A FURTHER NOTE ON LOTOTSKY-TYPE TRANSFORMATIONS

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1. In two recent papers by V. F. Cowling and C. L. Miracle (**1**; **2**), the regularity of generalized Lototsky transformations, as well as their application to the geometric series, has been investigated. The main interest of the papers centres around (**1**, Theorems 3.1 and 4.1) and (**2**, Theorem A). In (**1**, Theorem 3.1) and in (**2**, Theorem A), the authors stated and proved a set of sufficient conditions for the regularity of Lototsky-type transformations. In (**1**, Theorem 4.1), they proved under certain additional conditions that these transformations sum the geometric series $\sum z^n$ to $(1 - z)^{-1}$ if $\operatorname{Re} z < 1$.

In this note we shall show that (**1**, Theorem 3.1) is actually weaker than, and (**2**, Theorem A) is equivalent to (**4**, Theorem 3.C). We shall also include a slightly improved version of (**1**, Theorem 4.1) and a simplified proof for it.

For the definition of Lototsky-type (or $[F, d_n]$) transformations we refer to (**3**). The notation of this note will follow that of (**1**). For brevity, we shall also use the notation

$$(1.1) \qquad |1 + d_n| = r_n, \qquad n = 1, 2, \ldots .$$

2. For convenience we restate (**4**, Theorem 3.C).

THEOREM. *Suppose* $d_n \neq -1$ $(n = 1, 2, \ldots)$ *and*

$$(2.1) \qquad \prod_{n=1}^{\infty} (|d_n| + 1)|d_n + 1|^{-1} < +\infty .$$

Then a necessary and sufficient condition for the regularity of the $[F, d_n]$*-transformation is*

$$(2.2) \qquad \sum_{n=1}^{\infty} |d_n + 1|^{-1} = +\infty .$$

(**1**, Theorem 3.1) and (**2**, Theorem A) can be deduced from this result as follows:

A simple computation yields

$$(|d_n| + 1)^2|d_n + 1|^{-2} = 1 + 4\rho_n r_n^{-2} \sin^2(\theta_n/2).$$

Thus (2.1) is equivalent to

$$(2.3) \qquad \sum_{n=1}^{\infty} \rho_n r_n^{-2} \sin^2(\theta_n/2) < +\infty .$$

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Since the first assumption of (**1**, Theorem 3.1),

$$\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty,$$

obviously implies (and is implied by) (2.2), we see that (**2**, Theorem A) is equivalent to (**4**, Theorem 3.C).

Furthermore, it was shown in (**2**, Theorem 1) that the second assumption of (**1**, Theorem 3.1),

$$\sum_{n=1}^{\infty} \rho_n^{-1}\theta_n^{2} < +\infty,$$

implies (2.3), and therefore, as just mentioned, also (2.1). Thus we may conclude that (**1**, Theorem 3.1) is not stronger than (**4**, Theorem 3.C). That (**1**, Theorem 3.1) is actually properly included in (**4**, Theorem 3.C) follows from the simple example, given in (**5**),

$$d_n = i\cdot n^{-2}, \qquad n = 1, 2, \ldots,$$

where $i = \sqrt{(-1)}$.

3. The following is an improved version of (**1**, Theorem 4.1).

THEOREM. *Suppose the* $[F, d_n]$*-transformation is regular, the sequence* $\{d_n\}$ *satisfies* (2.1), *and*

$$\lim_{n\to\infty} \rho_n = +\infty. \tag{3.1}$$

Then the $[F, d_n]$*-transformation sums the geometric series* $\sum z^n$ *to* $(1 - z)^{-1}$ *if* $\text{Re}(z) < 1$, *and does not sum it if* $\text{Re}(z) > 1$ *and* $z \neq -d_n$ $(n = 1, 2, \ldots)$.

Proof. By (**1**, pp. 426–7), the $[F, d_n]$-transformation sums the geometric series to $(1 - z)^{-1}$ if and only if

$$z\cdot\prod_{n=1}^{N} (d_n + z)(d_n + 1)^{-1} = o(1), \qquad \text{as } N \to \infty. \tag{3.2}$$

(3.2) obviously holds if $z = 0$ or $z = -d_n$ for some n. For other values of z we obtain by an easy computation, setting $z = x + iy$,

$$|d_n + z|^2|d_n + 1|^{-2} = 1 + a_n + 2(x - 1)b_n + c_n \equiv 1 + t_n, \tag{3.3}$$

where

$$a_n = (x^2 + y^2 - 1)r_n^{-2}, \quad b_n = \rho_n r_n^{-2} \cos\theta_n, \quad 2y\rho_n r_n^{-2} \sin\theta_n.$$

From (**4**, Theorem 3.C) it follows that (2.2) holds for a regular $[F, d_n]$-transformation. Therefore by (3.1) we have also

$$\sum_{n=1}^{\infty} \rho_n r_n^{-2} = +\infty. \tag{3.4}$$

Thus (3.1) and (3.4) readily imply

$$\sum_{n=1}^{N} a_n = o(1)\cdot \sum_{n=1}^{N} \rho_n r_n^{-2}, \qquad \text{as } N \to \infty. \tag{3.5}$$

Since by assumption (2.1) and therefore also (2.3) holds, we have by (3.4)

$$\sum_{n=1}^{N} b_n = \sum_{n=1}^{N} \rho_n r_n^{-2}(1 - 2\sin^2(\theta_n/2)) \tag{3.6}$$

$$= [1 + o(1)]\cdot \sum_{n=1}^{N} \rho_n r_n^{-2}, \qquad \text{as } N \to \infty.$$

Using the inequality $|\sin\theta| \leqslant 2|\sin(\theta/2)|$ we obtain

$$\sum_{n=1}^{N} |c_n| \leqslant 4|y|\cdot \sum_{n=1}^{N} \rho_n r_n^{-2}|\sin(\theta_n/2)|.$$

By applying the Cauchy–Schwartz inequality to the right-hand side, we obtain

$$\sum_{n=1}^{N} |c_n| \leqslant 4|y|\cdot \left[\sum_{n=1}^{N} \rho_n r_n^{-2}\sin^2(\theta_n/2)\right]^{\frac{1}{2}}\cdot \left[\sum_{n=1}^{N} \rho_n r_n^{-2}\right]^{\frac{1}{2}},$$

which, in turn, by (2.3) and (3.4), implies that

$$\sum_{n=1}^{N} c_n = o(1)\cdot \sum_{n=1}^{N} \rho_n r_n^{-2} \qquad \text{as } N \to \infty. \tag{3.7}$$

Since $1 + t \leqslant \exp(t)$ for all real t, we have

$$\prod_{n=1}^{N} (1 + t_n) \leqslant \exp\left\{\sum_{n=1}^{N} t_n\right\},$$

and thus by (3.3) and (3.5)–(3.7),

$$\prod_{n=1}^{N} |z + d_n|^2 |1 + d_n|^{-2} \leqslant \exp\left\{[2(x-1) + o(1)]\cdot \sum_{n=1}^{N} \rho_n r_n^{-2}\right\},$$

as $N \to \infty$. Thus by (3.4), if $x < 1$,

$$\prod_{n=1}^{N} (z + d_n)(1 + d_n)^{-1} = o(1), \qquad \text{as } N \to \infty,$$

which proves that the $[F, d_n]$-transformation sums $\sum z^n$ to $(1 - z)^{-1}$ if $\mathrm{Re}(z) < 1$.

On the other hand, for $t > -1$,

$$1 + t \geqslant \exp[t - t^2(1 + t)^{-1}].$$

Therefore, if $z \neq -d_n$ $(n = 1, 2, \ldots)$,

$$\prod_{n=1}^{N} (1 + t_n) \geqslant \exp\left\{\sum_{n=1}^{N} [t_n - t_n^2(1 + t_n)^{-1}]\right\}. \tag{3.8}$$

Now, by (3.1) and (3.3),

$$t_n = O(r_n^{-1})$$

and thus, by (3.1) and (3.4),

$$(3.9) \quad \sum_{n=1}^{N} t_n^2(1 + t_n)^{-1} = O(1)\cdot \sum_{n=1}^{N} r_n^{-2} = o(1)\cdot \sum_{n=1}^{N} \rho_n r_n^{-2}, \qquad \text{as } N \to \infty.$$

By (3.3) and (3.5)–(3.9), we have, if $z \neq -d_n$ $(n = 1, 2, \ldots)$,

$$\prod_{n=1}^{N} |z + d_n|^2 |1 + d_n|^{-2} \geqslant \exp\left\{[2(x - 1) + o(1)]\cdot \sum_{n=1}^{N} \rho_n r_n^{-2}\right\}$$

as $N \to \infty$. By (3.4), this implies that (3.2) does not hold if $x > 1$. Thus the $[F, d_n]$-transformation does not sum $\sum z^n$ to $(1 - z)^{-1}$ if $\mathrm{Re}(z) > 1$ and $z \neq -d_n$ $(n = 1, 2, \ldots)$. This completes our proof.

References

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