x_5^2 + 630x_1^3x_2^2x_4^2 + 280x_1^3x_3^4 + 840x_1^3x_6^2 + 952x_1x_2^7 + 210x_1x_2^4x_3^2 + 700x_1x_2^4x_6 + 840x_1x_2^3x_4^2 + 504x_1x_2^2x_5^2 + 1008x_1x_2^2x_{10} + 1120x_1x_2x_6^2 + 420x_1x_3^2x_4^2 + 420x_1x_4^2x_6 + 720x_1x_{14}\}/2!\cdot 7!,$$

$$(1, 2, 6) \quad \{x_1^{20} + 15x_1^{14}x_2^3 + 40x_1^{11}x_3^3 + 45x_1^8x_2^6 + 90x_1^8x_4^3 + x_1^6x_2^7 + 120x_1^5x_2^3x_3^3 + 144x_1^5x_5^3 + 15x_1^4x_2^8 + 40x_1^3x_2^4x_3x_6 + 60x_1^2x_2^9 + 180x_1^2x_2^3x_4^3 + 40x_1^2x_3^6 + 120x_1^2x_6^3 + 120x_1x_2^5x_3x_6 + 144x_1x_2^2x_5x_{10} + 15x_2^{10} + 90x_2^4x_4^3 + 40x_2x_3^2x_6^2 + 120x_2x_6^3\}/2!\cdot 6!,$$

$$(1, 3, 5) \quad \{x_1^{23} + 10x_1^{15}x_2^4 + 3x_1^{11}x_2^6 + 20x_1^{11}x_3^4 + 45x_1^7x_2^8 + 30x_1^7x_4^4 + 60x_1^5x_2^3x_3^2x_6 + 2x_1^5x_3^6 + 45x_1^3x_2^{10} + 20x_1^3x_2^4x_3^4 + 90x_1^3x_2^2x_4^4 + 20x_1^3x_2x_3^4x_6 + 24x_1^3x_5^4 + 40x_1^2x_3^7 + 60x_1x_2^5x_3^2x_6 + 30x_1x_2^2x_3^2x_6^2 + 72x_1x_2x_5^2x_{10} + 60x_1x_3^2x_4x_{12} + 40x_2x_3^5x_6 + 48x_3x_5x_{15}\}/3!\cdot 5!,$$

$$(1,4,4)\quad \{x_1^{24} + 12x_1^{14}x_2^5 + 16x_1^9x_3^5 + 36x_1^8x_2^8 + 96x_1^5x_2^2x_3^3x_6 + 30x_1^4x_2^{10} + 12x_1^4x_4^5 + 64x_1^3x_3^7 + 36x_1^2x_2^{11} + 144x_1^2x_2^3x_4^4 + 72x_1^2x_2x_4^5 + 48x_1x_2^4x_3x_6^2 + 192x_1x_2x_3x_6^3 + 96x_1x_3x_4^2x_{12} + 9x_2^{12} + 36x_2^2x_4^5 + 108x_4^6 + 144x_8^3\}/2!(4!)^2,$$

$$(2,2,5)\quad \{x_1^{24} + 10x_1^{16}x_2^4 + 20x_1^{12}x_3^4 + 2x_1^{10}x_2^7 + 15x_1^8x_2^8 + 30x_1^8x_4^4 + 20x_1^6x_2^9 + 20x_1^4x_2^4x_3^4 + 40x_1^4x_2^4x_3^2x_6 + 24x_1^4x_5^4 + 82x_1^2x_2^{11} + 80x_1^2x_2^5x_6^2 + 120x_1^2x_2^3x_4^4 + 48x_1^2x_2x_{10}^2 + 26x_2^{12} + 40x_2^6x_3^2x_6 + 40x_2^6x_6^2 + 30x_2^4x_4^4 + 24x_2^2x_{10}^2 + 48x_2x_5^2x_{10} + 112x_4^6 + 80x_4^3x_{12} + 48x_4x_{20}\}/(2!)^3 5!,$$

$$(2,3,4)\quad \{x_1^{26} + 6x_1^{16}x_2^5 + 3x_1^{14}x_2^6 + x_1^{12}x_2^7 + 8x_1^{11}x_3^5 + 18x_1^8x_2^9 + 2x_1^8x_3^6 + 9x_1^6x_2^{10} + 6x_1^6x_4^5 + 24x_1^5x_2^3x_3^3x_6 + 3x_1^4x_2^9x_4 + 12x_1^4x_2^2x_3^4x_6 + 8x_1^3x_2^4x_3^3x_6 + 9x_1^2x_2^{12} + 18x_1^2x_2^{10}x_4 + 18x_1^2x_2^2x_4^5 + 16x_1^2x_3^8 + 24x_1x_2^3x_3x_4x_6^2 + 3x_2^{13} + 9x_2^{11}x_4 + 2x_2^4x_3^4x_6 + 18x_2^4x_3^2x_6^2 + 6x_2^4x_6^3 + 6x_2^3x_4^5 + 18x_2x_4^6 + 16x_2x_6^2x_3^4 + 12x_3^2x_4^2x_{12} + 12x_4^2x_6x_{12}\}/2!\cdot 3!\cdot 4!,$$

$$(3,3,3)\quad \{x_1^{27} + 9x_1^{15}x_2^6 + 6x_1^9x_3^6 + 27x_1^7x_2^{10} + 99x_1^3x_2^{12} + 36x_1^3x_2^3x_3^4x_6 + 36x_1^3x_2^3x_6^3 + 54x_1x_2^4x_3^2x_6^2 + 216x_1x_2^3x_4^5 + 108x_1x_4^2x_6x_{12} + 92x_3^9 + 36x_3^5x_6^2 + 432x_3x_6^4 + 144x_9^3\}/(3!)^4.$$

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