

A SIMPLE PROOF OF AN EXPANSION OF AN ETA-QUOTIENT AS A LAMBERT SERIES

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We give a simple proof of the identity

$$\prod_{n=1}^{\infty} \frac{(1 - q^{3n})^{10}}{(1 - q^n)^3(1 - q^{9n})^3} = 1 + 3 \sum_{\substack{n=1 \\ 9 \nmid n}}^{\infty} \frac{nq^n}{1 - q^n}.$$

The proof uses only a few well-known properties of the cubic theta functions $a(q)$, $b(q)$ and $c(q)$. We show this identity implies the interesting definite integral

$$\int_0^{e^{-2\pi/3}} \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^{10}}{(1 - q^n)^6} dq = \frac{1}{3\sqrt{3}}.$$

1. INTRODUCTION

The purpose of this article is to give a direct proof of the following identity.

THEOREM 1.1. *Let q be a complex number satisfying $|q| < 1$. Then*

$$\prod_{n=1}^{\infty} \frac{(1 - q^{3n})^{10}}{(1 - q^n)^3(1 - q^{9n})^3} = 1 + 3 \sum_{\substack{n=1 \\ 9 \nmid n}}^{\infty} \frac{nq^n}{1 - q^n}.$$

The summation is over all positive integers n excluding multiples of 9.

This result was discovered using symbolic computation by Borwein and Garvan [4], and it was used to produce a ninth order iteration that converges to $1/\pi$. The proof of Theorem 1.1 in [4] appeals to two entries in Ramanujan’s Notebook [16, Chapter 20 Entry 1(iv) and Chapter 21 Entry 7(i)]. The proofs of these entries in Berndt’s excellent book [1] take several pages, and appeal to several earlier results in Ramanujan’s Notebook.

Two proofs of Theorem 1.1 were given by Berndt, Chan, Liu and Yesilyurt [3]. The first is essentially the same as the one in [4]. The second proof in [3] uses less sophisticated machinery, but is more than three pages long, and depends on another entry in Ramanujan’s Notebook [16, Chapter 20 Entry 1(v)].

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Another proof of Theorem 1.1 of a completely different nature was obtained by Farkas and Kra [9, p. 307]. Their proof uses meromorphic functions defined on Riemann surfaces.

In view of the importance of Theorem 1.1, it is desirable to have as direct a proof as possible. We give such a proof, which depends only on the well-known properties satisfied by the cubic theta functions $a(q)$, $b(q)$ and $c(q)$ given in Lemma 2.1 below.

We conclude by showing that Theorem 1.1 leads to an evaluation of a definite integral. Three similar integrals were given by Fine [10, pp. 86–91].

2. PROOF

The three cubic theta functions are defined by

$$\begin{aligned}
 a(q) &= \sum_m \sum_n q^{m^2+mn+n^2}, \\
 b(q) &= \sum_m \sum_n q^{m^2+mn+n^2} \omega^{m-n}, \\
 c(q) &= \sum_m \sum_n q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2},
 \end{aligned}$$

where $\omega = \exp(2\pi i/3)$ and $q = e^{-2\pi t}$, $\text{Re}(t) > 0$. The summation indices m and n range over all integer values. The following are some well known properties of the cubic theta functions.

LEMMA 2.1.

(2.2) $a(q)^3 = b(q)^3 + c(q)^3,$

(2.3) $b(q) = \prod_{n=1}^{\infty} \frac{(1 - q^n)^3}{(1 - q^{3n})},$

(2.4) $c(q) = 3q^{1/3} \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^3}{(1 - q^n)},$

(2.5) $a(q) = a(q^3) + 2c(q^3),$

(2.6) $b(q) = a(q^3) - c(q^3),$

(2.7) $a(q) = 1 + 6 \sum_{n=1}^{\infty} \left(\frac{q^{3n-2}}{1 - q^{3n-2}} - \frac{q^{3n-1}}{1 - q^{3n-1}} \right),$

(2.8) $a(q)^2 = 1 + 12 \sum_{\substack{n=1 \\ 3 \nmid n}}^{\infty} \frac{nq^n}{1 - q^n}.$

Equation (2.2) was discovered and proved by Borwein and Borwein [5]. Additional proofs have since been given by Borwein, Borwein and Garvan [6], Chapman [7], Garvan [11], Hirschhorn, Garvan and Borwein [12], Liu [14] and Solé [18]. Proofs of (2.3)–(2.6)

can be found in [6, 11, 12]. Equation (2.7) was known to Lorenz and Ramanujan; see [13]. A beautiful and elementary proof of (2.8) using (2.7) was given by Ramanujan [15, equation 19].

LEMMA 2.9. *Let $x = c(q)^3/a(q)^3$, $z = a(q)$, $X = c(q^3)^3/a(q^3)^3$, $Z = a(q^3)$. Then*

$$\begin{aligned} x &= 1 - \left(\frac{1 - X^{1/3}}{1 + 2X^{1/3}} \right)^3, \\ z &= Z(1 + 2X^{1/3}), \\ X &= \left(\frac{1 - (1 - x)^{1/3}}{1 + 2(1 - x)^{1/3}} \right)^3, \\ Z &= \frac{z}{3}(1 + 2(1 - x)^{1/3}). \end{aligned}$$

PROOF: From Lemma 2.1 we have

$$\begin{aligned} 1 - x &= 1 - \frac{c(q)^3}{a(q)^3} \\ &= \frac{b(q)^3}{a(q)^3} \\ &= \left(\frac{a(q^3) - c(q^3)}{a(q^3) + 2c(q^3)} \right)^3 \\ &= \left(\frac{1 - X^{1/3}}{1 + 2X^{1/3}} \right)^3. \end{aligned}$$

This proves the first part. Similarly,

$$\begin{aligned} z &= a(q) \\ &= a(q^3) + 2c(q^3) \\ &= a(q^3) \left(1 + 2 \frac{c(q^3)}{a(q^3)} \right) \\ &= Z(1 + 2X^{1/3}). \end{aligned}$$

This proves the second part. The third and fourth parts are obtained by rearranging the first two parts and solving for X and Z . □

REMARK 2.10. The first two formulas in Lemma 2.9 are called the trimidiation formulas, and the last two are called the triplication formulas. See [2, pp. 101–102] for another proof and further explanation.

PROOF OF THEOREM 1.1: Using Lemmas 2.1 and 2.9, we have

$$\begin{aligned}
 1 + 3 \sum_{\substack{n=1 \\ 9 \nmid n}}^{\infty} \frac{nq^n}{1 - q^n} &= \frac{1}{4} (a(q)^2 + 3a(q^3)^2) \\
 &= \frac{1}{4} \left(z^2 + \frac{z^2}{3} (1 + 2(1 - x)^{1/3})^2 \right) \\
 &= \frac{z^2}{3} (1 + (1 - x)^{1/3} + (1 - x)^{2/3}) \\
 &= \frac{z^2 x}{3(1 - (1 - x)^{1/3})} \\
 &= \frac{c(q)^3}{3a(q)(1 - b(q)/a(q))} \\
 &= \frac{c(q)^3}{3(a(q) - b(q))} \\
 &= \frac{c(q)^3}{9c(q^3)} \\
 &= \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^{10}}{(1 - q^n)^3 (1 - q^{9n})^3}.
 \end{aligned}$$

□

3. A DEFINITE INTEGRAL

In this section we state and prove the value of an interesting definite integral. We use the same method of proof as Fine [10, pp. 86–91], who gave three similar integrals.

THEOREM 3.1.

$$\int_0^{e^{-2\pi/3}} \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^{10}}{(1 - q^n)^6} dq = \frac{1}{3\sqrt{3}}.$$

PROOF: From Theorem 1.1 we have

$$\prod_{n=1}^{\infty} \frac{(1 - q^{3n})^{10}}{(1 - q^n)^3 (1 - q^{9n})^3} = 1 + 3 \sum_{\substack{n=1 \\ 9 \nmid n}}^{\infty} \frac{nq^n}{1 - q^n}.$$

If we multiply by $\prod_{n=1}^{\infty} \frac{(1 - q^{9n})^3}{(1 - q^n)^3}$, we get

$$\begin{aligned}
 \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^{10}}{(1 - q^n)^6} &= \prod_{n=1}^{\infty} \frac{(1 - q^{9n})^3}{(1 - q^n)^3} \left\{ 1 + 3 \sum_{\substack{n=1 \\ 9 \nmid n}}^{\infty} \frac{nq^n}{1 - q^n} \right\} \\
 &= \frac{d}{dq} \left\{ q \prod_{n=1}^{\infty} \frac{(1 - q^{9n})^3}{(1 - q^n)^3} \right\},
 \end{aligned}$$

or equivalently

$$(3.2) \quad \int_0^q \prod_{n=1}^{\infty} \frac{(1-s^{3n})^{10}}{(1-s^n)^6} ds = q \prod_{n=1}^{\infty} \frac{(1-q^{9n})^3}{(1-q^n)^3}.$$

Recall the modular transformation for the Dedekind eta function, for example, see [8, Theorem 4.11], which may be written in the form

$$q^{1/24} \prod_{n=1}^{\infty} (1-q^n) = \frac{1}{\sqrt{t}} p^{1/24} \prod_{n=1}^{\infty} (1-p^n),$$

where $q = e^{-2\pi t}$, $p = e^{-2\pi/t}$, $\operatorname{Re}(t) > 0$. If we take $t = 1/3$, then $p = q^9$, and so in this case the modular transformation implies

$$\begin{aligned} q \prod_{n=1}^{\infty} \frac{(1-q^{9n})^3}{(1-q^n)^3} &= q \prod_{n=1}^{\infty} \frac{(1-p^n)^3}{(1-q^n)^3} \\ &= q \left(\frac{q}{p}\right)^{1/8} (\sqrt{t})^3 \\ &= t^{3/2} \\ &= \frac{1}{3\sqrt{3}}. \end{aligned}$$

Using this in (3.2) we complete the proof. □

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