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The Symplectic Geometry of Polygons in the 3-Sphere

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Abstract. We study the symplectic geometry of the moduli spaces $M_r = M_r(\mathbb{S}^3)$ of closed n-gons with fixed side-lengths in the 3-sphere. We prove that these moduli spaces have symplectic structures obtained by reduction of the fusion product of n conjugacy classes in $\mathrm{SU}(2)$ by the diagonal conjugation action of $\mathrm{SU}(2)$. Here the fusion product of n conjugacy classes is a Hamiltonian quasi-Poisson $\mathrm{SU}(2)$-manifold in the sense of [AKSM]. An integrable Hamiltonian system is constructed on M_r in which the Hamiltonian flows are given by bending polygons along a maximal collection of nonintersecting diagonals. Finally, we show the symplectic structure on M_r relates to the symplectic structure obtained from gauge-theoretic description of M_r. The results of this paper are analogues for the 3-sphere of results obtained for $M_r(\mathbb{H}^3)$, the moduli space of n-gons with fixed side-lengths in hyperbolic 3-space [KMT], and for $M_r(\mathbb{E}^3)$, the moduli space of n-gons with fixed side-lengths in $\mathbb{E}^3$ [KM1].

1 Introduction

This paper is an analogue to [KM1] and [KMT] which studied the symplectic geometry of moduli spaces of polygonal linkages with fixed side-lengths in Euclidean 3-space and hyperbolic 3-space, respectively. We obtain the moduli space of polygonal linkages with fixed side-lengths in the 3-sphere, $\mathbb{S}^3$, by the reduction of a nondegenerate Hamiltonian quasi-Poisson $\mathrm{SU}(2)$-manifold in the sense of [AKSM].

We will use the following definitions throughout this paper. An (open) n-gon P in $\mathbb{S}^3$ is an ordered $(n+1)$-tuple of points in $\mathbb{S}^3 \subset \mathbb{C}^2$, $P = [y_1, \dots, y_{n+1}]$, called the *vertices*. We join the vertex y_i to the vertex y_{i+1} by a shortest geodesic segment e_i, called the *i-th edge*. This puts the restriction on the length of $e_i \leq \pi$ for all $1 \leq i \leq n$. Let $\mathrm{Pol}_n(\mathbb{S}^3)$ denote the space of n-gons in $\mathbb{S}^3$.

An n-gon is said to be *closed* if $y_{n+1} = y_1$. We let $\mathrm{CPol}_n(\mathbb{S}^3)$ denote the space of closed n-gons in $\mathbb{S}^3$. Let $\mathrm{Isom}_+(\mathbb{S}^3)$ denote the group of orientation preserving isometries of $\mathbb{S}^3$. There exists a natural (diagonal) action of $\mathrm{Isom}_+(\mathbb{S}^3)$ on $\mathrm{Pol}_n(\mathbb{S}^3)$ by

$$g \cdot [y_1, \dots, y_n] = [g \cdot y_1, \dots, g \cdot y_{n+1}].$$

Two n-gons $P = [y_1, \dots, y_{n+1}]$ and $P' = [y'_1, \dots, y'_{n+1}]$ are said to be *equivalent* if there exists $g \in \mathrm{Isom}_+(\mathbb{S}^3)$ such that $g \cdot P = P'$.

Fix an n-tuple of strictly positive real numbers $r = (r_1, \dots, r_n) \in \mathbb{R}^n_+$. Denote the space of open n-gons with fixed side-lengths, $d(y_i, y_{i+1}) = r_i$, by $N_r(\mathbb{S}^3)$. Let $\mathrm{CN}_r(\mathbb{S}^3) = N_r(\mathbb{S}^3) \cap \mathrm{CPol}_n(\mathbb{S}^3)$, the space of closed polygons with fixed side-lengths. Finally, let $M_r(\mathbb{S}^3) = \mathrm{CN}_r(\mathbb{S}^3) / \mathrm{Isom}_+(\mathbb{S}^3)$. We study $M_r(\mathbb{S}^3)$, the moduli space

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closed n-gons with fixed side-lengths in $\mathbb{S}^3$ by the group of orientation preserving isometries.

Fix $* \in \mathbb{S}^3$. Denote by $\operatorname{Rot}(\mathbb{S}^3, *) \subset \operatorname{Isom}_+(\mathbb{S}^3)$ the group of rotations of $\mathbb{S}^3$ fixing $*$. Let $\operatorname{Pol}_n(\mathbb{S}^3, *)$ denote the space of n-gons in $\mathbb{S}^3$ such that $y_1 = *$. Let $\operatorname{CPol}_n(\mathbb{S}^3, *) = \operatorname{CPol}_n(\mathbb{S}^3) \cap \operatorname{Pol}_n(\mathbb{S}^3, *)$, $N_r(\mathbb{S}^3, *) = \operatorname{Pol}_n(\mathbb{S}^3, *) \cap N_r(\mathbb{S}^3)$, and $\operatorname{CN}_r(\mathbb{S}^3, *) = \operatorname{CPol}_n(\mathbb{S}^3) \cap N_r(\mathbb{S}^3, *)$.

It is easy to see the space $M_r(\mathbb{S}^3)$ can be identified with $\operatorname{CN}_r(\mathbb{S}^3, *)/\operatorname{Rot}(\mathbb{S}^3, *)$, the moduli space of closed, based n-gons with fixed side-lengths in $\mathbb{S}^3$ by the group of rotations about the first vertex $*$.

The group of orientations preserving isometries is given by $\operatorname{Isom}_+(\mathbb{S}^3) = \big(\operatorname{SU}(2) \times \operatorname{SU}(2)\big)/\{\pm I\}$. The group of rotations fixing the north and south poles is the diagonal subgroup, $K \simeq P\operatorname{SU}(2)$, and translations are given by $\operatorname{Isom}_+(\mathbb{S}^3)/K$ which we identify with $\operatorname{SU}(2)$.

In this paper, a symplectic structure is obtained on $M_r(\mathbb{S}^3)$ by reduction of a Hamiltonian quasi-Poisson $\operatorname{SU}(2)$-manifold. We are interested in finding an integrable system on $M_r(\mathbb{S}^3)$. Denote by d_{ij} a shortest geodesic segment connecting the vertices y_i and y_j (we assume $i < j$), which we call a diagonal. Let ℓ_{ij} be the length of the diagonal d_{ij}. Then ℓ_{ij} is a continuous function on $M_r(\mathbb{S}^3)$, but it is not smooth when either $\ell_{ij} = 0$ or $\ell_{ij} = \pi$. If d_{ij} and d_{km} are nonintersecting diagonals, then

$$\{\ell_{ij}, \ell_{km}\} = 0.$$

By considering a maximal collection of nonintersecting diagonals, we obtain $\frac{1}{2}\dim\big(M_r(\mathbb{S}^3)\big) = 2(n-3)$ Poisson commuting Hamiltonians.

The Hamiltonian flow Ψ^t_{ij} associated to a ℓ_{ij} has the following nice description. Separate the polygon into two pieces via the diagonal d_{ij}. The Hamiltonian flow is given by leaving one piece fixed while rotating the other piece about the diagonal at constant angular velocity with period 2π. The flow Ψ^t_{ij} is called the *bending flow* along the diagonal d_{ij} and defines a $\mathbb{T}^{n-3}$-action on $M_r(\mathbb{S}^3)$.

The paper is organized as follows:

In Section 2, we give the Lie group description of spherical polygons.

In Section 3, we give criteria for the moduli space $M_r(\mathbb{S}^3)$ to be smooth and nonempty.

In Section 4, we give the necessary background material on quasi-Poisson manifolds.

In Section 5, we obtain a symplectic structure on $M_r(\mathbb{S}_3)$ and study the Hamiltonians ℓ_{ij} and their associated Hamiltonian flows.

In Section 6, we obtain the an action of the pure braid group on $M_r(\mathbb{S}^3)$ given by the time 1 Hamiltonian flows of a certain family of functions.

In Section 7, we relate the symplectic form on $M_r(\mathbb{S}^3)$ to symplectic form given on the relative character varieties on n-punctured 2-spheres.

We note that the moduli spaces of polygons in the 3-spaces of constant curvature give examples of completely integrable systems obtained from the theory of Manin pairs associated to a compact simple Lie group [AMM2]. The Manin pairs corresponding to the various moduli spaces are:

- $\big(\mathrm{su}(2) \ltimes \mathrm{su}(2)^*, \mathrm{su}(2)\big)$ for polygons in the zero curvature space (Lie-Poisson theory);
- $\big(\mathrm{sl}_2(\mathbb{C}) = \mathrm{su}(2)^{\mathbb{C}}, \mathrm{su}(2)\big)$ for polygons in negative curvature space (Poisson-Lie theory);
- $\big(\mathrm{su}(2) \oplus \mathrm{su}(2), \mathrm{su}(2)\big)$ for polygons in positive curvature space (quasi-Poisson Lie theory).

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2 Lie Group Construction of Spherical Polygons

We identify $\mathbb{S}^3 \subset \mathbb{C}^2$ with $\mathrm{SU}(2)$ by the map which takes (z_1, z_2) to $\begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix}$. Fix $* \in \mathbb{S}^3$ to be the north pole of $\mathbb{S}^3$, $* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The group of orientation preserving isometries of $\mathbb{S}^3$, $\mathrm{Isom}_+(\mathbb{S}^3)$, is given by $G = \big(\mathrm{SU}(2) \times \mathrm{SU}(2)\big)/\{\pm I\}$ with the action of G on $\mathbb{S}^3$ given by

$$G \times \mathbb{S}^3 \to \mathbb{S}^3$$
$$\big((k_1, k_2), x\big) \mapsto k_1 x k_2^{-1}.$$

The diagonal subgroup, $K \simeq P\mathrm{SU}(2)$, of G acts as the group of rotations on $\mathbb{S}^3$ fixing the north and south poles. Translations are then given by G/K, which we identify with $\mathrm{SU}(2)$ by the map

$$G/K \to \mathrm{SU}(2)$$
$$(k_1, k_2) \mapsto k_1 k_2^{-1}.$$

Recall the definitions of the various polygon spaces given in Section 1. We have a diffeomorphism from n copies of $\mathrm{SU}(2)$ to $\mathrm{Pol}_n(\mathbb{S}^3, *)$, the space of n-gons in $\mathbb{S}^3$ based at the point $*$ given by

$$\Phi\colon\ \mathrm{SU}(2)^n \to \mathrm{Pol}_n(\mathbb{S}^3, *)$$
$$(k_1, k_2, \dots, k_n) \mapsto [*, k_1 *, k_1 k_2 *, \dots, k_1 k_2 \cdots k_n *].$$

The condition for a polygon to be closed is $k_1 k_2 \cdots k_n = I$. The map Φ restricts to a diffeomorphism

$$\Phi\colon \{(k_1, \dots, k_n) \in \mathrm{SU}(2)^n : k_1 \cdots k_n = I\} \to \mathrm{CPol}_n(\mathbb{S}^3, *).$$

It is easily seen that the map Φ is K-equivariant where the action on $\mathrm{SU}(2)^n$ is given by diagonal conjugation and and on $\mathrm{Pol}_n(\mathbb{S}^3, *)$ by the natural (diagonal) action.

We next see that fixing side-lengths for a polygon corresponds to restricting to conjugacy classes, $\mathcal{C}_\lambda \subset G$. Let $k, k' \in \mathcal{C}_\lambda$. If there exists $g \in K$ so that $g \cdot k = k'$, then

$$d(k \cdot *, *) = d(g \cdot k*, g \cdot *) = d(k' \cdot *, *)$$

We have the following lemma.

Lemma 2.1 *The map Φ induces a K-equivariant diffeomorphism between $\prod_{i=1}^n \mathcal{C}_{\lambda_i}$ and $N_r(\mathbb{S}^3, *)$, the configuration space of open based n-gon linkages with fixed side-lengths $r = (r_1, r_2, \dots, r_n) \in \mathbb{R}_+^n$, where $r_i = d(k_i*, *)$ for $k_i \in \mathcal{C}_{\lambda_i}$, $1 \le i \le n$.*

We now have the identification $\{(k_1, \dots, k_n) \in \prod_{i=1}^n \mathcal{C}_{\lambda_i} : k_1 \cdots k_n = I\}/\operatorname{SU}(2)$ with $M_r(\mathbb{S}^3)$ by Φ.

3 Criteria For Smoothness and Nonemptiness

In this section we give necessary and sufficient conditions for the moduli space $M_r(\mathbb{S}^3)$ to be nonempty and sufficient conditions for $M_r(\mathbb{S}^3)$ to be a smooth manifold.

Let $\Pi\colon \operatorname{CPol}_n(\mathbb{S}^3) \to \mathbb{R}_+^n$ be the map that assigns to an n-gon P its set of side-lengths, $\Pi(P) = (r_1, \dots, r_n)$ with $r_i = d(x_i, x_{i+1})$, $1 \le i \le n$. Let $I \subset \{1, 2, \dots, n\}$, $\bar{I}$ denote the complement of I, $|I|$ denote the cardinality of I, and $r_I = \sum_{i \in I} r_i$.

Lemma 3.1 *The image of Π is the closed polyhedron D_n defined by the inequalities*

$$0 \le r_i \le \pi,\ 1 \le i \le n, \quad \textit{and}$$
$$r_I \le r_{\bar{I}} + (|I| - 1)\pi,\ I \subset \{1, 2, \dots, n\}, \quad \textit{with } |I| \textit{ odd}.$$

Proof The proof for n-gons in $\mathbb{S}^2$ was given by Galitzer [Ga]. Since any n-gon, P, in $\mathbb{S}^m$ can obtained from a finite number of bends along diagonals of an n-gon, P', in $\mathbb{S}^2 \subset \mathbb{S}^n$, these inequalities hold for all $n \ge 2$. ∎

We next give sufficient conditions for $M_r(\mathbb{S}^3)$ to be a smooth manifold. We will use two results and the notation from Section 5.1 (the reader will check that no circular reasoning is involved here). By Theorem 5.2 we find that $M_r(\mathbb{S}^3)$ is a symplectic manifold obtained by the reduction of a non-degenerate Hamiltonian quasi-Poisson manifold,

$$M_r(\mathbb{S}^3) \cong (\mu|_{N_r(\mathbb{S}^3, *)})^{-1}(1)/\operatorname{SU}(2).$$

By Lemma 5.1, 1 is a regular value of μ unless there exists $P \in \operatorname{CN}_r(\mathbb{S}^3, *)$ such that the infinitesimal isotropy $\mathrm{su}_2|_P = \{X \in \mathrm{su}_2 : X_{\operatorname{CN}_r(\mathbb{S}^3, *)}(P) = 0\}$ is nonzero.

Definition 3.2 An n-gon P is degenerate if it is contained in a geodesic.

We now have the following lemma due to Galitzer [Ga], also see [KM3].

Lemma 3.3 *$M_r(\mathbb{S}^3)$ is singular only if there exists a partition $\{1,\dots,n\} = I \amalg J$ with $\#(I) > 1, \#(J) > 1$ and $m \in \mathbb{Z}$ such that*

$$\sum_{i \in I} r_i = \sum_{j \in J} r_J + 2m\pi.$$

Proof Clearly $\mathrm{su}_2|_P = 0$ unless P is degenerate. But if P is degenerate there exists a partition $\{1,\dots,n\} = I \amalg J$ as above (I corresponds to the back-tracks and J to the forward-tracks of P). ■

Remark 3.4 In the terminology of [KM1], $M_r(\mathbb{S}^3)$ is smooth unless r is on a wall of D_n.

4 Quasi-Poisson Manifolds

We have seen that $N_r(\mathbb{S}^3, *)$ can be identified with the product of n conjugacy classes, $\prod_{i=1}^{n} \mathcal{C}_{\lambda_i}$. In this section, we introduce the machinery needed to construct quasi-Poisson bivectors on these spaces. We begin by reviewing basic definitions and results for quasi-Poisson K-spaces. For a complete treatment of quasi-Poisson manifolds see [AKS] and [AKSM].

4.1 Basic Definitions and the Moment Map

Let K be a Lie group whose Lie algebra $\mathfrak{k}$ is equipped with an invariant nondegenerate bilinear form. Let $\{e_i\}$ be an orthonormal basis with respect to the bilinear form on $\mathfrak{k}$. We define $\varphi \in \wedge^3\mathfrak{g}$ by

$$\varphi = \sum f_{ij}^{k} e_i \wedge e_j \wedge e_k,$$

where $[e_i, e_j] = \sum_k f_{ij}^k e_k$.

We denote by the subscript M, x_M, the vector field (resp. multivector field) on M induced by the action of K on M and $x \in \mathfrak{k}$ (resp. $x \in \wedge^j\mathfrak{k}$) satisfying

$$(x_M f)(m) = \frac{d}{dt}\Big|_{t=0} f\big(\exp(-tx)\cdot m\big) \tag{1}$$

where $f \in C^\infty(M)$ and $m \in M$. This is a Lie algebra homomorphism, *i.e.* $[x_M, y_M] = [x, y]_M$ for $x, y \in \mathfrak{k}$.

Definition 4.1 A quasi-Poisson manifold is a K-manifold M, equipped with an invariant bivector field $\pi_M \in C^\infty(M, \wedge^2 TM)$ such that the Schouten bracket of π_M is the invariant trivector field,

$$[\pi_M, \pi_M] = \varphi_M.$$

We next define the notion of a K-valued moment map.

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Definition 4.2 An Ad-invariant map $\mu\colon M \to K$ is called a *moment map* for a quasi-Poisson K-manifold (M, π_M) if

$$\pi_M^\sharp\big(d(\mu^* f)\big) = (\mu^*(\frac{1}{2}(e^\lambda + e^\rho)f)_M,$$

for all functions $f \in C^\infty(K)$. The triple (M, π_M, μ) is then called a *Hamiltonian quasi-Poisson K-manifold.*

The following lemma gives us the formulation of the moment map most useful for this paper.

Lemma 4.3 *Let (M, π_M) be a quasi-Poisson K-manifold. An* Ad*-equivariant map $\mu\colon M \to K$ is a moment map if and only if*

$$\pi_M^\sharp(\mu^*\langle\theta, X\rangle) = \frac{1}{2}\Big((1 + \mathrm{Ad}_{\mu^{-1}})X\Big)_M,$$

for all $X \in \mathfrak{k}$ and $\theta = k^{-1}dk$ the left-invariant Maurer-Cartan form on K.

Proof See [AKSM]. ■

Although the Schouten bracket of the bivector field on a Hamiltonian quasi-Poisson manifold is in general a nonzero invariant trivector field, we may still define a notion of reduction to obtain a symplectic manifold.

Lemma 4.4 **(quasi-Poisson reduction)** *Let (M, π_M, μ) be a non-degenerate Hamiltonian quasi-Poisson manifold. Let M_* be the subset of M on which the K-action is free. Let $1 \in K$ be a regular value for $\mu\colon M \to K$. Then intersection of $\mu^{-1}(1)/K$ with M_*/K is a symplectic submanifold.*

Proof See [AKSM]. ■

4.2 Conjugacy Classes as Quasi-Poisson Manifolds

The basic example of a Hamiltonian quasi-Poisson K-manifold is (K, π_K, μ) where the action is given by conjugation, the moment map $\mu = id_K$ is the identity map on K, and the bivector π_K is given by

$$\pi_K(k) = \frac{1}{2}\sum_i dR_k e_i \wedge dL_k e_i.$$

The bivector π_K restricts to a nondegenerate quasi-Poisson bivector on conjugacy classes $\mathcal{C} \subset K$. The triple $(\mathcal{C}, \pi_K|_{\mathcal{C}}, \mu|_{\mathcal{C}})$ is a Hamiltonian quasi-Poisson K-manifold.

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4.3 Fusion Product of Quasi-Poisson Manifolds

Given Hamiltonian quasi-Poisson K-manifolds (M_1, π_1, μ_1) and (M_2, π_2, μ_2), it is not true that $M_1 \times M_2$ with the product bivector is a Hamiltonian quasi-Poisson K-space for the diagonal action of K on $M_1 \times M_2$. We must construct a new bivector, π_{fus}, on $M_1 \times M_2$ for the diagonal action to be a quasi-Poisson action. $M_1 \times M_2$ with this bivector is called the *fusion product* and denoted $M_1 \circledast M_2$. This construction is due to [AKSM].

As defined in Section 4.1, the subscript M denotes the vector field, or multivector field, induced by the action of K on M. We define $\psi \in \wedge(\mathfrak{k} \oplus \mathfrak{k})$ to be

$$\psi = \frac{1}{2} \sum_i e_i^1 \wedge e_i^2$$

where $\{e_i\}$ is a basis of $\mathfrak{k}$ and the superscripts refer to the respective $\mathfrak{k}$-summand.

Proposition 4.5 *Let (M, π) be a quasi-Poisson $K \times K \times H$-manifold. Then*

$$\pi_{\text{fus}} = \pi - \psi_M$$

defines a quasi-Poisson structure on M for the diagonal $K \times H$-action. Moreover, if $(\mu_1, \mu_2, \mu_H)\colon M \to K \times K \times H$ is a moment map for the action, then the point-wise product $(\mu_1\mu_2, \mu_H)$ is a moment map for the diagonal $K \times H$-action.

Proof See [AKSM, Proposition 5.1].

In the previous section, we showed a conjugacy class $\mathcal{C} \subset K$ was a Hamiltonian quasi-Poisson manifold. For this paper, the Hamiltonian quasi-Poisson spaces we are most interested in are the fusion products of n conjugacy classes in K.

Example 4.6 Let $(\mathcal{C}_{\lambda_1}, \pi_1, \mu_1)$ and $(\mathcal{C}_{\lambda_2}, \pi_2, \mu_2)$ be conjugacy classes in K. Then $\mathcal{C}_{\lambda_1} \times \mathcal{C}_{\lambda_2}$ with the the bivector

$$\pi_{\text{fus}}(k_1, k_2) = \pi_1(k_1) + \pi_2(k_2) - \sum_i (dL_{k_1} e_i^1 - dR_{k_1} e_i^1) \wedge (dL_{k_2} e_i^2 - dR_{k_2} e_i^2),$$

where the superscripts denote the conjugacy class on which e_i acts, is a Hamiltonian quasi-Poisson K-space where the action is given by diagonal conjugation, $k \cdot (k_1, k_2) = (kk_1k^{-1}, kk_2k^{-1})$. The moment map associated to this action is the product $\mu(k_1, k_2) = k_1k_2$.

We are now in the position to give a formula for quasi-Poisson bivector on the product of n conjugacy classes given by fusion. The fusion product $\circledast_{i=1}^n C_{\lambda_i}$ is a Hamiltonian quasi-Poisson K-space with action given by the diagonal conjugation and moment map $\mu\colon \circledast_{i=1}^n C_{\lambda_i} \to K$ given by the product, $\mu(k_1, k_2, \dots, k_n) = k_1k_2 \cdots k_n$. The quasi-Poisson bivector on this space is given by

$$\pi_{\text{fus}} = \frac{1}{2} \sum_i \sum_l (dR_{k_i} e_l^i \wedge dL_{k_i} e_l^i) - \frac{1}{2} \sum_{i<j} \sum_l (dL_{k_i} e_l^i - dR_{k_i} e_l^i) \wedge (dL_{k_j} e_l^j - dR_{k_j} e_l^j)$$

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where the superscripts i, j denote the conjugacy class $\mathcal{C}_{\lambda_i}, \mathcal{C}_{\lambda_j} \subset \circledast_{i=1}^n \mathcal{C}_{\lambda_i}$ on which $e_l \in \mathfrak{k}$ acts.

The following remark from [AKS, Example 5.5.4] gives the nondegeneracy of $\circledast_{i=1}^n C_{\lambda_i}$.

Remark 4.7 Let (M, π, μ) be a Hamiltonian quasi-Poisson K space. Then (M, π, μ) is nondegenerate if and only if for each $m \in M$,

$$\ker\big(\pi^\sharp(m)\big) = \{\mu^*(x, \theta) : x \in \ker(1 + \mathrm{Ad}_{\mu(m)})\}$$

where $x \in \mathfrak{k}$.

4.4 Poisson Bracket on $C^\infty(\circledast_{i=1}^n \mathcal{C}_{\lambda_i})^K$

For a general quasi-Poisson manifold (M, π_M), we can define a bracket on $C^\infty(M)$ by $\{\psi_1, \psi_2\} = \pi_M(d\psi_1, d\psi_2)$, where $\psi_1, \psi_2 \in C^\infty(M)$. This bracket is not, in general, a Poisson bracket. This is because the Schouten bracket of π_M is an invariant trivector field, φ_M, not necessarily zero. The bracket does however define a Poisson bracket when we restrict to $C^\infty(M)^K$, the smooth K-invariant functions on M.

We will next find a formula for the Poisson bracket on the K-invariant functions on the fusion product of n conjugacy classes in K. Let $\mathfrak{k}_i \subset \bigoplus_{i=1}^n \mathfrak{k}$ denote the i-th summand, $\nu_i \in \mathfrak{k}_i$, and $k = (k_1, \dots, k_n) \in \circledast_{i=1}^n \mathcal{C}_{\lambda_i}$. For $\psi \in C^\infty(\circledast_{i=1}^n \mathcal{C}_{\lambda_i})$ we define

$$D_i\psi \colon \circledast_{i=1}^n \mathcal{C}_{\lambda_i} \to \mathfrak{k}_i, \quad D_i'\psi \colon \circledast_{i=1}^n \mathcal{C}_{\lambda_i} \to \mathfrak{k}_i$$

by

$$\langle D_i\psi(k), \nu_i\rangle = \frac{d}{dt}\Big|_{t=0} \psi(k_1, \dots, e^{t\nu_i}k_i, \dots, k_n)$$

$$\langle D_i'\psi(k), \nu_i\rangle = \frac{d}{dt}\Big|_{t=0} \psi(k_1, \dots, k_i e^{t\nu_i}, \dots, k_n).$$

Here $\langle\,,\rangle$ is the nondegenerate invariant bilinear form on $\mathfrak{k}$ extended to $\bigoplus_{i=1}^n \mathfrak{k}$ by $\langle x, y\rangle = \sum_{i=1}^n \langle x_i, y_i\rangle$ for $x = (x_1, \dots, x_n) \in \bigoplus_{i=1}^n \mathfrak{k}$ and $y = (y_1, \dots, y_n) \in \bigoplus_{i=1}^n \mathfrak{k}$.

Lemma 4.8

$$\mathrm{Ad}_{k_i} D_i'\psi(k) = D_i\psi(k).$$

Proof

$$\begin{aligned}
\langle \mathrm{Ad}_{k_i} D_i'\psi(k), \nu_i\rangle &= \langle D_i'\psi(k), \mathrm{Ad}_{k_i^{-1}} \nu_i\rangle \\
&= \frac{d}{dt}\Big|_{t=0} \psi(k_1, \dots, k_i \,\mathrm{Ad}_{k_i^{-1}} e^{t\nu_i}, \dots, k_n) \\
&= \frac{d}{dt}\Big|_{t=0} \psi(k_1, \dots, e^{t\nu_i}k_i, \dots, k_n) \\
&= \langle D_i\psi(k), \nu_i\rangle.
\end{aligned}$$

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We also define

$$\Psi_j(k) = \sum_{i=1}^{j-1} [D_i\psi(k) - D'_i\psi(k)] + D_j\psi(k)$$

We now give a formula for the Poisson bracket on $C^\infty(\circledast_{i=1}^n \mathcal{C}_{\lambda_i})^K$.

Proposition 4.9 *Let $\phi, \psi \in C^\infty(\circledast_{i=1}^n \mathcal{C}_{\lambda_i})^K$ then*

$$\{\phi, \psi\}(k) = \sum_{j=1}^{n} \langle D'_j\phi(k) - D_j\phi(k), \Psi_j(k)\rangle.$$

Proof Let us first note that $\sum_l \langle x, e_l\rangle\langle y, e_l\rangle = -\langle x, y\rangle$ for $x, y \in \mathfrak{k}$ and $\{e_l\}$ an orthonormal basis of $\mathfrak{k}$. Now,

$$\begin{aligned}
\{\phi, \psi\}(k) &= \pi_{\text{fus}}(d\phi, d\psi) \\
&= \frac{1}{2}\sum_i \sum_l (dR_{k_i}e^i_l \wedge dL_{k_i}e^i_l)(d\phi, d\psi) \\
&\quad - \frac{1}{2}\sum_{i<j}^{n} \sum_l \big((dL_{k_i}e^i_l - dR_{k_i}e^i_l) \wedge (dL_{k_j}e^j_l - dR_{k_j}e^j_l)\big)(d\phi, d\psi) \\
&= \frac{1}{2}\sum_i \sum_l \langle D_i\phi, e^i_l\rangle\langle D'_i\psi, e^i_l\rangle - \langle D'_i\phi, e^i_l\rangle\langle D_i\psi, e^i_l\rangle \\
&\quad - \frac{1}{2}\sum_{i<j} \sum_l \langle D'_i\phi - D_i\phi, e^i_l\rangle\langle D'_j\psi - D_j\psi, e^j_l\rangle \\
&\quad + \frac{1}{2}\sum_{i<j} \sum_l \langle D'_j\phi - D_j\phi, e^j_l\rangle\langle D'_i\psi - D_i\psi, e^i_l\rangle \\
&= \frac{1}{2}\sum_i \langle D'_i\phi, D_i\psi\rangle - \langle D_i\phi, D'_i\psi\rangle \\
&\quad + \frac{1}{2}\sum_{i<j} \langle D'_i\phi - D_i\phi, D'_j\psi - D_j\psi\rangle - \langle D'_j\phi - D_j\phi, D'_i\psi - D_i\psi\rangle \\
&= \frac{1}{2}\sum_i \langle D'_i\phi, D_i\psi\rangle - \langle D_i\phi, D'_i\psi\rangle \\
&\quad + \frac{1}{2}\sum_{i<j} \langle D'_i\phi + D_i\phi, D'_j\psi - D_j\psi\rangle - \sum_{i>j} \langle D'_i\phi - D_i\phi, D'_j\psi - D_j\psi\rangle.
\end{aligned}$$

But since $\psi \in C^\infty(\circledast_{i=1}^n \mathcal{C}_{\lambda_i})^K$ is K-invariant, a quick calculation shows

$$\sum_i [D_i\psi - D'_i\psi] = 0.$$

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Using this fact and also that $\langle D_i'\phi, D_i'\psi\rangle = \langle D_i\phi, D_i\psi\rangle$ for all i, we can rewrite the above as,

$$\begin{aligned}\{\phi,\psi\} &= \frac{1}{2}\sum_i \langle D_i'\phi - D_i\phi, D_i\psi + D_i'\psi\rangle \\ &\quad - \frac{1}{2}\sum_{i\geq j}\langle D_i'\phi - D_i\phi, D_j'\psi - D_j\psi\rangle - \frac{1}{2}\sum_{i>j}\langle D_i'\phi - D_i\phi, D_j'\psi - D_j\psi\rangle \\ &= \sum_i \langle D_i'\varphi - D_i\varphi, \Psi_i\rangle. \qquad \blacksquare\end{aligned}$$

From the above proposition we can also define the Hamiltonian vector field X_ψ associated to $\psi \in C^\infty(\circledast_{i=1}^n \mathcal{C}_{\lambda_i})^K$ by $X_\psi = \pi^\sharp(d\psi)$.

Corollary 4.10 *The Hamiltonian vector field $X_\psi(k) = \big(X_1(k),\dots,X_n(k)\big)$ associated to the K-invariant function $\psi \in C^\infty(\circledast_{i=1}^n \mathcal{C}_{\lambda_i})^K$ is given by*

$$X_j(k) = dL_{k_j}\Psi_j - dR_{k_j}\Psi_j, \quad 1 \leq j \leq n.$$

Proof We use the convention $\{\phi,\psi\} = d\phi(X_\psi) = \sum_{j=1}^n d_j\phi\big(X_j(k)\big)$. Proposition 4.9 gives us

$$\begin{aligned} d\phi\big(X_\psi(k)\big) &= \{\phi,\psi\} \\ &= \sum_{j=1}^n \langle D_j'\phi - D_j\phi, \Psi_j\rangle \\ &= \sum_{j=1}^n d_j\phi(dL_{k_j}\Psi_j) - d_j\phi(dR_{k_j}\Psi_j) \\ &= \sum_{j=1}^n d_j\phi(dL_{k_j}\Psi_j - dR_{k_j}\Psi_j). \qquad \blacksquare\end{aligned}$$

5 Integrable Systems on $M_r(\mathbb{S}^3)$

We restrict to the case which gives rise to $M_r(\mathbb{S}^3)$, that is $K = \mathrm{SU}(2)$ and $\langle\,,\,\rangle = -\frac{1}{2}\,\mathrm{Trace}()$, although most of the results of this section follow for $K = \mathrm{SU}(n)$ and $\langle\,,\,\rangle$ the Killing form.

5.1 $M_r(\mathbb{S}^3)$ as a Symplectic Manifold

In Section 2, we constructed the K-equivariant diffeomorphism $\Phi\colon \prod_{i=1}^n \mathcal{C}_{\lambda_i} \to N_r(\mathbb{S}^3,*)$, defined by

$$\Phi(k_1,\dots,k_n) = [*, k_1*, k_1k_2*, \dots, k_1k_2\cdots k_n*].$$

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In Section 4.4, we constructed a nondegenerate quasi-Poisson structure on $\prod_{i=1}^n \mathcal{C}_{\lambda_i}$ with moment map associated to diagonal conjugation given by $\mu(k_1, k_2, \dots, k_n) = k_1 k_2 \cdots k_n$. By Lemma 4.4, if 1 is a regular value of μ we obtain a symplectic structure on $\mu^{-1}(1)/K$ given by reduction.

Lemma 5.1 *1 is a regular value of μ if and only if $\{x \in \mathfrak{k} : x_{\prod_{i=1}^n \mathcal{C}_{\lambda_i}} = 0\} = 0$ for all $k \in \mu^{-1}(1)$.*

Proof We refer to Lemma 4.3. Let $x \in \mathfrak{k}$ and $k \in \prod_{i=1}^n \mathcal{C}_{\lambda_i}$. Then

$$\begin{aligned} x \in \left(\operatorname{Im}(d\mu|_k)\right)^{\perp} &\Leftrightarrow \mu^*\langle x, \theta\rangle = 0 \\ &\Leftrightarrow 0 = \pi^\sharp(\mu^*\langle x, \theta\rangle) = x_{\prod_{i=1}^n \mathcal{C}_{\lambda_i}}. \end{aligned}$$ ■

By Lemma 3.3, a polygon is said to be *degenerate* if it can be contained in a geodesic of $\mathbb{S}^3$. It follows from the above lemma that if there does not exist $k \in \mu^{-1}(1) \subset \prod_{i=1}^n \mathcal{C}_{\lambda_i}$ such that $\Phi(k)$ is a degenerate polygon, then 1 is a regular value of μ.

We can therefore construct a symplectic structure on $\mu^{-1}(1)/K$ by quasi-Poisson reduction.

Theorem 5.2 *The moduli space $M_r(\mathbb{S}^3)$ containing no degenerate polygons has a symplectic structure which is obtained from the symplectic structure on $\mu^{-1}(1)/K$ via the diffeomorphism Φ.*

In [AKSM], the authors prove the correspondence between nondegenerate quasi-Poisson K-manifolds and quasi-Hamiltonian K-manifolds in the sense of [AMM1]. In Section 7, we need a formula for the 2-form on $\prod_{i=1}^n \mathcal{C}_{\lambda_i}$ which corresponds to π_{fus}. We will relate this 2-form to the 2-form obtained from the gauge-theoretic description of $M_r(\mathbb{S}^3)$.

Remark 5.3 The 2-form on $\prod_{i=1}^n \mathcal{C}_{\lambda_i}$ which corresponds to π_{fus} is given by

$$\tilde{\omega} = \sum_{i=1}^n \omega_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=i+1}^n (\operatorname{Ad}_{k_1 \cdots k_{i-1}} \bar{\theta}_i \wedge_b \operatorname{Ad}_{k_1 \cdots k_{j-1}} \bar{\theta}_j).$$

where ω_i is the quasi-Hamiltonian 2-form on the conjugacy class $C_i \subset \mathrm{SU}(2)$, see [AMM1], and $\bar{\theta}_i$ is the right-invariant Maurer-Cartan form on $C_i \subset \mathrm{SU}(2)$. We denote by $\wedge_b$ the wedge product together with the Killing form on K.

5.2 Hamiltonian Vector Fields

Let d_i denote the diagonal connecting the 1-st vertex with the $(i+1)$-th vertex. Let ℓ_i be the function giving the length of d_i. We show that $\{\ell_i\}_{i=2}^{n-1}$ give us an integrable system on $M_r(\mathbb{S}^3)$. We first consider the functions

$$f_i(k) = \operatorname{tr}(k_1 \cdots k_i), \quad 1 \leq i \leq n.$$

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They are related to ℓ_i by

$$\ell_i = \cos^{-1}\Big(-\frac{1}{2} f_i\Big).$$

In this section we compute the Hamiltonian vector fields X_{f_i} associated to the functions f_i.

See Section 4.4 for the definition of the Poisson bracket on $C^\infty(\circledast_{i=1}^n \mathcal{C}_{\lambda_i})^K$. We leave it to the reader to verify the following lemma.

Lemma 5.4

$$D_{i+1} f_j(k) = D_i' f_j(k), \quad 1 \le i \le j-1$$
$$D_1 f_j(k) = D_j' f_j(k)$$

for all $1 \le j \le n$.

We define $F_j \colon \circledast_{i=1}^n \mathcal{C}_{\lambda_i} \to \mathfrak{k}$ by

$$F_j(k) = \big((k_1 \cdots k_j) - (k_1 \cdots k_j)^{-1}\big).$$

We then have the following lemma.

Lemma 5.5 $F_j(k) = -D_1 f_j(k)$

Proof For $k \in \circledast_{i=1}^n \mathcal{C}_{\lambda_i}$ and $X \in \mathfrak{k}$

$$\begin{aligned}\langle D_1 f_j(k), X\rangle &= \frac{d}{dt}\Big|_{t=0} \operatorname{tr}(e^{tX} k_1 k_2 \cdots k_j) \\ &= \operatorname{tr}(X k_1 k_2 \cdots k_j) \\ &= \operatorname{tr}(k_1 k_2 \cdots k_j X)\end{aligned}$$

but since

$$\operatorname{tr}\big((k_1 k_2 \cdots k_j)^{-1} X\big) = \operatorname{tr}\big((k_1 \cdots k_j)^* X\big) = \operatorname{tr}(X^* k_1 \cdots k_j) = -\operatorname{tr}(k_1 \cdots k_j X)$$

it follows that

$$\begin{aligned}\operatorname{tr}(k_1 k_2 \cdots k_j X) &= \frac{1}{2} \operatorname{tr}\Big(\big((k_1 k_2 \cdots k_j) - (k_1 \cdots k_j)^{-1}\big) X\Big) \\ &= \langle -(k_1 \cdots k_j) + (k_1 \cdots k_j)^{-1}, X\rangle.\end{aligned}$$

Since $-(k_1 \cdots k_j) + (k_1 \cdots k_j)^{-1} \in \mathfrak{k}$ and $\langle\, , \rangle$ is a nondegenerate bilinear form, we have $D_1 f_j(k) = -\big((k_1 \cdots k_j) - (k_1 \cdots k_j)^{-1}\big) = -F_j(k)$. ∎

We have the following formula of the Hamiltonian vector fields X_{f_i}.

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Theorem 5.6 *The Hamiltonian vector field X_{f_i} has an i-th component given by*

$$\big(X_{f_j}(k)\big)_i = dR_{k_i}F_j(k) - dL_{k_i}F_j(k), \quad 1 \le i \le j,$$

$$\big(X_{f_j}(k)\big)_i = 0, \quad j < i \le n$$

Proof Recall from Corollary 4.10 that for $\psi \in C^\infty(\circledast_{i=1}^n \mathcal{C}_{\lambda_i})^K$, $X_\psi(k)$ is given by

$$\big(X_\psi(k)\big)_i = dL_{k_i}\Psi_i(k) - dR_{k_i}\Psi_i(k)$$

where $\Psi_i(k) = D_1\psi(k) - D_1'\psi(k) + D_2\psi(k) - \cdots - D_{i-1}\psi(k) + D_i\psi(k)$. This together with Lemma 5.4 gives us

$$\big(X_{f_j}(k)\big)_i = dL_{k_i}D_1 f_j(k) - dR_{k_i}D_1 f_j(k), \quad 1 \le i \le j$$

and

$$\big(X_{f_j}(k)\big)_i = 0, \quad j < i \le n.$$

In Lemma 5.5 we obtained $-F_j(k) = D_1 f_j(k)$ completing the proof. ∎

5.3 Commuting Hamiltonians

In this section we will show the family of Hamiltonians under consideration, $\{f_j\}_{j=1}^n$, Poisson commute.

Proposition 5.7 $\{f_i, f_j\} \equiv 0$ *for all* i, j.

Proof Without loss of generality we may assume $i < j$, then by Proposition 4.9

$$\begin{aligned}\{f_i, f_j\}(k) &= \sum_{k=1}^{j} \langle D_k' f_i(k) - D_k f_i(k), F_j(k)\rangle \\ &= -\Big\langle \sum_{k=1}^{j} \big(D_k' f_i(k) - D_k f_i(k)\big), F_j(k)\Big\rangle \\ &= \langle 0, F_j(k)\rangle \\ &= 0\end{aligned}$$

Here we use $\sum_{k=1}^{i}(D_k f_j - D_k' f_j) = 0$. ∎

5.4 Hamiltonian Flow

In this section we will calculate the Hamiltonian flow, ϱ_j^t, associated to f_j. We will see that these flows are the bending flows described in the introduction. The Hamiltonian flow is the solution to the ODE

$$\frac{dk_i}{dt} = dR_{k_i}F_j(k) - dL_{k_i}F_j(k) = [F_j(k), k_i], \quad 1 \le i \le j, \tag{2}$$

$$\frac{dk_i}{dt} = 0, \quad j < i \le n.$$

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Since we are working with matrix groups, we use the matrix commutator [,] in the above equation.

Lemma 5.8 *$F_j(k)$ is invariant along solution curves of (2).*

Proof To prove the lemma, it suffices to show that $\tilde{\varrho}_j^t(k) = k_1(t)\cdots k_j(t)$ is invariant along solution curves of (2).

$$\begin{aligned}
\frac{d}{dt}\tilde{\varrho}_j^t(k) &= \frac{d}{dt}\big(k_1(t)k_2(t)\cdots k_j(t)\big)\\
&= \frac{dk_1}{dt}(t)k_2(t)\cdots k_{j-1}(t)k_j(t) + \cdots + k_1(t)k_2(t)\cdots\frac{dk_j}{dt}(t)\\
&= \big[F_j\big(k(t)\big), k_1(t)\big]\, k_2(t)\cdots k_j(t) + k_1(t)k_2(t)\cdots\big[F_j\big(k(t)\big), k_j(t)\big]\\
&= F_j\big(k(t)\big)\, k_1(t)\cdots k_j(t) - k_1(t)\cdots k_j(t)F_j\big(k(t)\big)\\
&= 0.
\end{aligned}$$

■

Lemma 5.9 *The curve* $\exp\big(tF_j(k)\big) \subset K$ *is periodic with period* $2\pi/\sqrt{4-f_j^2}$.

Proof To simplify notation, let $X = F_j(k) \in \mathfrak{k}$. Then

$$X^{-1} = -\frac{1}{\det(X)}X$$

giving us

$$X^2 = -\big(\det(X)\big)X^{-1}X = -\det(X)I.$$

So,

$$\begin{aligned}
\exp tX &= \sum_{n=0}^{\infty}\frac{t^nX^n}{n!}\\
&= \sum_{n=1}^{\infty}\frac{(-1)^n\big(t\det(X)\big)^n}{(2n)!}I + \sum_{n=1}^{\infty}\frac{(-1)^n\big(t\det(X)\big)^n}{(2n+1)!}\frac{X}{\sqrt{\det(X)}}\\
&= \cos\Big(t\sqrt{\det(X)}\Big)I + \frac{\sin\big(t\sqrt{\det(X)}\big)}{\sqrt{\det(X)}}X\\
&= \cos\Big(t\sqrt{4-f_j(k)^2}\Big)1 + \frac{\sin\big(t\sqrt{4-f_j(k)^2}\big)}{\sqrt{4-f_j(k)^2}}F_j(k).
\end{aligned}$$

Therefore the curve is periodic with period $2\pi/\sqrt{4-f_j(b)^2}$. ■

We are now able to calculate the formula for the Hamiltonian flow ϱ_j^t.

Theorem 5.10 *Suppose $P \in M_r(\mathbb{S}^3)$ has vertices given by $[*, k_1*, \dots, k_1 \cdots k_n*]$. Then the Hamiltonian flow, $P(t) = \varrho_j^t(P)$, associated to the Hamiltonian f_j has vertices given by $P(t) = [*, \tilde{k}_1(t)*, \dots, \tilde{k}_n(t)*]$ where*

$$\tilde{k}_i(t) = \begin{cases} \mathrm{Ad}_{\exp(tF_j(k))}(k_1 \cdots k_i), & 1 \le i < j \\ k_1 \cdots k_i, & j \le i \le n. \end{cases}$$

The flow is periodic with period $2\pi / \sqrt{4 - f_j^2}$.

The flow $\varrho_j^t(P)$ has the following geometric description. Let d_j be the diagonal connecting the first vertex with the $(j+1)$-th vertex, that is $*$ with $k_1 \cdots k_j*$. Then $\varrho_j^t(P)$ rotates the first j vertices, $k_1 \cdots k_{i-1}*$, for $2 \le i \le 2$, about the diagonal d_j at constant angular velocity. The flows $\{\varrho_j^t\}$, $1 < j < n$, do not give rise to a torus action on $M_r(\mathbb{S}^3)$ since they do not have constant period. For example, as the length of a diagonal goes to zero, the period of flow about that diagonal goes to infinity.

To get a torus action on $M_r(\mathbb{S}^3)$, we need to look instead at the length functions $\ell_j(k) = \cos^{-1}\bigl(-\frac{1}{2} f_j(k)\bigr)$. Then

$$d\ell_j = \frac{1}{\sqrt{4 - f_j^2}} df_j$$

and

$$X_{\ell_j} = \frac{1}{\sqrt{4 - f_j^2}} X_{f_j}.$$

It is not difficult to see that the family of functions $\{\ell_j\}_{j=2}^{n-2}$ also Poisson commute, although the Hamiltonian flows for these functions are not everywhere defined on $M_r(\mathbb{S}^3)$. We restrict to the space $M_r'(\mathbb{S}^3)$ such $\ell_j \neq 0$ or $\ell_j \neq \pi$ for all $1 \le j \le n$. The Hamiltonian flows $\{\Psi_j^t\}$ on $M_r'(\mathbb{S}^3)$ associated to $\{\ell_j\}$ are periodic with constant period 2π and constant angular velocity. These flows define a Hamiltonian $(n-3)$-torus action on the space $M_r'(\mathbb{S}^3)$.

6 Braid Action on $M_r(\mathbb{S}^3)$

There exists an action of the pure braid group $\mathcal{P}_n$ on the manifold $M_r(\mathbb{S}^3)$ which preserves the symplectic structure. In this section, we show that the generators of the pure braid group arise as the time 1 Hamiltonian flows of the family of functions h_{ij}, $1 \le i < j \le n-1$ where $h_{ij} \in C^\infty\bigl(M_r(\mathbb{S}^3)\bigr)^K$ is defined by,

$$h_{ij}(k) = \frac{1}{2}\Bigl(\cos^{-1}\bigl(-\frac{1}{2}\,\mathrm{tr}(k_i k_j)\bigr)\Bigr)^2.$$

Let π_{12} denote the quasi-Poisson bivector on $\mathcal{C}_1 \circledast \mathcal{C}_2$. We have the following proposition.

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Proposition 6.1 *[AKSM, Proposition 5.7] The diffeomorphism* $R\colon C_1 \circledast C_2 \to C_2 \circledast C_1$ *given by* $R(k_1, k_2) = (\mathrm{Ad}_{k_1} k_2, k_1)$ *is a bivector map taking* π_{12} *to* π_{21}.

A similar proof gives us,

Proposition 6.2 *The diffeomorphism* $R'\colon C_1 \circledast C_2 \to C_2 \circledast C_1$ *given by* $R'(k_1, k_2) = (k_2, \mathrm{Ad}_{k_2^{-1}} k_1)$ *is also a bivector map taking* π_{12} *to* π_{21}.

Remark 6.3 $R \circ R' = \mathrm{Id}_{C_1 \circledast C_2} = R' \circ R$

We now define $R_i\colon C_1 \circledast \cdots \circledast (C_i \circledast C_{i+1}) \circledast \cdots \circledast C_n \to C_1 \circledast \cdots \circledast (C_{i+1} \circledast C_i) \circledast \cdots \circledast C_n$ to be the map given by

$$R_i(k_1, \ldots, k_i, k_{i+1}, \ldots k_n) = (k_1, \ldots, \mathrm{Ad}_{k_i} k_{i+1}, k_i, \ldots, k_n)$$

that is, R applied to the i-th and $(i+1)$-th term of $M_r(\mathbb{S}^3)$. R_i' can be defined in a similar way. See [Bi] for definitions of the full braid group, $\mathcal{B}_n$, and the pure braid group, $\mathcal{P}_n$.

Lemma 6.4 *The full braid group* $\mathcal{B}_n$ *has a faithful representation as a group of automorphism of the closed n-gons in* $\mathbb{S}^3$ *in which side-lengths are fixed but the order of the sides is not fixed. The generators of* $\mathcal{B}_n$ *are given by* R_i, $1 \le i \le n-1$.

We now restrict $\mathcal{B}_n$ to $\mathcal{P}_n$ to get an action of the pure braid group on $\circledast_{i=1}^n \mathcal{C}_{\lambda_i}$. This action induces a symplectomorphism on the moduli space $M_r(\mathbb{S}^3)$.

Corollary 6.5 *Let* $A_{ij} = R_{j-1} \circ \cdots \circ R_{i+1} \circ R_i^2 \circ R_{i+1}' \circ \cdots \circ R_{j-1}'$, $1 \le i < j \le n$. A_{ij} *induces a symplectomorphism from* $M_r(\mathbb{S}^3)$ *to itself. The* A_{ij}, $1 \le i < j \le n$ *are generators of* $\mathcal{P}_n$ *which has a faithful representation as a group of automorphisms of* $M_r(\mathbb{S}^3)$.

We will now show that the braid group actions A_{ij} can be realized as the time one Hamiltonian flows of the Hamiltonians h_{ij} given at the beginning of this section. We first study the Hamiltonian flows associated to the functions $f_{ij} \in C^\infty(\circledast_{i=1}^n \mathcal{C}_{\lambda_i})^K$ given by $f_{ij}(k) = \mathrm{tr}(k_i k_j)$. Define $F_{ij}\colon \circledast_{i=1}^n \mathcal{C}_{\lambda_i} \to \mathfrak{k}$ by $F_{ij}(k) = \big((k_i k_j) - (k_i k_j)^{-1}\big)$.

The Hamiltonian flow associated to f_{ij} is given by $\Phi_{ij}^t(k) = \big(\widehat{k_1}(t), \ldots, \widehat{k_n}(t)\big)$ where

$$\widehat{k_l}(t) = \begin{cases} k_l, & 0 < l < i \text{ and } j < l < n+1 \\ \mathrm{Ad}\Big(\exp\big(tF_{ij}(k)\big)\Big) k_l, & l = i, j \\ \mathrm{Ad}\Big(\exp\big(tF_{ij}(k)\big) k_j \exp\big(-tF_{ij}(k)\big) k_j^{-1}\Big) k_l, & i < l < j. \end{cases}$$

Following the proof of Lemma 5.9, we obtain

Lemma 6.6

$$\exp\left(\frac{\cos^{-1}\big(-\frac{1}{2}\,\mathrm{tr}(k)\big)}{\sqrt{4 - \mathrm{tr}^2(k)}}(k - k^{-1})\right) = k$$

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We now notice that for time $t = \frac{\cos^{-1}\left(-\frac{1}{2}f_{ij}(k)\right)}{\sqrt{4-f_{ij}^2(k)}}$,

$$\Phi_{ij}^t = A_{ij}.$$

The time for which Φ_{ij}^t flows depends on the point in $M_r(\mathbb{S}^3)$ at which flow begins. We would like this time to be independent of the starting point. This can be achieved by taking functions $h_{ij} = \frac{1}{2}\left(\cos^{-1}(-\frac{1}{2}f_{ij})\right)^2$. The Hamiltonian flow $\tilde{\Phi}_{ij}^t$ associated to h_{ij} is the renormalization of the flow Φ_{ij}^t so that

$$\tilde{\Phi}_{ij}^1 = A_{ij}.$$

We can see the pure braid group as the integer points in the Hamiltonian flows $\tilde{\Phi}_{ij}^t$, $1 \le i < j \le n$.

7 Connection With Symplectic Forms on Relative Character Varieties of n-Punctured 2-Spheres

In this section, we relate the symplectic form on $M_r(\mathbb{S}^3)$ given in Remark 5.3 to the symplectic form of Goldman type obtained from the description of $M_r(\mathbb{S}^3)$ as the moduli space of flat connections on an n-punctured 2-sphere. We follow the arguments of Kapovich and Millson [KM1, Section 5] which considers the analogous question for $M_r(\mathbb{E}^3)$. As a consequence, we obtain, using a result of L. Jeffrey, a symplectomorphism from $M_r(\mathbb{E}^3)$ and $M_r(\mathbb{S}^3)$ for sufficiently small side-lengths.

We begin with the general case in which G is any Lie group with Lie algebra $\mathfrak{g}$ which admits a nondegenerate, G-invariant, symmetric, bilinear form.

7.1 Relative Characteristic Varieties and Parabolic Cohomology

Let $\Sigma = \mathbb{S}^2 - \{p_1, \ldots, p_n\}$ denote the n-punctured 2-sphere and $U_1, \ldots, U_n$ be disjoint open disc neighborhoods of $p_1, \ldots, p_n$, respectively. Further, Γ is the fundamental group of Σ with generators γ_i and $T = \{\Gamma_1, \ldots, \Gamma_n\}$ is the collection of subgroups of Γ with Γ_i the cyclic subgroup generated by γ_i.

Fix $\rho_0 \in \mathrm{Hom}(\Gamma, G)$ a representation. In [KM1], the relative representation variety $\mathrm{Hom}(\Gamma, T; G)$ is defined as the representations $\rho\colon \Gamma \to G$ such that $\rho|_{\Gamma_i}$ is contained in the closure of the conjugacy class of $\rho_0|_{\Gamma_i}$.

Remark 7.1 If $G = \mathrm{SU}(2)$, there exists a ρ_0 such that the relative character variety $\mathrm{Hom}(\Gamma, T; G)/G$ is isomorphic to $M_r(\mathbb{S}^3)$. We will make this isomorphism explicit later on.

Let $\rho \in \mathrm{Hom}(\Gamma, T; G)$. Then ρ induces a flat principal G-bundle over Σ. The associated flat Lie algebra bundle will be denoted by ad P.

We define the parabolic cohomology, $H^1_{\mathrm{par}}(\Sigma, \mathrm{ad}\, P)$ to be the subspace of the de Rham cohomology classes in $H^1_{\mathrm{DR}}(\Sigma, \mathrm{ad}\, P)$ whose restrictions to each U_i are trivial.

7.2 Group Cohomology Construction of the Symplectic Form

Let b be the nondegenerate, G-invariant, symmetric, bilinear form on $\mathfrak{g}$. A skew symmetric bilinear form

$$B\colon H^1_{\mathrm{par}}(\Sigma, \mathrm{ad}\, P) \times H^1_{\mathrm{par}}(\Sigma, \mathrm{ad}\, P) \to H^2(\Sigma, U; \mathbb{R})$$

is defined by taking the wedge product together with the bilinear form b. Evaluating on the relative fundamental class of Σ gives the skew symmetric form,

$$A\colon H^1_{\mathrm{par}}(\Sigma, \mathrm{ad}\, P) \times H^1_{\mathrm{par}}(\Sigma, \mathrm{ad}\, P) \to \mathbb{R}.$$

Poincaré duality gives us nondegeneracy of A, so A is a symplectic form on $\mathrm{Hom}(\Gamma, T; G)$. We will show A corresponds to the symplectic form $\tilde{\omega}$ given in Remark 5.3.

We first pass through the group cohomology description of $H^1_{\mathrm{par}}(\Sigma, \mathrm{ad}\, P)$ to make this correspondence explicit.

We identify the universal cover of Σ, denoted $\tilde{\Sigma}$, with the hyperbolic plane, $\mathbb{H}^2$. Let $p\colon \tilde{\Sigma} \to \Sigma$ be the covering projection. We identify the $\mathcal{A}^\bullet(\tilde{\Sigma}, p^*\, \mathrm{ad}\, P)$ with $\mathcal{A}^\bullet(\tilde{\Sigma}, \mathfrak{g})$ by parallel translation from a point x_0. Given $[\eta] \in H^1(\Sigma, \mathrm{ad}\, P)$ choose a representing closed 1-form $\eta \in \mathcal{A}^1(\Sigma, \mathrm{ad}\, P)$. Let $\tilde{\eta} = p^*\eta$. Then there is a unique function $f\colon \tilde{\Sigma} \to \mathfrak{g}$ satisfying:

- $f(x_0) = 0$
- $df = \tilde{\eta}$

A 1-cochain $h(\eta) \in C^1(\Gamma, \mathfrak{g})$ is defined by

$$h(\eta)(\gamma) = f(x) - \mathrm{Ad}_{\rho(\gamma)}\, f(\gamma^{-1}x).$$

This induces an isomorphism from $H^1(\Sigma, \mathrm{ad}\, P)$ to $H^1(\Gamma, \mathfrak{g})$. It can be seen that $[\eta] \in H^1_{\mathrm{par}}(\Sigma, \mathrm{ad}\, P)$ if and only if $h(\eta)$ restricted to Γ_i is exact for all i. That is, there exists an $x_i \in \mathfrak{g}$ such that $h(\eta)(\gamma_i^k) = x_i - \mathrm{Ad}_{\rho(\gamma_i^k)}\, x_i$ for each γ_i a generator of Γ.

We construct the fundamental domain $\mathcal{D}$ for Γ operating on $\mathbb{H}^2$ as in [KM1]. Choose x_0 on Σ and make cuts along geodesics from x_0 to the cusps. The resulting fundamental domain $\mathcal{D}$ is a geodesic $2n$-gon with vertices $v_1, \ldots, v_n$ and cusps $v_1^\infty, \ldots, v_n^\infty$ ordered so that as we proceed clockwise around $\partial\mathcal{D}$ we see $v_1, v_1^\infty, \ldots, v_n, v_n^\infty$. The generator γ_i fixes v_i^∞ and satisfies $\gamma_i v_{i+1} = v_i$. Let e_i be the oriented edge joining v_i to v_i^∞ and $\hat{e}_i$ be the oriented edge joining v_i^∞ to v_{i+1}. Then $\gamma_i \hat{e}_i = -e_i$.

Let $\rho \in \mathrm{Hom}(\Gamma, T; G)$ and $c, c' \in T_\rho\big(\mathrm{Hom}(\Gamma, T; G)/G\big) \simeq H^1_{\mathrm{par}}(\Gamma, \mathfrak{g})$ be tangent vectors at ρ. The corresponding elements in $H^1_{\mathrm{par}}(\Sigma, \mathrm{ad}\, P)$ are denoted α and α'. So $f\colon \Sigma \to \mathfrak{g}$ which satisfies $df = \tilde{\alpha}$ and $f_i(x_0) = 0$. Let $f(v_i^\infty) = x_i$. Then

$$\begin{aligned} c(\gamma_i) &= f(x) - \mathrm{Ad}_{\rho(\gamma_i)}\, f(\gamma_i^{-1}x) \\ &= f(v_i^\infty) - \mathrm{Ad}_{\rho(\gamma_i)}\, f(\gamma_i^{-1}v_i^\infty) \\ &= f(v_i^\infty) - \mathrm{Ad}_{\rho(\gamma_i)}\, f(v_i^\infty) \\ &= x_i - \mathrm{Ad}_{\rho(\gamma_i)}\, x_i. \end{aligned}$$

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There is an equivalent formulas for c', α', and f' with $f'(v_i^\infty) = x_i'$.

Let $B_\bullet(\Gamma)$ be the bar resolution of Γ. Thus $B_k(\Gamma)$ is the free $\mathbb{Z}[\Gamma]$-module on the symbols $[\gamma_1|\gamma_2|\cdots|\gamma_k]$ with

$$\partial[\gamma_1|\gamma_2|\cdots|\gamma_k] \\ = \gamma_1[\gamma_2|\cdots|\gamma_k] + \sum_{i=1}^{k-1}(-1)^i[\gamma_1|\cdots|\gamma_i\gamma_{i+1}|\cdots|\gamma_k] + (-1)^k[\gamma_1|\cdots|\gamma_{k-1}].$$

Let $C_k(\Gamma) = B_k(\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathbb{Z}$ with $\mathbb{Z}[\Gamma]$ acting on $\mathbb{Z}$ by the homomorphism ϵ defined by

$$\epsilon(\sum_{i=1}^{m} a_i\gamma_i) = \sum_{i=1}^{m} a_i.$$

Then $C_k(\gamma)$ is the free abelian group on the symbols $(\gamma_1|\cdots|\gamma_k) = [\gamma_1|\gamma_2|\cdots|\gamma_k]\otimes 1$ with

$$\partial(\gamma_1|\gamma_2|\cdots|\gamma_k) \\ = (\gamma_2|\cdots|\gamma_k) + \sum_{i=1}^{k-1}(-1)^i(\gamma_1|\cdots|\gamma_i\gamma_{i+1}|\cdots|\gamma_k) + (-1)^k(\gamma_1|\cdots|\gamma_{k-1}).$$

A relative fundamental class $F \in C_2(\Gamma)$ is defined by the property

$$\partial F = \sum_{i=1}^{n}(\gamma_i).$$

Let $[\Gamma, \partial\Gamma] = \sum_{i=2}^{n}(\gamma_1\cdots\gamma_{i-1}|\gamma_i) \in C_2(\Gamma)$, then

Lemma 7.2 $[\Gamma, \partial\Gamma]$ *is a relative fundamental class.*

Proof The proof is left to the reader.

We will now give the symplectic form A in terms of group cohomology. We denote by $\cup_b$ the cup product of Eilenberg-MacLane cochains using the form b on the coefficients.

Proposition 7.3

$$A(\alpha, \alpha') = \sum_{i=1}^{n}\langle c \cup_b x_i'), (\gamma_i)\rangle - \langle c \cup_b c', [\Gamma, \partial\Gamma]\rangle$$

We will use the next Lemmas to prove Proposition 7.3.

Lemma 7.4

$$\int_{e_i} B(f, \tilde{\alpha}') + \int_{\hat{e}_i} B(f, \tilde{\alpha}') = b\big(c(\gamma_i), f'(v_i^\infty)\big) - b\big(c(\gamma_i), f'(v_i)\big)$$

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Proof Recall $\gamma_i \hat{e}_i = -e_i$, so that $\hat{e}_i = -\gamma_i^{-1} e_i$. We then have

$$\begin{aligned}
\int_{e_i} B(f,\tilde{\alpha}') + \int_{\hat{e}_i} B(f,\tilde{\alpha}') &= \int_{e_i} B(f,\tilde{\alpha}') + \int_{\gamma_i^{-1} e_i} B(f,\tilde{\alpha}') \\
&= \int_{e_i} B(f,\tilde{\alpha}') + \int_{e_i} (\gamma_i^{-1})^* B(f,\tilde{\alpha}') \\
&= \int_{e_i} B(f,\tilde{\alpha}') + \int_{e_i} B\big((\gamma_i^{-1})^* f, (\gamma_i^{-1})^* \tilde{\alpha}'\big) \\
&= \int_{e_i} B(f,\tilde{\alpha}') + \int_{e_i} B\big(\mathrm{Ad}_{\rho(\gamma_i)} (\gamma_i^{-1})^* f, \mathrm{Ad}_{\rho(\gamma_i)} (\gamma_i^{-1})^* \tilde{\alpha}'\big) \\
&= \int_{e_i} B\big(f - \mathrm{Ad}_{\rho(\gamma_i)} (\gamma_i^{-1})^* f, \tilde{\alpha}'\big) \\
&= \int_{e_i} B\big(c(\gamma_i), \tilde{\alpha}'\big) \\
&= b\big(c(\gamma_i), f'(v_i^\infty)\big) - b\big(c(\gamma_i), f'(v_i)\big). \qquad \blacksquare
\end{aligned}$$

Lemma 7.5

$$\sum_{i=1}^n b\big(c(\gamma_i), f'(v_i)\big) = \sum_{i=1}^n b\big(c(\gamma_i), f'(v_i^\infty)\big) - \sum_{i=1}^n \langle c \cup_b y_i, (\gamma_i)\rangle + \langle c \cup_b c', [\Gamma, \partial\Gamma]\rangle.$$

Proof By definition, for any $x \in \mathbb{H}^2$ and $\gamma \in \Gamma$ we have

$$c'(\gamma) = f'(x) - \mathrm{Ad}_{\rho(\gamma)} f'(\gamma^{-1} x).$$

Let $\gamma = \gamma_i$ and $x = v_i$, then

$$c'(\gamma_i) = f'(v_i) - \mathrm{Ad}_{\rho(\gamma_i)} f'(v_{i+1}).$$

Using $f'(v_1) = 0$, we obtain

$$\begin{aligned}
c'(\gamma_1 \cdots \gamma_i) &= f'(v_1) - \mathrm{Ad}_{\rho(\gamma_1 \cdots \gamma_i)} f'(\gamma_i^{-1} \cdots \gamma_1^{-1} v_1) \\
&= -\mathrm{Ad}_{\rho(\gamma_1 \cdots \gamma_i)} f'(v_{i+1}).
\end{aligned}$$

We will also need

$$\begin{aligned}
c'(\gamma_1 \cdots \gamma_i) &= c'(\gamma_1 \cdots \gamma_{i-1}) + \mathrm{Ad}_{\rho(\gamma_1 \cdots \gamma_{i-1})} c'(\gamma_i) \\
&= c'(\gamma_1) + \mathrm{Ad}_{\rho(\gamma_1)} c'(\gamma_2) + \cdots + \mathrm{Ad}_{\rho(\gamma_1 \cdots \gamma_{i-1})} c'(\gamma_i).
\end{aligned}$$

and, since $\gamma_1 \cdots \gamma_n = 1$,

$$0 = c'(\gamma_1 \cdots \gamma_n) = c'(\gamma_1) + \mathrm{Ad}_{\rho(\gamma_1)} c'(\gamma_2) + \cdots + \mathrm{Ad}_{\rho(\gamma_1 \cdots \gamma_{n-1})} c'(\gamma_n).$$

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We then have,

$$
\begin{aligned}
&\sum_{i=1}^{n} b\big(c(\gamma_i), f'(v_i)\big) \\
&\quad = -\sum_{i=1}^{n} b\big(c(\gamma_i), \mathrm{Ad}_{\rho(\gamma_1\cdots\gamma_i)^{-1}}\, c'(\gamma_1\cdots\gamma_{i-1})\big) \\
&\quad = -\sum_{i=1}^{n} b\Big(\mathrm{Ad}_{\rho(\gamma_1\cdots\gamma_{i-1})}\, c(\gamma_i), \sum_{j=1}^{i-1} \mathrm{Ad}_{\rho(\gamma_1\cdots\gamma_{j-1})}\, c'(\gamma_j)\Big) \\
&\quad = -\sum_{i=1}^{n}\sum_{j=1}^{i-1} b\big(\mathrm{Ad}_{\rho(\gamma_1\cdots\gamma_{i-1})}\, c(\gamma_i), \mathrm{Ad}_{\rho(\gamma_1\cdots\gamma_{j-1})}\, c'(\gamma_j)\big) \\
&\quad = -\sum_{j=1}^{n}\sum_{i=j+1}^{n} b\big(\mathrm{Ad}_{\rho(\gamma_1\cdots\gamma_{i-1})}\, c(\gamma_i), \mathrm{Ad}_{\rho(\gamma_1\cdots\gamma_{j-1})}\, c'(\gamma_j)\big) \\
&\quad = \sum_{j=1}^{n}\sum_{i=1}^{j} b\big(\mathrm{Ad}_{\rho(\gamma_1\cdots\gamma_{i-1})}\, c(\gamma_i), \mathrm{Ad}_{\rho(\gamma_1\cdots\gamma_{j-1})}\, c'(\gamma_j)\big) \\
&\quad = \sum_{j=1}^{n} b\big(c(\gamma_1\cdots\gamma_j), \mathrm{Ad}_{\rho(\gamma_1\cdots\gamma_{j-1})}\, c'(\gamma_j)\big) \\
&\quad = \sum_{j=1}^{n} b\big(c(\gamma_1\cdots\gamma_{j-1}) + \mathrm{Ad}_{\rho(\gamma_1\cdots\gamma_{j-1})}\, c(\gamma_j), \mathrm{Ad}_{\rho(\gamma_1\cdots\gamma_{j-1})}\, c'(\gamma_j)\big) \\
&\quad = \sum_{j=1}^{n} b\big(c(\gamma_1\cdots\gamma_{j-1}), \mathrm{Ad}_{\rho(\gamma_1\cdots\gamma_{j-1})}\, c'(\gamma_j)\big) + \sum_{j=1}^{n} b\big(c(\gamma_j), c'(\gamma_j)\big) \\
&\quad = \langle c \cup_b c'), [\Gamma, \partial\Gamma]\rangle + \sum_{j=1}^{n} b\big(c(\gamma_j), f'(v_j^{\infty}) - \mathrm{Ad}_{\rho(\gamma_j)}\, f'(v_j^{\infty})\big) \\
&\quad = \langle c \cup_b c', [\Gamma, \partial\Gamma]\rangle + \sum_{j=1}^{n} b\big(c(\gamma_j), f'(v_j^{\infty})\big) - \sum_{j=1}^{n} \langle B(c, y_j'), (\gamma_j)\rangle \quad \blacksquare
\end{aligned}
$$

Proof of Proposition 7.3

$$
\begin{aligned}
A(\alpha, \alpha') &= \int_{\Sigma} B(\alpha, \alpha') \\
&= \int_{\mathcal{D}} B(\tilde{\alpha}, \tilde{\alpha}') \\
&= \int_{\partial\mathcal{D}} B(\tilde{\alpha}, f')
\end{aligned}
$$

$$= \sum_{i=1}^{n} \Big(\int_{e_i} B(\tilde{\alpha}, f') + \int_{\hat{e}_i} B(\tilde{\alpha}, f') \Big)$$

$$= \sum_{j=1}^{n} \langle c \cup_b x'_j), (\gamma_j) \rangle - \langle c \cup_b c', [\Gamma, \partial\Gamma] \rangle. \qquad \blacksquare$$

7.3 Relating $\mathrm{Hom}\big(\Gamma, T; \mathrm{SU}(2)\big) / \mathrm{SU}(2)$ and $M_r(\mathbb{S}^3)$

We now restrict to the case $G = \mathrm{SU}(2)$. We define the isomorphism

$$\Upsilon \colon \mathrm{Hom}\big(\Gamma, T; \mathrm{SU}(2)\big) \to \mathrm{CN}_r(\mathbb{S}^3, *),$$

where $\mathrm{CN}_r(\mathbb{S}^3, *)$ is the closed polygonal linkages in $\mathbb{S}^3$ based at a point, by

$$\Upsilon(\rho) = \big(\rho(\gamma_1), \dots, \rho(\gamma_n)\big).$$

This induces an isomorphism, which we also denote by Υ,

$$\Upsilon \colon \mathrm{Hom}\big(\Gamma, T; \mathrm{SU}(2)\big) / \mathrm{SU}(2) \to M_r(\mathbb{S}^3).$$

The differential $d\Upsilon_\rho \colon T_\rho\Big(\mathrm{Hom}\big(\Gamma, T; \mathrm{SU}(2)\big) / \mathrm{SU}(2)\Big) \to T_{\Upsilon(\rho)} M_r(\mathbb{S}^3)$ is then defined by

$$d\Upsilon_\rho(c) = \big(dR_{\rho(\gamma_1)} c(\gamma_1), \dots, dR_{\rho(\gamma_n)} c(\gamma_n)\big).$$

Here $T_\rho\Big(\mathrm{Hom}\big(\Gamma, T; \mathrm{SU}(2)\big) / \mathrm{SU}(2)\Big)$ is identified with an element of $\mathbb{Z}^1_{\mathrm{par}}(\Gamma, \mathrm{su}_2)$. We have

$$d\Upsilon_\rho(c) = (dR_{k_1} x_1 - dL_{k_1} x_1, \dots, dR_{k_n} x_n - dL_{k_n} x_n)$$

and

$$d\Upsilon_\rho(c') = (dR_{k_1} x'_1 - dL_{k_1} x'_1, \dots, dR_{k_n} x'_n - dL_{k_n} x'_n).$$

Recall, the symplectic form on $M_r(\mathbb{S}^3)$ is given by

$$\tilde{\omega} = \sum_{i=1}^{n} \omega_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \big(\mathrm{Ad}_{k_1 \cdots k_{i-1}} \bar{\theta}_i \wedge_b \mathrm{Ad}_{k_1 \cdots k_{j-1}} \bar{\theta}_j\big).$$

We can now prove the main result of this section.

Theorem 7.6 $\Upsilon^* \tilde{\omega} = A$

Proof First we note that

$$\Upsilon^* \bar{\theta}_i(c) = c(\gamma_i)$$

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and

$$
\begin{aligned}
(\Upsilon^*\omega_i)(c,c') &= \omega_i\big(dR_{k_i}c(\gamma_i), dR_{k_i}c'(\gamma_i)\big)\\
&= -\frac{1}{2}\big(\operatorname{Ad}_{k_i^{-1}} c(\gamma_i) + c(\gamma_i), x_i'\big)\\
&= -\frac{1}{2}\big(c(\gamma_i), \operatorname{Ad}_{k_i} x_i' + x_i'\big)\\
&= -\frac{1}{2}\big(c(\gamma_i), c'(\gamma_i)\big) - \big(c(\gamma_i), \operatorname{Ad}_{k_i} x_i'\big)\\
&= -\frac{1}{2}\big(\operatorname{Ad}_{k_1\cdots k_{i-1}} c(\gamma_i), \operatorname{Ad}_{k_1\cdots k_{i-1}} c'(\gamma_i)\big) + \langle c\cup_b x_i'), (\gamma_i)\rangle.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(\Upsilon^*\tilde{\omega})(c,c') &= \sum_{i=1}^{n}(\Upsilon^*\omega_i)(c,c')\\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\Upsilon^*\big(\operatorname{Ad}_{k_1\cdots k_{i-1}}\bar{\theta}_i \wedge_b \operatorname{Ad}_{k_1\cdots k_{j-1}}\bar{\theta}_j\big)(c,c')\\
&= \sum_{i=1}^{n}\langle c\cup_b x_i'), (\gamma_i)\rangle - \sum_{i=1}^{n}\frac{1}{2}\big(\operatorname{Ad}_{k_1\cdots k_{i-1}} c(\gamma_i), \operatorname{Ad}_{k_1\cdots k_{i-1}} c'(\gamma_i)\big)\\
&\quad + \sum_{i=1}^{n}\sum_{j=i+1}^{n}\big(\operatorname{Ad}_{k_1\cdots k_{i-1}} c(\gamma_i), \operatorname{Ad}_{k_1\cdots k_{j-1}} c'(\gamma_j)\big)\\
&\quad - \sum_{i=1}^{n}\sum_{j=i+1}^{n}\big(\operatorname{Ad}_{k_1\cdots k_{i-1}} c'(\gamma_i), \operatorname{Ad}_{k_1\cdots k_{j-1}} c(\gamma_j)\big)\\
&= \sum_{i=1}^{n}\langle c\cup_b x_i', (\gamma_i)\rangle - \sum_{i=1}^{n}\frac{1}{2}\big(\operatorname{Ad}_{k_1\cdots k_{i-1}} c(\gamma_i), \operatorname{Ad}_{k_1\cdots k_{i-1}} c'(\gamma_i)\big)\\
&\quad + \sum_{j=2}^{n}\sum_{i=1}^{j-1}\big(\operatorname{Ad}_{k_1\cdots k_{i-1}} c(\gamma_i), \operatorname{Ad}_{k_1\cdots k_{j-1}} c'(\gamma_j)\big)\\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{i}\big(\operatorname{Ad}_{k_1\cdots k_{i-1}} c'(\gamma_i), \operatorname{Ad}_{k_1\cdots k_{j-1}} c(\gamma_j)\big)\\
&= \sum_{i=1}^{n}\langle c\cup_b x_i', (\gamma_i)\rangle + \sum_{j=2}^{n}\sum_{i=1}^{j-1}\big(\operatorname{Ad}_{k_1\cdots k_{i-1}} c(\gamma_i), \operatorname{Ad}_{k_1\cdots k_{j-1}} c'(\gamma_j)\big)
\end{aligned}
$$

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$$= \sum_{i=1}^{n} \langle c \cup_b x_i'), (\gamma_i) \rangle + \sum_{j=2}^{n} \big(\mathrm{Ad}_{k_1 \cdots k_{i-1}} c'(\gamma_i), c(\gamma_1 \cdots \gamma_{i-1}) \big)$$

$$= \sum_{i=1}^{n} \langle c \cup_b x_i', (\gamma_i) \rangle - \langle c \cup_b c', [\Gamma, \partial\Gamma] \rangle$$

$$= A(\alpha, \alpha'). \qquad \blacksquare$$

It is easily seen that the functions ℓ_i from Section 5.3 corresponds to the following Goldman functions. Let $\phi\colon \mathrm{SU}(2) \to \mathbb{R}$ be defined by $\phi(g) = \cos^{-1}\big(-\frac{1}{2}\operatorname{tr}(g)\big)$. We then defined the function $\phi_\gamma\colon \mathrm{Hom}\big(\Gamma, T; \mathrm{SU}(2)\big) / \mathrm{SU}(2) \to \mathbb{R}$ by $\phi_\gamma(\rho) = \phi\big(\rho(\gamma)\big)$. We see that

$$\Upsilon^* \ell_i = \phi_{\gamma_1 \cdots \gamma_i}.$$

Then choosing an maximal collection of nonintersecting diagonal on $M_r(\mathbb{S}^3)$ corresponds to a pair of pants decomposition on Σ.

7.4 Symplectomorphism of $M_r(\mathbb{S}^3)$ and $M_r(\mathbb{E}^3)$

We now use the following result due to L. Jeffrey.

Lemma 7.7 *There exists an open neighborhood U of 0 in $\mathfrak{g}^n$ such that if $\bar{\lambda} = (\lambda_1, \dots, \lambda_n) \in U$ then the moduli space of parabolic bundles on n-punctured surface with weights $\lambda_1, \dots, \lambda_n$ is symplectomorphic to the symplectic reduced space $\{(X_1, \dots, X_n) \in \mathcal{O}_{\lambda_1} \times \cdots \times \mathcal{O}_{\lambda_n} : X_1 + \cdots + X_n = 0\}/G$.*

Proof See [Je, Theorem 6.6]. ■

We can identify the moduli space of parabolic bundles on n-punctured surface with weights $\lambda_1, \dots, \lambda_n$ with $M_r(\mathbb{S}^3)$. Also, it was shown in [KM1] that $\{(X_1, \dots, X_n) \in \mathcal{O}_{\lambda_1} \times \cdots \times \mathcal{O}_{\lambda_n} \subset \mathfrak{g}^n : X_1 + \cdots + X_n = 0\}/G$ can be identified with $M_r(\mathbb{E}^3)$. We have the following corollary.

Corollary 7.8 *For sufficiently small side-lengths, $M_r(\mathbb{S}^3)$ is symplectomorphic to $M_r(\mathbb{E}^3)$.*

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