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$\mathcal{P}$-adic modular forms over Shimura curves over totally real fields

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Dedicated to S. Shahshahani

Abstract

We set up the basic theory of $\mathcal{P}$-adic modular forms over certain unitary PEL Shimura curves $M'_{K'}$. For any PEL abelian scheme classified by $M'_{K'}$, which is not 'too supersingular', we construct a canonical subgroup which is essentially a lifting of the kernel of Frobenius from characteristic p. Using this construction we define the U and Frob operators in this context. Following Coleman, we study the spectral theory of the action of U on families of overconvergent $\mathcal{P}$-adic modular forms and prove that the dimension of overconvergent eigenforms of U of a given slope is a locally constant function of the weight.

1. Introduction

The theory of p-adic modular forms started with the work of J. P. Serre, B. Dwork and N. Katz. The original motivation for this theory was the problem of p-adic interpolation of special values of the Riemann zeta function. Serre [Ser73] defined p-adic modular forms as p-adic limits of q-expansions of classical modular forms of varying weights and he constructed p-adic L-functions by using his families of p-adic modular forms.

Katz [Kat73] gave a modular definition of Serre's p-adic modular forms of integral weight. They are defined as certain functions on the moduli space of test objects consisting of an ordinary elliptic curve with a level structure. He also gave modular descriptions of the action of Hecke operators on these modular forms, including the analogue of the classical U_p operator of Atkin, which is called the U operator. This operator takes a p-adic modular form with q-expansion $\sum_n a_n q^n$ to $\sum_n a_{np} q^n$.

Studying the action of the U operator on p-adic modular functions, Dwork introduced the notion of *growth condition* and noted that U was a completely continuous operator in his case. Katz showed that the subspace of p-adic modular forms with growth condition r (which is an element of the p-adically complete base ring R_0) can be defined as certain functions on the moduli of test objects which are *not too supersingular*, in the sense that the value of E_{p-1} (Eisenstein series of weight $p-1$) at the test object has p-adic valuation at most equal to that of r. If $r=1$ (or a unit), this amounts to being ordinary and hence growth condition $r=1$ retrieves the full space of p-adic modular forms. When r is not a unit in R_0, the modular forms of growth condition r are called *overconvergent* modular forms. Using the work of Lubin on the theory of canonical subgroups of formal groups of dimension one, Katz showed that the subspace of the overconvergent modular forms of growth condition r is invariant under the U operator. The significance of the concept of

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growth condition lies in the fact that for a non-unit r, U is a completely continuous operator on the subspace of overconvergent modular forms of growth condition r. As a result, one can use Serre's Fredholm theory [Ser62] to study the spectral theory of U.

Hida [Hid86a, Hid86b] constructed and extensively studied the ordinary (or slope zero) part of the space of p-adic modular forms (and of their Hecke algebra). This space is topologically generated by the eigenforms of U whose eigenvalues are p-adic units (or have valuation zero). Hida showed that ordinary modular forms are all overconvergent. He also constructed p-adic families of Galois representations attached to families of ordinary p-adic modular forms. As a result of this work, Hida could prove that the number of normalized ordinary eigenforms is a locally constant function of the weight. Gouvêa and Mazur [GM92] conjectured similar results for a general slope $\beta \in \mathbb{Q}$. These questions were almost settled by Coleman [Col97, Col96]. Coleman uses rigid analytic geometry to analyze the properties of overconvergent modular forms. He considers rigid analytic affinoids over L_0, the fraction field of R_0, obtained by deleting supersingular discs of different sizes from the modular curve $X_1(N)$ and defines overconvergent modular forms as sections of certain line bundles on these affinoids. These modular forms can be described in terms of Katz's overconvergent modular forms. Using the work of Katz, Coleman defines the completely continuous action of U on these modular forms. He also generalizes Serre's Fredholm theory for completely continuous operators of Banach modules over Banach algebras [Col96]. He then applies this theory to families of overconvergent modular forms parameterized by certain rigid affinoids and obtains results about the overconvergent eigenforms of U. This leads, among other things, to his proofs of (slightly weaker versions) of the Gouvêa–Mazur conjectures [Col96].

The current work was inspired by Coleman's elegant method. One would like to prove similar results for automorphic forms on GL_2 of a totally real field F. In order to avoid complications arising from high dimension of Hilbert modular varieties one could switch to certain quaternionic Shimura curves M_K. In certain cases the automorphic forms over these Shimura curves correspond to automorphic forms for GL_2 over F via the Jacquet–Langlands correspondence. The automorphic forms on M_K are, in turn, closely related to the automorphic forms on certain unitary Shimura curves $M'_{K'}$ defined over F. In this work we develop a $\mathcal{P}$-adic theory for modular forms defined over the Shimura curves $M'_{K'}$, and show how Coleman's method can be generalized to produce results about the dimension of various spaces of $\mathcal{P}$-adic eigenforms in this context. As mentioned above, one of the potential applications of this work is to understanding of the p-adic deformations of (certain) Hilbert modular forms. The key to passage from our results to results about Hilbert modular forms is in proving a criterion to decide which $\mathcal{P}$-adic eigenforms are classical. This will make it possible to restate the results for spaces of classical eigenforms on $M'_{K'}$. The relationship between the curves M_K and $M'_{K'}$ (for example as in Theorem 4.2) allows one to compare dimensions of spaces of classical eigenforms on these curves and obtain results concerning the dimension of spaces of eigenforms on the quaternionic Shimura curves. Finally, replacing (normalized) eigenforms with their corresponding automorphic representations (on both the Hilbert modular variety and the quaternionic Shimura curve), one could incorporate the Jacquet–Langlands correspondence (when applicable) to obtain results on the dimension of spaces of Hilbert modular eigenforms.

Coleman [Col96] has proven such a classicality criterion in the context of modular curves. Work on a similar criterion in this context is in progress.

Let us introduce some notation. Assume F has degree d over $\mathbb{Q}$. Let $\mathcal{P}$ be a prime ideal of the ring of integers $\mathcal{O}_F$, which lies over p. Let $F_{\mathcal{P}}$ and $\mathcal{O}_{\mathcal{P}}$ denote the completion of F and its ring of integers at $\mathcal{P}$. Let π be a uniformizer of $\mathcal{O}_{\mathcal{P}}$. Let κ denote the residue field of $\mathcal{P}$ of cardinality q. Let R_0 denote a complete discrete valuation ring containing $\mathcal{O}_{\mathcal{P}}$, with valuation v, such that $v(\pi) = 1$. Let B be a quaternion algebra defined over F which splits at $\mathcal{P}$. Assume also that B splits at exactly one infinite place $\tau : F \hookrightarrow \mathbb{R}$. Let $G = \mathrm{Res}_{F/\mathbb{Q}}(B^*)$ be the Weil restriction from F to $\mathbb{Q}$ of B^*. To G, one can associate a projective system of Shimura curves M_K indexed by open compact subgroups of $G(\mathbb{A}^f)$. The connected components of these curves over $\mathbb{C}$ can be described as the quotient of the complex upper half plane by congruence subgroups of B^*. By Shimura's work these curves have canonical smooth and proper models over F. Carayol [Car86] explains how to construct canonical integral models for these Shimura curves. These models are constructed via the integral models for the Shimura curves $M'_{K'}$ which are associated to a unitary group G' obtained from an involution of the second kind on $D = B \otimes_F E$, where E is a quadratic imaginary extension of F. The curves $M'_{K'}$ and their canonical integral models can be described as moduli spaces of abelian schemes with PEL structures. Carayol [Car86] records the observation of Deligne and Shimura that every connected component of $M_K \otimes F_{\mathcal{P}}^{\mathrm{nr}}$ is isomorphic to a connected component of some $M'_{K'} \otimes F_{\mathcal{P}}^{\mathrm{nr}}$. This is used to construct the canonical integral models of M_K. It also establishes the aforementioned close connection between modular forms over M_K and modular forms over $M'_{K'}$. As hinted before, we will develop the theory entirely for the Shimura curves $M'_{K'}$ which are more suitable for our constructions.

In §§ 2, 3, and 4, we review some background on Shimura curves M_K and $M'_{K'}$. In § 5 we define modular forms on $M'_{K'}$ which, unlike the classical case, do not have q-expansions. Consequently, we don't have a canonical Eisenstein series of weight $q-1$. In § 6 we construct a modular form $\mathbf{H}$ over κ, the Hasse invariant, which vanishes exactly at the supersingular points of $M'_{K'}$. In § 7 we show that (when $q > 3$) there is a lifting of this modular form to characteristic zero which, for our purposes, is as good as the Eisenstein series in the classical context. This lifting will serve to limit the supersingularity of a test object in our moduli space. Indeed in § 12 we prove that our theory is independent of the choice of this lifting. This is essentially a corollary of Proposition 6.3 which states that the Hasse invariant has simple zeros in this case. Using this lifting, in § 8 following Katz, we define the spaces of $\mathcal{P}$-adic modular forms over $M'_{K'}$ of growth condition $r \in R_0$. When $r \in R_0$ is not a unit, we call these modular forms overconvergent.

In § 9 we describe $\mathcal{P}$-adic modular forms using rigid affinoids and formal schemes. In § 10 we use the Lubin–Katz method to construct the canonical subgroup of a 'not too supersingular' test object in our moduli space. We also explain how to measure the supersingularity of the quotient of the test object by its canonical subgroup.

The canonical subgroup is essentially a canonical lifting to characteristic zero of the kernel of Fr_q in the abelian scheme. In § 11 we use the canonical subgroup to define Frob, the Frobenius morphism of $\mathcal{P}$-adic modular forms. We also use the rigid theoretic description of $\mathcal{P}$-adic modular forms to study the properties of Frob. This allows us to define the U operator in § 12 essentially as a trace of Frob, and to prove that the space of overconvergent modular forms is invariant under U. In § 13 we study the continuity properties of U and show that U is a completely continuous operator when r is not a unit in R_0. In the last section, § 14, we use Coleman's method [Col96] to study the overconvergent eigenforms of U. An overconvergent modular form f is called a generalized eigenform of U of slope $\beta \in \mathbb{Q}$ if there is a polynomial $Q(T) \in L_0[T]$ such that all of its roots (in $\bar{L}_0$) have valuation β and $Q(\mathrm{U})(f) = 0$. Let $d(K', k, \beta)$ denote the dimension of space of overconvergent generalized eigenforms of slope β, level K', and weight k. We prove the following theorem.

THEOREM 1.1. *There exists an $M > 0$ depending only on β, K', and D, such that if for integers k, k', we have $k \equiv k' \bmod p^M(q-1)$, then*

$$d(K', k, \beta) = d(K', k', \beta).$$

Moreover $d(K', k, \beta)$ is uniformly bounded for all $k \in \mathbb{Z}$.

We shall mention that the $\mathcal{P}$-adic modular forms defined here are in level 'away from $\mathcal{P}$'. These include all modular forms of level 'divisible' by $\mathcal{P}$, but with trivial character at $\mathcal{P}$. We will address the case of 'level divisible by $\mathcal{P}$' in a future work.

Despite the lack of q-expansions in this setting, we can define a notion of congruence modulo powers of π for $\mathcal{P}$-adic modular forms (of possibly different weights). Our setting is quite suitable to study such congruences. Indeed in this paper we already prove a congruence modulo π between two $\mathcal{P}$-adic modular forms (of the same weight), the analogue of which in the classical setting is proven by the q-expansion principle (see Theorem 14.1). We define and study this notion of congruence in an upcoming article where we prove an analogue of Theorem A of [Col97] in this context.

2. Notation and setup

Our reference for this section is [Car86]. Let p be a prime number. Let F be a totally real field of degree $d > 1$ with $\tau_i : F \to \mathbb{R}$ for $1 \leqslant i \leqslant d$ its embeddings in $\mathbb{R}$ (the case $F = \mathbb{Q}$ is done in [Kas99]). We will denote τ_1 simply by τ. The primes of F which lie above p are denoted by $\mathcal{P}_1, \dots, \mathcal{P}_m$, and we will call $\mathcal{P}_1$ simply $\mathcal{P}$. Let $F_{\mathcal{P}}$ denote the completion of F at $\mathcal{P}$. Let $\mathcal{O}_{\mathcal{P}}$ be the ring of integers of $F_{\mathcal{P}}$ with a uniformizer π and residue field κ of order q. We will assume that $q > 3$.

Let B be a quaternion algebra over F which is split at τ and ramified at all other infinite places. We also assume that B is split at $\mathcal{P}$.

Let $\lambda < 0$ be a rational number such that $\mathbb{Q}(\sqrt{\lambda})$ splits at p. Define $E = F(\sqrt{\lambda})$. By choosing a square root of λ in $\mathbb{C}$, the embeddings $\tau_i : F \to \mathbb{R}$ can be extended to embeddings $\tau_i : E \to \mathbb{C}$ for $1 \leqslant i \leqslant d$.

We always consider E to be a subfield of $\mathbb{C}$ via $\tau_1 = \tau$. Choose a square root μ of λ in $\mathbb{Q}_p$. The morphism $E \to F_p \oplus F_p$ which sends $x + y\sqrt{\lambda}$ to $(x + y\mu, x - y\mu)$ extends to an isomorphism

$$E \otimes \mathbb{Q}_p \xrightarrow{\sim} F_p \oplus F_p \xrightarrow{\sim} (F_{\mathcal{P}_1} \oplus \cdots \oplus F_{\mathcal{P}_m}) \oplus (F_{\mathcal{P}_1} \oplus \cdots \oplus F_{\mathcal{P}_m}),$$

which gives an inclusion of E in $F_{\mathcal{P}}$ via the first projection

$$E \hookrightarrow E \otimes \mathbb{Q}_p \xrightarrow{\sim} F_p \oplus F_p \xrightarrow{\mathrm{pr}_1} F_p \xrightarrow{\mathrm{pr}_1} F_{\mathcal{P}}.$$

Let $z \mapsto \bar{z}$ denote the conjugation with respect to F of E. Define $D = B \otimes_F E$ and denote by $l \mapsto \bar{l}$ the product of the canonical involution of B with the conjugation of E over F. Let V denote the underlying $\mathbb{Q}$-vector space of D with left action of D. Choose $\delta \in D$ such that $\bar{\delta} = \delta$ and define an involution on D by $l^* = \delta^{-1}\bar{l}\delta$. Choose $\alpha \in E$ such that $\bar{\alpha} = -\alpha$. One can define a symplectic form Ψ on V: for $v, w \in V$ define

$$\Psi(v, w) = \mathrm{tr}_{E/\mathbb{Q}}(\alpha\, \mathrm{tr}_{D/E}(v\delta w^*)).$$

The symplectic form Ψ is an alternating nondegenerate form on V and satisfies

$$\Psi(lv, w) = \Psi(v, l^*w).$$

Let G' be the reductive algebraic group over $\mathbb{Q}$ such that for any $\mathbb{Q}$-algebra R, $G'(R)$ is group of D-linear symplectic similitudes of $(V \otimes_{\mathbb{Q}} R, \Psi \otimes_{\mathbb{Q}} R)$. Let $\mathbf{S}$ denote $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$. In [Car86] a morphism $h' : \mathbf{S} \to G'_{\mathbb{R}}$ is defined such that X', the $G'(\mathbb{R})$-conjugacy class of h', can be identified with the complex upper half plane, $\mathcal{H}^+$, and the composition $\mathbf{S} \xrightarrow{h'} G'_{\mathbb{R}} \to \mathrm{GL}(V_{\mathbb{R}})$ defines a Hodge structure on $V_{\mathbb{R}}$ which is of type $\{(-1, 0), (0, -1)\}$. One can choose δ in such a way that Ψ becomes a polarization for this Hodge structure, which is to say that the form on $V_{\mathbb{R}}$ defined by $(x, y) \to \Psi(x, yh'(i)^{-1})$ is positive definite.

Let $\mathcal{O}_B$ be a fixed maximal order of B and fix an isomorphism $\mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_{\mathcal{P}} \xrightarrow{\sim} \mathrm{M}_2(\mathcal{O}_{\mathcal{P}})$. Let $\mathcal{O}_D$ be a maximal order of D. Let $V_{\mathbb{Z}}$ denote the corresponding lattice in V. The above mentioned decomposition of $E \otimes \mathbb{Q}_p$ induces the following decompositions of $D \otimes \mathbb{Q}_p$ and $\mathcal{O}_D \otimes \mathbb{Z}_p$:

$$\begin{array}{ccccccccccccc} \mathcal{O}_D \otimes \mathbb{Z}_p & = & \mathcal{O}_{D_1^1} & \oplus & \cdots & \oplus & \mathcal{O}_{D_m^1} & \oplus & \mathcal{O}_{D_1^2} & \oplus & \cdots & \oplus & \mathcal{O}_{D_m^2} \\ \bigcap & & \bigcap & & & & \bigcap & & \bigcap & & & & \bigcap \\ D \otimes \mathbb{Q}_p & = & D_1^1 & \oplus & \cdots & \oplus & D_m^1 & \oplus & D_1^2 & \oplus & \cdots & \oplus & D_m^2 \end{array}$$

where each D_j^k is an $F_{\mathcal{P}_j}$-algebra isomorphic to $B \otimes_F F_{\mathcal{P}_j}$. In particular, D_1^1 and D_1^2 are isomorphic to $\mathrm{M}_2(F_{\mathcal{P}})$, and $l \mapsto l^*$ switches D_j^1 and D_j^2. One can choose $\mathcal{O}_D, \alpha, \delta$ in such a way that the following conditions are satisfied:

i) $\mathcal{O}_D$ is stable under the involution $l \mapsto l^*$;

ii) each $\mathcal{O}_{D_j^k}$ is a maximal order in D_j^k and $\mathcal{O}_{D_1^2} \hookrightarrow D_1^2 = \mathrm{M}_2(F_{\mathcal{P}})$ identifies with $\mathrm{M}_2(\mathcal{O}_{\mathcal{P}})$;

iii) Ψ takes integer values on $V_{\mathbb{Z}}$;

iv) Ψ induces a perfect pairing Ψ_p on $V_{\mathbb{Z}_p} = V_{\mathbb{Z}} \otimes \mathbb{Z}_p$.

Each $\mathcal{O}_D \otimes \mathbb{Z}_p$-module Λ admits a decomposition similar to that of $\mathcal{O}_D \otimes \mathbb{Z}_p$,

$$\Lambda = \Lambda_1^1 \oplus \cdots \oplus \Lambda_m^1 \oplus \Lambda_1^2 \oplus \cdots \oplus \Lambda_m^2,$$

such that each Λ_j^k is an $\mathcal{O}_{D_j^k}$-module. The $\mathrm{M}_2(\mathcal{O}_{\mathcal{P}})$-module Λ_1^2 decomposes further as the direct sum of two $\mathcal{O}_{\mathcal{P}}$-modules $\Lambda_1^{2,1}$ and $\Lambda_1^{2,2}$ by choosing idempotents e and $f = 1 - e$ in $\mathrm{M}_2(\mathcal{O}_{\mathcal{P}})$.

The finite adelic points of G' can be described as

$$G'(\mathbb{A}^f) = \mathbb{Q}_p^* \times \mathrm{GL}_2(F_{\mathcal{P}}) \times (B \otimes_F F_{\mathcal{P}_2})^* \times \cdots \times (B \otimes_F F_{\mathcal{P}_m})^* \times G'(\mathbb{A}^{f,p}).$$

As was mentioned earlier, $G'(\mathbb{A}^f)$ is the group of D-linear symplectic similitudes of $(V \otimes \mathbb{A}^f, \psi \otimes \mathbb{A}^f)$. We describe how it acts on $V \otimes \mathbb{A}^f$ at p. Let

$$V \otimes \mathbb{Q}_p = V_1^1 \oplus \cdots \oplus V_m^1 \oplus V_1^2 \oplus \cdots \oplus V_m^2$$

be the decomposition of $V \otimes \mathbb{Q}_p$ as a $D \otimes \mathbb{Q}_p$-module. It turns out that V_j^i and V_l^k are orthogonal with respect to Ψ unless $i \neq k$ and $j = l$. Now if $(\lambda, g_1, g_2, \ldots, g_m, \gamma) \in G'(\mathbb{A}^f)$, then it acts on V_j^2 by multiplication with g_j for $1 \leqslant j \leqslant m$. One can extend the action of g_j on V_j^2 to V_j^1 by the rule $\Psi(g_j x_j^1, g_j x_j^2) = \lambda \Psi(x_j^1, x_j^2)$ for $x_j^1 \in V_j^1$ and $x_j^2 \in V_j^2$. Finally γ acts on $V \otimes \mathbb{A}^{f,p}$.

3. Quaternionic Shimura curves over F

In this section we will review some basic facts about certain quaternionic Shimura curves over F. Our reference is [Car86]. Let $G = \mathrm{Res}_{F/\mathbb{Q}}(B^*)$. Then $G(\mathbb{Q}) = B^*$ and

$$G(\mathbb{R}) \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{R}) \times (\mathbb{H}^*)^{d-1},$$

where $\mathbb{H}$ denotes the Hamiltonian quaternions. The $G(\mathbb{R})$-conjugacy class of $h : \mathbf{S} \to G_{\mathbb{R}}$ defined by $h(x+iy) = \left(\left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)^{-1}, 1, \ldots, 1\right)$ can be identified with $\mathcal{H}^{\pm} = \mathbb{C} \setminus \mathbb{R}$. Consider the projective system of Shimura curves over $\mathbb{C}$ associated to the pair (G, X), indexed by open-compact subgroups K of $G(\mathbb{A}^f) = (B \otimes_F \mathbb{A}_F^f)^*$,

$$M_K(\mathbb{C}) = G(\mathbb{Q}) \backslash G(\mathbb{A}^f) \times \mathcal{H}^{\pm} / K.$$

Here K acts trivially on $\mathcal{H}^{\pm}$ and by right multiplication on $G(\mathbb{A}^f)$. The action of $G(\mathbb{Q})$ on $\mathcal{H}^{\pm}$ is via $G(\mathbb{Q}) \hookrightarrow G(\mathbb{R}) \xrightarrow{\mathrm{pr}_1} \mathrm{GL}_2(\mathbb{R})$. There is a canonical model over F for $M_K(\mathbb{C})$ denoted by M_K which is smooth and proper. The curves M_K are not PEL Shimura curves.

We consider a special class of subgroups K. Let $K_{\mathcal{P}}^n \subset \mathrm{GL}_2(F_{\mathcal{P}}) \xrightarrow{\sim} (B \otimes_F F_{\mathcal{P}})^*$ be the subgroup consisting of elements of $\mathrm{GL}_2(\mathcal{O}_{\mathcal{P}})$ which are congruent to identity modulo π^n. Let Γ be the restricted

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product of $(B \otimes_F F_\nu)^*$ for all finite places ν of F but $\mathcal{P}$. Consider subgroups of $G(\mathbb{A}^f)$ of the form

$$K = K_{\mathcal{P}}^n \times H,$$

where H is an open compact subgroup of Γ. For such K denote M_K by $M_{n,H}$.

If H is small enough, then $M_{0,H}$ has a smooth and proper model $\mathbb{M}_{0,H}$ defined over $\mathcal{O}_{(\mathcal{P})}$. The $\mathbb{M}_{0,H}$ for all possible H form a projective system all of whose transition morphisms are étale. These models are constructed from canonical integral models of certain unitary Shimura curves which are moduli spaces of PEL abelian schemes. In the next section we will review these curves.

4. Unitary Shimura curves $M'_{K'}$

Let X' be the $G'(\mathbb{R})$-conjugacy class of $h' : \mathbf{S} \to G'_{\mathbb{R}}$. One associates to (G', X') and any subgroup $K' \subset G'(\mathbb{A}^f)$ which is open and compact, a Shimura curve over $\mathbb{C}$,

$$M'_{K'}(\mathbb{C}) = G'(\mathbb{Q}) \backslash G'(\mathbb{A}^f) \times X'/K',$$

which is a compact Riemann surface. Here K' acts on $G'(\mathbb{A}^f)$ by multiplication and $G'(\mathbb{Q})$ acts on X' by conjugation. Contrary to the case of modular curves, these curves do not have cusps.

The Riemann surface $M'_{K'}(\mathbb{C})$ has a canonical smooth and proper model $M'_{K'}$ which is defined over E. We will use the injection of E in $F_{\mathcal{P}}$ defined in § 2 to base change $M'_{K'}$ to $F_{\mathcal{P}}$. We will denote this by $M'_{K'}$ again. The transition morphisms of the projective system $\{M'_{K'}\}_{K'}$ which is indexed by open compact subgroups of $G'(\mathbb{A}^f)$ are étale morphisms.

4.1 Moduli problem over $F_{\mathcal{P}}$

We describe a moduli problem over $F_{\mathcal{P}}$ which is represented by $M'_{K'}$. We assume K' to be small enough to keep the lattice $V_{\hat{\mathbb{Z}}} = V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}} \subset V \otimes \mathbb{A}^f$ invariant.

Let A be an abelian scheme defined over an $F_{\mathcal{P}}$-algebra R which has an action of $\mathcal{O}_D$. Then $\mathrm{Lie}(A)$ is an $\mathcal{O}_D \otimes \mathbb{Z}_p$-module and hence, from § 2, it admits a decomposition

$$\mathrm{Lie}(A) = \mathrm{Lie}_1^1(A) \oplus \cdots \oplus \mathrm{Lie}_m^1(A) \oplus \mathrm{Lie}_1^2(A) \oplus \cdots \oplus \mathrm{Lie}_m^2(A),$$

where $\mathrm{Lie}_j^k(A)$ is a projective R-module with an action of $\mathcal{O}_{D_j^k}$. Furthermore $\mathrm{Lie}_1^2(A)$ admits a decomposition

$$\mathrm{Lie}_1^2(A) = \mathrm{Lie}_1^{2,1}(A) \oplus \mathrm{Lie}_1^{2,2}(A)$$

to two isomorphic projective R-modules which have an action of $\mathcal{O}_{\mathcal{P}}$. Similarly, any finite flat subgroup scheme C of A which is killed by a power of p admits an action of $\mathcal{O}_D \otimes \mathbb{Z}_p$ and hence it decomposes as $C_1^1 \oplus \cdots \oplus C_m^1 \oplus C_1^2 \oplus \cdots \oplus C_m^2$. The part C_1^2 admits a decomposition $C_1^{2,1} \oplus C_1^{2,2}$ into two isomorphic finite flat subgroups with an $\mathcal{O}_{\mathcal{P}}$-action. In particular, $(A[p^n])_j^k$ is defined for $1 \leqslant j \leqslant m$, and $k = 1, 2$, and we have a decomposition into a sum of two $\mathcal{O}_{\mathcal{P}}$-modules of $(A[p^n])_1^2 = (A[p^n])_1^{2,1} \oplus (A[p^n])_1^{2,2}$. We will let $(A[\pi^n])_1^{2,k}$ denote the π^n-torsion in $(A[p^n])_1^{2,k}$ and define $(A[\pi^n])_1^2 := (A[\pi^n])_1^{2,1} \oplus (A[\pi^n])_1^{2,2}$.

The Shimura curve $M'_{K'}$ represents the functor

$$\mathcal{M}_{K'} : ((F_{\mathcal{P}}\text{-algebras})) \longrightarrow ((\text{sets}))$$

where for any $F_{\mathcal{P}}$-algebra R, $\mathcal{M}_{K'}(R)$ consists of the isomorphism classes of all quadruples $(A, i, \theta, \bar{\alpha})$ such that:

i) A is an abelian scheme of relative dimension $4d$ over R with an action of $\mathcal{O}_D$ via $i : \mathcal{O}_D \hookrightarrow \mathrm{End}_R(A)$, which satisfies
   a) the projective R-module $\mathrm{Lie}_1^{2,1}(A)$ has rank one and $\mathcal{O}_{\mathcal{P}}$ acts on it via $O_{\mathcal{P}} \hookrightarrow R$;

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b) for $j \geqslant 2$, we have $\mathrm{Lie}^2_j(A) = 0$;

ii) θ is a polarization of A (of degree prime to p) such that the corresponding Rosati involution sends $i(l)$ to $i(l^*)$;

iii) $\bar{\alpha}$ is a K' level structure which is a class modulo K' of symplectic $\mathcal{O}_D$-linear isomorphisms $\alpha : \hat{T}(A) \xrightarrow{\sim} V_{\hat{\mathbb{Z}}}$ (locally in étale topology).

Here $\hat{T}(A) = \prod_p T_p(A)$ denotes $\varprojlim_n A[n]$ as a sheaf over $\mathrm{Spec}(R)$ in the étale topology and the symplectic form on $\hat{T}(A)$ comes from the Weil pairing composed with the polarization θ. Note that the condition on the Lie algebra is an equivalent form of the trace condition (coming from h') when the base is $F_{\mathcal{P}}$.

Let $\Gamma' = G'(\mathbb{A}^{f,p}) \times (B \otimes_F F_{\mathcal{P}_2})^* \times \cdots \times (B \otimes_F F_{\mathcal{P}_m})^*$. As shown in § 2, we have $G'(\mathbb{A}^f) = \mathbb{Q}_p^* \times \mathrm{GL}_2(F_{\mathcal{P}}) \times \Gamma'$. We will only consider subgroups of the form

$$K' = \mathbb{Z}_p^* \times K^n_{\mathcal{P}} \times H' \hookrightarrow \mathbb{Q}_p^* \times \mathrm{GL}_2(F_{\mathcal{P}}) \times \Gamma',$$

where $K^n_{\mathcal{P}}$ is defined in § 3 and H' is an open compact subgroup of Γ'. We sometimes denote such $M'_{K'}$ by $M'_{n,H'}$. For this particular choice of K' the above moduli problem can be stated differently. Let $\hat{T}^p(A)$ denote $\prod_{l \neq p} T_l(A)$ and

$$T_p(A) = (T_p(A))^1_1 \oplus \cdots \oplus (T_p(A))^1_m \oplus (T_p(A))^2_1 \oplus \cdots \oplus (T_p(A))^2_m$$

be the decomposition of $T_p(A)$ as an $\mathcal{O}_D \otimes \mathbb{Z}_p$-module. Denote $(T_p(A))^2_2 \oplus \cdots \oplus (T_p(A))^2_m$ by $T^{\mathcal{P}}_p$. Let $\hat{W}^p$ denote $V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}^p$ and $W^{\mathcal{P}}_p$ denote the direct sum $(V_{\mathbb{Z}_p})^2_2 \oplus \cdots \oplus (V_{\mathbb{Z}_p})^2_m$. The level structure $\bar{\alpha}$ in the description of the functor $\mathcal{M}_{K'}$ can be replaced with the following data:

1) $\alpha_{\mathcal{P}}$ is an isomorphism of $\mathcal{O}_{\mathcal{P}}/\pi^n$-modules $\alpha_{\mathcal{P}} : (A[\pi^n])^{2,1}_1 \xrightarrow{\sim} (\pi^{-n}\mathcal{O}_{\mathcal{P}}/\mathcal{O}_{\mathcal{P}})^2$;
2) $\bar{\alpha}^{\mathcal{P}}$ is a class of isomorphisms $\alpha^{\mathcal{P}} = \alpha^{\mathcal{P}}_p \oplus \alpha^p : T^{\mathcal{P}}_p(A) \oplus \hat{T}^p(A) \xrightarrow{\sim} W^{\mathcal{P}}_p \oplus \hat{W}^p$ modulo H', with $\alpha^{\mathcal{P}}_p$ linear and α^p symplectic.

When $n = 0$ the condition 1 disappears.

4.2 Moduli problem over $\mathcal{O}_{\mathcal{P}}$

Let $K' = \mathbb{Z}_p^* \times K^0_{\mathcal{P}} \times H'$ be an open compact subgroup of $G'(\mathbb{A}^f)$. When H' is small enough there is a smooth and proper scheme $\mathbb{M}'_{K'} = \mathbb{M}'_{0,H'}$ defined over $\mathcal{O}_{\mathcal{P}}$ such that $\mathbb{M}'_{0,H'} \otimes F_{\mathcal{P}} \xrightarrow{\sim} M'_{0,H'}$ and $\mathbb{M}'_{0,H'}$ represents the functor

$$\mathcal{M}_{0,H'} : ((\mathcal{O}_{\mathcal{P}}\text{-algebras})) \longrightarrow ((\text{sets}))$$

such that for any $\mathcal{O}_{\mathcal{P}}$-algebra R, $\mathcal{M}_{0,H'}(R)$ is the set of all isomorphism classes of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ such that:

i) A is an abelian scheme over R of relative dimension $4d$ equipped with an action of $\mathcal{O}_D$ given by $i : \mathcal{O}_D \hookrightarrow \mathrm{End}_R(A)$ such that
   a) the projective R-module $\mathrm{Lie}^{2,1}_1(A)$ has rank one and $\mathcal{O}_{\mathcal{P}}$ acts on it via $O_{\mathcal{P}} \hookrightarrow R$,
   b) for $j \geqslant 2$, we have $\mathrm{Lie}^2_j(A) = 0$;

ii) θ is a polarization of A of degree prime to p such that the corresponding Rosati involution sends $i(l)$ to $i(l^*)$;

iii) $\bar{\alpha}^{\mathcal{P}}$ is a class of isomorphisms $\alpha^{\mathcal{P}} = \alpha^{\mathcal{P}}_p \oplus \alpha^p : T^{\mathcal{P}}_p(A) \oplus \hat{T}^p(A) \xrightarrow{\sim} W^{\mathcal{P}}_p \oplus \hat{W}^p$ modulo H', with $\alpha^{\mathcal{P}}_p$ linear and α^p symplectic.

There is a universal $(\mathbb{A}'_{K'} = \mathbb{A}'_{0,H'}, i, \theta, \bar{\alpha}^{\mathcal{P}})$ defined over $\mathbb{M}'_{0,H'}$ such that any $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ over an $\mathcal{O}_{\mathcal{P}}$-algebra R is obtained by pulling back the universal quadruple via the corresponding morphism $\mathrm{Spec}(R) \to \mathbb{M}'_{0,H'}$. Let $\epsilon : \mathbb{A}'_{0,H'} \to \mathbb{M}'_{0,H'}$ denote the structure map. The $\mathcal{O}_{\mathbb{M}'_{0,H'}}$-module

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$\epsilon_*\Omega^1_{\mathbb{A}'_{0,H'}/\mathbb{M}'_{0,H'}}$ is an $\mathcal{O}_D \otimes \mathbb{Z}_p$-module. Let $\underline{\omega} = \underline{\omega}_{\mathbb{A}'_{0,H'}/\mathbb{M}'_{0,H'}}$ denote $(\epsilon_*\Omega^1_{\mathbb{A}'_{0,H'}/\mathbb{M}'_{0,H'}})^{2,1}_1$. A similar construction could be done for any $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ defining $\underline{\omega}_{A/R}$, which could also be obtained by pulling back $\underline{\omega}_{\mathbb{A}'_{0,H'}/\mathbb{M}'_{0,H'}}$ via the morphism $\mathrm{Spec}(R) \to \mathbb{M}'_{0,H'}$. We usually drop the subscript and use $\underline{\omega}$ whenever there is no confusion. The fact that $\mathrm{Lie}^{2,1}_1(\mathbb{A}'_{0,H'})$ is locally free of rank one shows that $\underline{\omega}$ is indeed a line bundle over $\mathbb{M}'_{0,H'}$. We have the following Kodaira–Spencer isomorphism in this case.

Proposition 4.1.

i) $\underline{\omega}_{\mathbb{A}'_{0,H'}/\mathbb{M}'_{0,H'}} \otimes \underline{\omega}_{(\mathbb{A}'_{0,H'})^\vee/\mathbb{M}'_{0,H'}} \xrightarrow{\sim} \Omega^1_{\mathbb{M}'_{0,H'}/\mathcal{O}_{\mathcal{P}}}$.

ii) *There is a noncanonical isomorphism* $(\underline{\omega}_{\mathbb{A}'_{0,H'}/\mathbb{M}'_{0,H'}})^{\otimes 2} \xrightarrow{\sim} \Omega^1_{\mathbb{M}'_{0,H'}/\mathcal{O}_{\mathcal{P}}}$.

Proof. For part i adopt the proof of Lemma 7 in [DT94]. For part ii note that since θ is prime to p, the rank of the kernel of θ is invertible on $\mathbb{M}'_{0,H'}$ and hence θ is étale. This gives an isomorphism $\underline{\omega}_{\mathbb{A}'_{0,H'}/\mathbb{M}'_{0,H'}} \xrightarrow{\sim} \underline{\omega}_{(\mathbb{A}'_{0,H'})^\vee/\mathbb{M}'_{0,H'}}$. ☐

Finally, we state Corollary 4.5.4 of [Car86] which relates the Shimura curves $M'_{0,H'}$ and $M_{0,H}$. Let $F^{\mathrm{nr}}_{\mathcal{P}}$ denote the maximal unramified extension of $F_{\mathcal{P}}$.

Theorem 4.2. *Let $H \subset \Gamma$ be a small enough open compact subgroup, and N_H a connected component of $M_{0,H} \otimes F^{\mathrm{nr}}_{\mathcal{P}}$. Then there exists an open compact subgroup $H' \subset \Gamma'$, and a connected component of $N'_{H'}$ of $M'_{0,H'} \otimes F^{\mathrm{nr}}_{\mathcal{P}}$, such that N_H and $N'_{H'}$ are isomorphic over $F^{\mathrm{nr}}_{\mathcal{P}}$.*

As we explained in § 1, this establishes a close connection between automorphic forms on the two types of Shimura curves. From now on we will confine our attention to Shimura curves $M'_{K'}$.

4.3 Formal $\mathcal{O}_{\mathcal{P}}$-module and π-divisible subgroup of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$

Let R be a π-adically complete $\mathcal{O}_{\mathcal{P}}$-algebra. Let $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ over R be as in § 4.2. Let $A[p^\infty] = \varinjlim A[p^n]$ be the p-divisible subgroup of A. Then

$$A[p^\infty] = (A[p^\infty])^1_1 \oplus \cdots \oplus (A[p^\infty])^1_m \oplus (A[p^\infty])^2_1 \oplus \cdots \oplus (A[p^\infty])^2_m$$

and we have a decomposition $(A[p^\infty])^2_1 = (A[p^\infty])^{2,1}_1 \oplus (A[p^\infty])^{2,2}_1$. We call $(A[\pi^\infty])^{2,1}_1 := \varinjlim (A[\pi^n])^{2,1}_1$ the π-divisible subgroup of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ (see § 4.1).

Since θ is prime to p, $\theta : A[p^\infty] \to (A[p^\infty])^\vee$ is an isomorphism. Since the involution $*$ interchanges $\mathcal{O}_{D^1_j}$ and $\mathcal{O}_{D^2_j}$, θ induces isomorphisms

$$(A[p^\infty])^1_j \xrightarrow{\theta} ((A[p^\infty])^2_j)^\vee.$$

Since $\mathrm{Lie}^2_j(A) = 0$ for $j \geqslant 2$, we know that $A[p^\infty]^2_j$ is étale for $j \geqslant 2$.

4.3.1 *Formal $\mathcal{O}_{\mathcal{P}}$-modules.* By a formal $\mathcal{O}_{\mathcal{P}}$-module G over R we mean a formal group law G of dimension one with an action of $\mathcal{O}_{\mathcal{P}}$ where 1 acts as identity, and the action of $\mathcal{O}_{\mathcal{P}}$ on $\mathrm{Lie}(G)$ is via the structural morphism $\mathcal{O}_{\mathcal{P}} \to R$. The height of a formal $\mathcal{O}_{\mathcal{P}}$-module G is an integer $\mathrm{ht}(G)$ such that $\mathrm{rk}(G[\pi^j]) = q^{j\,\mathrm{ht}(G)}$ for any $j \geqslant 0$. Let $[\pi]_{\bar{G}}$ denote the power series giving the multiplication by π in the reduction of G modulo π. Then $\mathrm{ht}(G)$ can be characterized as the largest integer n such that $[\pi]_{\bar{G}}$ is a power series in x^{q^n}. If $\pi = 0$ in R, then there are morphisms $\mathrm{Fr}_q : G \to G^{(q)}$ and $\mathrm{V} : G^{(q)} \to G$ such that $\mathrm{Fr}_q(x) = x^q$ and $\mathrm{V}(\mathrm{Fr}_q(x)) = [\pi]_G$ and $\mathrm{Fr}_q(\mathrm{V}(x)) = [\pi]_{G^{(q)}}$.

Let G_A denote the formal completion of A at its identity section. Zariski locally on the base G_A is a formal group law with an action of $\mathcal{O}_D$. Since G_A has also an action of $\mathbb{Z}_p$ it decomposes as an

$\mathcal{O}_D \otimes \mathbb{Z}_p$-module. By definition of the moduli problem in § 4.2 the component $(G_A)_1^{2,1}$ is a formal $\mathcal{O}_{\mathcal{P}}$-module of dimension one.

PROPOSITION 4.3. *Let R be an $\mathcal{O}_{\mathcal{P}}$-algebra and G a formal $\mathcal{O}_{\mathcal{P}}$-module over R. There exists a coordinate x on G such that for each $(q-1)$th root of unity $\zeta \in \mathcal{O}_{\mathcal{P}}$ we have $[\zeta](x) = \zeta x$ (where $[\zeta]$ denotes the power series in x giving the action of $\zeta \in \mathcal{O}_{\mathcal{P}}$ on G). In this coordinate the action of π takes the special form*

$$[\pi](x) = \pi x + a x^q + \sum_{j=2}^{\infty} c_j x^{j(q-1)+1},$$

where $a, c_j (j \geqslant 2) \in R$ and $c_j \in \pi R$ unless $j \equiv 1 \bmod q$.

Proof. This can be proven in exactly the same way as Proposition 3.6.6 in [Kat73] by replacing F, V by $\mathrm{Fr}_q, \mathrm{V}$ in this context. □

4.4 Quotient of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ by a finite flat subgroup of A

We will define the quotient of a test object $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ as in § 4.2 by certain finite flat subgroups of A. Let $C \subset A$ be a finite flat subgroup scheme. We assume C to satisfy the following properties:

i) C is killed by q, is $\mathcal{O}_D$-invariant, and has rank $q^{\dim(A)}$;

ii) the isomorphism $\theta : A[q] \xrightarrow{\sim} A[q]^\vee$ takes $C \subset A[q]$ to $(A[q]/C)^\vee \subset A[q]^\vee$;

iii) $C_p^{\mathcal{P}} := C_2^2 \oplus \cdots \oplus C_m^2 = 0$ or $C_p^{\mathcal{P}} = (A[q])_2^2 \oplus \cdots \oplus (A[q])_m^2$.

If C satisfies these conditions, we say C is of type 1 if $C_p^{\mathcal{P}} = 0$ and of type 2 if $C_p^{\mathcal{P}} = (A[q])_2^2 \oplus \cdots \oplus (A[q])_m^2$. Note that any C of type 1 or 2 is uniquely determined by $C_1^{2,1}$.

Since C is $\mathcal{O}_D$-invariant $A' = A/C$ inherits an action of $\mathcal{O}_D$ which makes the natural projection $A \to A'$ $\mathcal{O}_D$-equivariant. We will denote this action by i'.

4.4.1 *Definition of θ'.* Let (A, i) be an abelian scheme with an action of $\mathcal{O}_D$ and θ a polarization of A. We will say that θ is compatible with the action of $\mathcal{O}_D$ if the associated Rosati involution coincides with $l \mapsto l^*$ on $\mathcal{O}_D$.

LEMMA 4.4. *Let A and A' be abelian schemes over R each equipped with an action of $\mathcal{O}_D$ and $f : A \to A'$ be an $\mathcal{O}_D$-equivariant isogeny of kernel C and degree $q^{\dim(A)}$. Let $\theta : A \to A^\vee$ be a polarization compatible with the action of $\mathcal{O}_D$. If θ takes $C \subset A$ onto $(A[q]/C)^\vee \subset A^\vee$, then there is a unique polarization $\theta' : A' \to (A')^\vee$ which is compatible with the action of $\mathcal{O}_D$ and such that the following diagram is commutative:*

$$\begin{tikzcd} A' \arrow[r, "g"] \arrow[d, "\theta'"] & A \arrow[d, "\theta"] \\ (A')^\vee \arrow[r, "f^\vee"] & A^\vee \end{tikzcd}$$

where g is the unique isogeny such that $g \circ f = [q]$ on A. Furthermore $\deg(\theta) = \deg(\theta')$.

Proof. Note that f identifies A' with A/C. Since $\mathrm{Ker}(g) \xrightarrow{\sim} A[q]/C$, we know that the kernel of $g^\vee : A^\vee \to (A')^\vee$ is $(A[q]/C)^\vee \subset A^\vee$. This identifies $(A')^\vee$ with $A^\vee/(A[q]/C)^\vee$. Now since θ takes C onto $(A[q]/C)^\vee$, it defines an isogeny (indeed a polarization) $\theta' : A' \to (A')^\vee$ for which the above diagram is commutative. The compatibility of θ' with the action of $\mathcal{O}_D$ is the result of the same fact for θ and the fact that f is $\mathcal{O}_D$-equivariant. Uniqueness is clear from construction. The fact that $\mathrm{rk}(C) = q^{\dim(A)}$ implies $\deg(\theta) = \deg(\theta')$. □

4.4.2 *Definition of* $(\bar{\alpha}^{\mathcal{P}})'$. Let $f : A \to A'$ and $g : A' \to A$ be as above. Since $\mathrm{rk}(C)$ is relatively prime to any prime number different from p, the map $T^p(g) : T^p(A') \to T^p(A)$ induced by g is an isomorphism. Define

$$(\alpha^p)' = \alpha^p \circ T^p(g).$$

We define $(\alpha_p^{\mathcal{P}})'$ depending on the type of C. Let C be of type 1. Then f induces an isomorphism $T_p^{\mathcal{P}}(f) : T_p^{\mathcal{P}}(A) \xrightarrow{\sim} T_p^{\mathcal{P}}(A')$ as $(\mathrm{Ker}(f))_p^{\mathcal{P}} = 0$. We define

$$(\alpha_p^{\mathcal{P}})' := \alpha_p^{\mathcal{P}} \circ (T_p^{\mathcal{P}}(f))^{-1}.$$

If C is of type 2, then $\mathrm{Ker}(g)$ is of type 1 and therefore g induces an isomorphism $T_p^{\mathcal{P}}(g) : T_p^{\mathcal{P}}(A') \xrightarrow{\sim} T_p^{\mathcal{P}}(A)$. We define

$$(\alpha_p^{\mathcal{P}})' := \alpha_p^{\mathcal{P}} \circ T_p^{\mathcal{P}}(g).$$

Finally $(\bar{\alpha}^{\mathcal{P}})'$ is defined as the class of $(\alpha_p^{\mathcal{P}})' \oplus (\alpha^p)'$ modulo K'. Note that

$$((\bar{\alpha}_p^{\mathcal{P}} \oplus \bar{\alpha}^p)')' = \bar{\alpha}_p^{\mathcal{P}} \oplus q\bar{\alpha}^p.$$

In the rest of this section we will define a pullback morphism from $\underline{\omega}_{A/R}^{\otimes k}$ to $\underline{\omega}_{A'/R}^{\otimes k}$. The pullback morphism g^*, defined via $g : A' \to A$, satisfies $g^* \circ f^* = q^k$. We will however need a more refined pullback.

Proposition 4.5. *Let C be of type 1 or 2. Assume $C_1^{2,1} \subset (A[\pi])_1^{2,1}$. Let $f : A \to A/C$ be the projection. Then for each $k \geqslant 0$ there is a functorial R-morphism*

$$(f')^* : \underline{\omega}_{A/R}^{\otimes k} \to \underline{\omega}_{(A/C)/R}^{\otimes k}$$

such that $(f')^* \circ f^* = \pi^k$ and $f^* \circ (f')^* = \pi^k$. Furthermore if $\pi = 0$ in R and $f = \mathrm{Fr}_q$, then $(f')^* = \mathrm{V}^*$.

Proof. It is enough to construct $(f')^*$ locally. The $\mathcal{O}_D$-isogeny $f : A \to A/C$ induces a morphism of formal $\mathcal{O}_{\mathcal{P}}$-modules, $[f] : (G_A)_1^{2,1} \to (G_{A/C})_1^{2,1}$, whose kernel is the intersection of $C_1^{2,1}$ with $(G_A)_1^{2,1}$, and hence we have $\mathrm{Ker}([f]) \subset (G_A)_1^{2,1}[\pi] = \mathrm{Ker}[\pi]$. Therefore, there is a morphism of formal $\mathcal{O}_{\mathcal{P}}$-modules $[f']$ making the following diagram commutative:

$$\begin{tikzcd}
(G_A)_1^{2,1} \arrow[r, "{[\pi]}"] \arrow[dr, "{[f]}"'] & (G_A)_1^{2,1} \\
& (G_{A/C})_1^{2,1} \arrow[u, "{[f']}"']
\end{tikzcd}$$

Therefore, $[f'] \circ [f] = [\pi]$. Now just define $(f')^*$ to be $[f']^* : \underline{\omega}_{(G_A)_1^{2,1}} \to \underline{\omega}_{(G_{A/C})_1^{2,1}}$ noting that $\underline{\omega}_{A/R} = \underline{\omega}_{(G_A)_1^{2,1}}$. Now $(f')^*$ satisfies the desired property since the action of $[\pi]$ on $\underline{\omega}$ is via multiplication by π. If $f = \mathrm{Fr}_q$, then $f' = \mathrm{V}$ and hence we have $(f')^* = \mathrm{V}^*$ (see § 4.3.1). $\square$

5. Modular forms with respect to D

Let H' be an open compact subgroup of $\Gamma' = G'(\mathbb{A}^{f,p}) \times (B \otimes_F F_{\mathcal{P}_2})^* \times \cdots \times (B \otimes_F F_{\mathcal{P}_m})^*$ and $K' = \mathbb{Z}_p^* \times K_{\mathcal{P}}^0 \times H'$ which is an open compact subgroup of $G'(\mathbb{A}^f)$. Let R_0 be an $\mathcal{O}_{\mathcal{P}}$-algebra.

5.1 Modular forms of level $K' = \mathbb{Z}_p{}^* \times K^0_{\mathcal{P}} \times H'$

A modular form with respect to D of weight $k \in \mathbb{Z}$ and level K' over R_0 is a rule which assigns to any $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega)$ over R, where

i) R is an R_0-algebra,
ii) $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ is as in § 4.2,
iii) ω is a basis for $\underline{\omega}_{A/R}$,

an element $f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega) \in R$ such that

i) $f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega)$ only depends on the R-isomorphism class of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega)$,
ii) the formation of $f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega)$ commutes with arbitrary base change of R_0-algebras,
iii) for any $\lambda \in R^*$ we have

$$f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, \lambda\omega) = \lambda^{-k} f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega).$$

Alternatively one can define a modular form with respect to D of weight $k \in \mathbb{Z}$ and level K' over R_0 to be a rule which assigns to any quadruple $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ as in § 4.2 a section $f(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ of $\underline{\omega}^{\otimes k}_{A/R}$ over $\mathrm{Spec}(R)$ such that

i) $f(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ depends only on the isomorphism class of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ over R,
ii) the formation of $f(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ commutes with arbitrary change of base over R_0.

The two definitions are related by $f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}) = f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega)\omega^{\otimes k}$. We denote the space of modular forms of weight k and level K' over R_0 by $S^D(R_0, K', k)$. From the definition it is immediate that if K' is small enough then

$$S^D(R_0, K', k) = H^0(\mathbb{M}'_{K'} \otimes R_0, \underline{\omega}^{\otimes k}).$$

6. The Hasse invariant

Let $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ over $\bar{\kappa}$ be as in § 4.2, and $(G_A)^{2,1}_1$ be the associated formal $\mathcal{O}_{\mathcal{P}}$-module. The height of $(G_A)^{2,1}_1$ as a formal $\mathcal{O}_{\mathcal{P}}$-module is either one or two.

We say that $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ is *supersingular* if $\mathrm{ht}((G_A)^{2,1}_1) = 2$. We will construct the *Hasse invariant* $\mathbf{H}$ which is a modular form of weight $q-1$ over $\mathbb{M}'_{K'} \otimes \kappa$ and vanishes exactly at the supersingular points. We will construct $\mathbf{H}$ as a section of $\underline{\omega}^{\otimes q-1}$ over $\mathbb{M}'_{K'} \otimes \kappa$.

Let $W = \mathrm{Spec}(R)$ be an open affine subset of $\mathbb{M}'_{K'} \otimes \kappa$. Let A be the restriction of $\mathbb{A}'_{K'} \otimes \kappa$ to W. Choose a nonvanishing section of $\underline{\omega}_{A/R}$, say ω, on W. Choose a coordinate on $(G_A)^{2,1}_1$ such that $\omega = (1 + a_1 x + a_2 x^2 + \dots)\, dx$. From § 4.3.1 we have $[\pi](x) = \mathrm{V}(x^q) = ax^q + \dots$. Define

$$\mathbf{H}|_W := a\omega^{\otimes q-1}.$$

We show that this definition is independent of the choice of ω and the dual coordinate. Replacing ω by $\omega' = \lambda\omega$ where λ is a unit of R amounts to replacing the coordinate x with another coordinate $y = \lambda x + \dots$. This in turn amounts to replacing a with $a' = \lambda a \lambda^{-q} = a\lambda^{1-q}$. But then

$$a'\omega'^{\otimes q-1} = a\lambda^{1-q}(\lambda\omega)^{\otimes q-1} = a\omega^{\otimes q-1}.$$

This argument shows that the locally defined sections of $\underline{\omega}^{\otimes q-1}$ glue together to give a global section $\mathbf{H}$ of this line bundle on $\mathbb{M}'_{K'} \otimes \kappa$ which we define to be the Hasse invariant.

Assume that $\mathbf{H}$ vanishes at a geometric point $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ of $\mathbb{M}'_{K'} \otimes \kappa$. Then for a choice of coordinate x on $(G_A)^{2,1}_1$ we have $[\pi](x) = \mathrm{V}(x^q)$ with $\mathrm{V}(0) = \mathrm{V}'(0) = 0$. This implies $\mathrm{rk}((G_A)^{2,1}_1[\pi]) \neq q$ which means that $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ is supersingular. We have shown the following.

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Proposition 6.1. *If $\pi = 0$ in R_0, then there is an $\mathbf{H} \in S^D(R_0, K', q-1)$ which vanishes at a geometric point $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ of $\mathbb{M}'_{K'} \otimes R_0$ exactly when $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ is supersingular.*

Following Proposition 4.3 we can now show the following.

Proposition 6.2. *Let R be an $\mathcal{O}_{\mathcal{P}}$-algebra and $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ be as in § 4.2. Let x be a coordinate on $(G_A)_1^{2,1}$ as in Proposition 4.3 and ω a differential dual to x. Then*

$$[\pi](x) = \pi x + a x^q + \sum_{j=2}^{\infty} c_j x^{j(q-1)+1},$$

where $a, c_j (j \geqslant 2) \in R$ and $c_j \in \pi R$ unless $j \equiv 1 \bmod q$. Furthermore,

$$a \equiv \mathbf{H}(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega) \mod \pi.$$

Proof. Reducing $[\pi](x)$ modulo π and noting that over R/π we have $[\pi](x) = \mathrm{V}(x^q)$, we find out that $a \equiv \mathrm{V}'(0) \bmod \pi$. Hence, by the definition of $\mathbf{H}$ we get the desired congruence. □

As is the case for the classical Hasse invariant, we have the following important property of $\mathbf{H}$. We will use this in Lemma 8.2 which is itself needed in the proof of Proposition 13.1 and hence Corollary 13.2.

Proposition 6.3. *$\mathbf{H}$ has simple zeros on $\mathbb{M}'_{K'} \otimes \bar{\kappa}$.*

Proof. Let $\bar{x} = (A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ be a geometric point of $\mathbb{M}'_{K'} \otimes \bar{\kappa}$ at which $\mathbf{H}$ has a zero of multiplicity bigger than one. Let $\mathcal{O}_{\bar{x}}$ denote the local ring of $\mathbb{M}'_{K'} \otimes \bar{\kappa}$ at $\bar{x}$ and $\mathcal{M}$ denote its maximal ideal. Let G denote the universal formal $\mathcal{O}_{\mathcal{P}}$-module at $\mathcal{O}_{\bar{x}}$. Then by assumption $\mathbf{H} = 0$ in $G_2 := G \otimes (\mathcal{O}_{\bar{x}}/\mathcal{M}^2)$. So G_2 has height two. This implies that $\mathrm{V}(x) = L(x^q)$ with $L'(0) \neq 0$. So $\mathrm{V}(x) = x^q \cdot$ (unit in $(\mathcal{O}_{\bar{x}}/\mathcal{M}^2)[[x]]$). Hence, $\mathrm{V} : G_2^{(q)} \to G_2$ and $\mathrm{Fr}_q : G_2^{(q)} \to G_2^{(q^2)}$ (which is given by $\mathrm{Fr}_q(x) = x^q$) have the same kernel. This implies that

$$G_2 \xrightarrow{\sim} G_2^{(q)}/\mathrm{Ker}(\mathrm{V}) \xrightarrow{\sim} G_2^{(q)}/\mathrm{Ker}(\mathrm{Fr}_q) \xrightarrow{\sim} G_2^{(q^2)}.$$

On the other hand, the q^2th power map $\mathcal{O}_{\bar{x}}/\mathcal{M}^2 \xrightarrow{q^2} \mathcal{O}_{\bar{x}}/\mathcal{M}^2$ kills $\mathcal{M}$ and hence factors through $\bar{\kappa}$. This means that $G_2^{(q^2)}$ and hence G_2 is defined over $\bar{\kappa}$. However, this is impossible since G_2 is reduction modulo $\mathcal{M}^2$ of the universal G. □

7. Lifting the Hasse invariant

In the classical setting the Eisenstein series of weight $p-1$, $E_{p-1}(q)$, which is a modular form of weight $p-1$ and level 1 over $\mathbb{Z}_p$ gives a canonical lifting of the Hasse invariant to $\mathbb{Z}_p$ ($p > 3$). Because of the lack of q-expansions there are no Eisenstein series in this context. However, $\mathbf{H}$ can be noncanonically lifted to a modular form of weight $q-1$ and level K' over $\mathcal{O}_{\mathcal{P}}$. This lifting will help us define $\mathcal{P}$-adic modular forms over our Shimura curves by giving a way to measure the supersingularity of a test object $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$. Here $K' = \mathbb{Z}_p^* \times K_{\mathcal{P}}^0 \times H'$ is an open compact subgroup of $G'(\mathbb{A}^f)$.

Lemma 7.1. *If H is small enough, then $H^1(\mathbb{M}'_{K'}, \underline{\omega}^{\otimes k}) = 0$ for $k \geqslant 3$.*

Proof. Assume that H' is small enough so that $M'_{K'}$ has genus bigger than zero. First note that by Proposition 4.1 we have $H^1(\mathbb{M}'_{K'} \otimes \bar{F_{\mathcal{P}}}, \underline{\omega}^{\otimes k}) = H^0(\mathbb{M}'_{K'} \otimes \bar{F_{\mathcal{P}}}, \underline{\omega}^{\otimes 2-k})^{\vee} = 0$ when $k \geqslant 3$. This shows that

$$H^1(\mathbb{M}'_{K'}, \underline{\omega}^{\otimes k}) \otimes F_{\mathcal{P}} = 0$$

and therefore $\pi^N H^1(\mathbb{M}'_{K'}, \underline{\omega}^{\otimes k}) = 0$ for some $N \geqslant 0$. With the same reasoning we get

$$H^1(\mathbb{M}'_{K'} \otimes \bar{\kappa}, \underline{\omega}^{\otimes k}) = 0.$$

Therefore,

$$0 = H^1(\mathbb{M}'_{K'} \otimes \kappa, \underline{\omega}^{\otimes k}) = H^1(\mathbb{M}'_{K'}, \underline{\omega}^{\otimes k}) \otimes \kappa.$$

This implies $\pi H^1(\mathbb{M}'_{K'}, \underline{\omega}^{\otimes k}) = H^1(\mathbb{M}'_{K'}, \underline{\omega}^{\otimes k})$. However, some power of π kills $H^1(\mathbb{M}'_{K'}, \underline{\omega}^{\otimes k})$. This proves the lemma. □

PROPOSITION 7.2. *If H is small enough and $q > 3$, then there is an E^D_{q-1} in $S^D(\mathcal{O}_{\mathcal{P}}, K', q-1)$ such that*

$$E^D_{q-1} \equiv \mathbf{H} \mod \pi.$$

Proof. Let $\underline{\omega}$ denote $\underline{\omega}_{\mathbb{A}'_{K'}/\mathbb{M}'_{K'}}$. The obstruction to lifting sections of $\underline{\omega}^{\otimes q-1} \otimes \kappa$ to sections of $\underline{\omega}^{\otimes q-1}$ is $H^1(\mathbb{M}'_{K'}, \underline{\omega}^{\otimes q-1})$ which by the above lemma vanishes under the assumptions. □

Fix a lifting once and for all. When there is no confusion we simply denote E^D_{q-1} by E_{q-1}. Later, in Corollary 13.2 we will show that our theory is independent of the choice of E_{q-1}.

8. $\mathcal{P}$-adic modular forms with respect to D

Assume $q > 3$ and let R_0 be a π-adically complete $\mathcal{O}_{\mathcal{P}}$-algebra. Let $r \in R_0$. Following Katz [Kat73] we define the R_0-module $S^D(R_0, r, K', k)$ of $\mathcal{P}$-adic modular forms (with respect to D) over R_0 of growth condition r, level $K' = \mathbb{Z}_p^* \times K^0_{\mathcal{P}} \times H'$, and weight $k \in \mathbb{Z}$. Here H' is as in § 4.1 (corresponding to level structure away from $\mathcal{P}$).

DEFINITION 8.1. An element $f \in S^D(R_0, r, K', k)$ is a rule which assigns to any r^q-test object $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y, \omega)$ where:

- i) R is an R_0-algebra in which π is nilpotent;
- ii) $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ is as in § 4.2;
- iii) ω is a basis for $\underline{\omega}_{A/R}$;
- iv) Y is an element of R such that $Y E_{q-1}(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega) = r$;

an element $f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y, \omega) \in R$ such that

- i) $f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y, \omega)$ depends only on the isomorphism class of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y, \omega)$ over R;
- ii) the formation of $f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y, \omega)$ commutes with arbitrary base change of R_0-algebras in which π is nilpotent;
- iii) for any $\lambda \in R^*$ we have

$$f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, \lambda^{q-1} Y, \lambda\omega) = \lambda^{-k} f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y, \omega).$$

We may equivalently think of f as a rule which assigns to any r-test object $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$, where $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ is as in § 4.2 over an R_0-scheme S on which π is nilpotent and Y is a section of $\underline{\omega}^{\otimes 1-q}_{A/S}$ such that $Y E_{q-1} = r$, a section $f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$ of $\underline{\omega}^{\otimes k}_{A/S}$ over S such that

- i) $f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$ depends only on the isomorphism class of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$ over S;
- ii) the formation of $f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$ commutes with arbitrary base change of R_0-schemes on which π is nilpotent;

It is easy to see that the two definitions are linked by

$$f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y w^{\otimes 1-q}) = f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y, \omega)\omega^{\otimes k}.$$

From the definition it is evident that

$$S^D(R_0, r, K', k) = \varprojlim_{n \geqslant 1} S^D(R_0/\pi^n, r, K', k).$$

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One can make the same definitions by allowing R to vary in π-adically complete R_0-algebras. Any modular form f of weight k and level K' with respect to D determines an element $f^\sharp$ of $S^D(R_0, r, K', k)$ by defining

$$f^\sharp(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y) = f(A, i, \theta, \bar{\alpha}^{\mathcal{P}}),$$

which we usually denote by f again.

The functor $F : ((\text{Schemes}/R_0)) \to ((\text{Sets}))$, which associates to any scheme S over R_0, the set of all isomorphism classes of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$ over S with $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ as in § 4.2 and Y a section of $\underline{\omega}^{\otimes 1-q}$ on S which satisfies $YE_{q-1} = r$, is representable by $\mathbb{Y}_r \otimes R_0$ where

$$\mathbb{Y}_r \otimes R_0 = \mathrm{Spec}_{\mathbb{M}'_{K'} \otimes R_0}(\mathrm{Symm}(\underline{\omega}^{\otimes q-1})/(E_{q-1} - r)).$$

This can be easily seen from the moduli theoretic description of $\mathbb{M}'_{K'} \otimes R_0$. (The R_0 in the notation is to stress the ring of definition, and it is not to suggest that $\mathbb{Y}_r \otimes R_0$ is obtained by base extension of a scheme over $\mathcal{O}_{\mathcal{P}}$.)

The universal object over $\mathbb{Y}_r \otimes R_0$ is $(\mathbb{B}_r \otimes R_0, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y_r)$, where $(\mathbb{B}_r \otimes R_0, i, \theta, \bar{\alpha}^{\mathcal{P}})$ is the pullback of $(\mathbb{A}'_{K'} \otimes R_0, i, \theta, \bar{\alpha}^{\mathcal{P}})$ under the natural projection $\mathbb{Y}_r \otimes R_0 \xrightarrow{\mu} \mathbb{M}'_{K'} \otimes R_0$ and Y_r is the restriction of the canonical section of $\mu^*\underline{\omega}^{\otimes 1-q}$ on $\mathrm{Spec}_{\mathbb{M}'_{K'} \otimes R_0}(\mathrm{Symm}(\underline{\omega}^{\otimes q-1}))$ to $\mathbb{Y}_r \otimes R_0$. We often denote $\mu^*\underline{\omega}_{\mathbb{A}'_{K'}/\mathbb{M}'_{K'}}$ by $\underline{\omega}$ or sometimes by $\underline{\omega}_{\mathbb{B}_r \otimes R_0/\mathbb{Y}_r \otimes R_0}$ if we need to be more specific.

We will use the following lemma a few times.

Lemma 8.2. *Assume that R_0 is an $\mathcal{O}_{\mathcal{P}}$-algebra with a maximal ideal generated by π_0, and residue field κ_0. The reduction modulo π_0 of $\mathbb{Y}_r \otimes R_0$, namely $\mathbb{Y}_r \otimes \kappa_0$, is reduced.*

Proof. This is clear when r is a unit. Assume $v(r) > 0$. Let $V = \mathrm{Spec}(R)$ be an affine inside $\mathbb{M}'_{K'} \otimes R_0$ such that ω is a nonvanishing section of $\underline{\omega}$ on V and $E_{q-1}|_V = a\omega^{\otimes q-1}$. Then the pullback of V under μ in $\mathbb{Y}_r \otimes R_0$ is isomorphic to $\mathrm{Spec}(R[x]/(ax - r))$ and hence the reduction modulo π_0 of it is $(R/\pi_0)[x]/(\bar{a}x)$ where $^-$ denotes reduction modulo π_0. Assume that $f(x)^n = 0$ in $(R/\pi_0)[x]/(\bar{a}x)$. We can assume $f(x)$ to be of minimal degree representing its class. Let $b \in R/\pi_0$ denote the leading coefficient of $f(x)$. Then $\bar{a}|b^n$. However, $\bar{a}\omega^{\otimes q-1}$ is the restriction to $V \otimes R/\pi_0$ of the Hasse invariant which implies that $\bar{a}$ has simple zeros by Proposition 6.3. Therefore, we have $\bar{a}|b$ which shows that $f(x)$ can be represented by a polynomial of lower degree unless $f(x) = 0$. □

Proposition 8.3. *If π is nilpotent in R_0, then there is a canonical isomorphism*

$$S^D(R_0, r, K', k) \xrightarrow{\sim} H^0\left(\mathbb{M}'_{K'} \otimes R_0, \bigoplus_{j \geqslant 0} \underline{\omega}^{\otimes k + j(q-1)}/(E_{q-1} - r)\right).$$

Proof. Since π is nilpotent in R_0, any R_0-algebra is π-adically complete. This implies

$$S^D(R_0, r, K', k) \xrightarrow{\sim} H^0(\mathbb{Y}_r \otimes R_0, \underline{\omega}^{\otimes k}).$$

Therefore, we have

$$\begin{aligned} H^0(\mathbb{Y}_r \otimes R_0, \underline{\omega}^{\otimes k}) &= H^0(\mathrm{Spec}_{\mathbb{M}'_{K'} \otimes R_0}(\mathrm{Symm}(\underline{\omega}^{\otimes q-1})/(E_{q-1} - r)), \underline{\omega}^{\otimes k}) \\ &= H^0\left(\mathbb{M}'_{K'} \otimes R_0, \left(\bigoplus_{j \geqslant 0} \underline{\omega}^{\otimes j(q-1)}/(E_{q-1} - r)\right) \otimes \underline{\omega}^{\otimes k}\right) \\ &= H^0\left(\mathbb{M}'_{K'} \otimes R_0, \bigoplus_{j \geqslant 0} \underline{\omega}^{\otimes k + j(q-1)}/(E_{q-1} - r)\right), \end{aligned}$$

which proves the desired result. □

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PROPOSITION 8.4. *Let R_0 be a π-adically complete $\mathcal{O}_{\mathcal{P}}$-algebra flat over $\mathcal{O}_{\mathcal{P}}$ and $r \in R_0$ not a zero divisor. Assume $q > 3$ and either $k = 0$ or $k \geqslant 3$. Then the natural map*

$$\varprojlim_{n \geqslant 0} \left(\bigoplus_{j \geqslant 0} S^D(\mathcal{O}_{\mathcal{P}}, K', k + j(q-1)) \right) \bigotimes_{\mathcal{O}_{\mathcal{P}}} R_0/\pi^n/(E_{q-1} - r) \xrightarrow{\sim} S^D(R_0, r, K', k)$$

is an isomorphism. The above map is induced by taking inverse limit of the natural maps

$$\left(H^0\left(\mathbb{M}'_{K'}, \bigoplus_{j \geqslant 0} \underline{\omega}^{(k+j(q-1))} \right) \otimes R_0/\pi^n \right) / (E_{q-1} - r) \to S^D(R_0/\pi^n, r, K', k).$$

Proof. Use Lemma 7.1 and adopt the proof of Theorem 2.5.1 in [Kat73]. □

9. $\mathcal{P}$-adic modular forms in the rigid setting

In § 8 we derived a description of $\mathcal{P}$-adic modular forms as sections of line bundles over certain schemes when π is nilpotent in the ground ring. To give a similar description for more general ground rings we have to work with formal schemes and rigid analytic varieties.

We recall some general facts. Let L_0 be the field of fractions of R_0, an $\mathcal{O}_{\mathcal{P}}$-algebra which is a complete valuation ring with normalized valuation such that $v(\pi) = 1$. Let π_0 denote a uniformizer and define $| \; |_v = (1/q)^v$. There exists a functor

$$\begin{aligned} \mathrm{an} : ((\text{Algebraic Varieties}/L_0)) &\to ((\text{Rigid Spaces}/L_0)) \\ X &\mapsto X^{\mathrm{an}}. \end{aligned}$$

This functor takes any sheaf of modules $\mathcal{F}$ on X to a sheaf of modules $\mathcal{F}^{\mathrm{an}}$ on X^{an}. We have the following rigid GAGA theorem for this functor.

THEOREM 9.1. *If X is a closed subscheme of $\mathbb{P}^n$, then*

i) *$\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ is an equivalence of categories between algebraic and rigid coherent sheaves;*
ii) *$H^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$ for each i;*
iii) *the functor* 'an' *is a fully faithful functor.*

Therefore, we can think of any algebraic variety over L_0 as a rigid analytic space via an. Often we will let X denote its analytification X^{an}. There are, of course, many rigid analytic spaces which are not algebraic. Raynaud constructed a functor which associates a rigid analytic space to any admissible formal scheme over R_0. A formal scheme is admissible if it is flat over R_0 and is locally topologically finitely generated. This functor is described in [BL93].

$$\begin{aligned} \mathrm{rig} : ((\text{Admissible Formal Schemes}/R_0)) &\to ((\text{Rigid Analytic Spaces}/L_0)) \\ X &\mapsto X^{\mathrm{rig}}. \end{aligned}$$

One can associate to any sheaf of modules $\mathcal{F}$ on X a sheaf of modules $\mathcal{F}^{\mathrm{rig}}$ on X^{rig}.

THEOREM 9.2.

i) *The functor* rig *is a faithful functor.*
ii) *Any rigid algebraic variety over L_0 is in the image of this functor.*
iii) *If X is a variety over R_0 and $\tilde{X}$ is its completion in the maximal ideal of R_0, then there is an open immersion $\tilde{X}^{\mathrm{rig}} \hookrightarrow (X \otimes L_0)^{\mathrm{an}}$ which is an isomorphism when X is proper over R_0.*

9.1 Formal schemes setting

Recall that in § 8 for any $r \in R_0$ we defined $\mathbb{Y}_r \otimes R_0$ to be $\mathrm{Spec}_{\mathbb{M}'_{K'} \otimes R_0}(\mathrm{Symm}(\underline{\omega}^{\otimes q-1})/(E_{q-1} - r))$. If $r = 1$, this is an affine scheme. This is a result of existence of supersingular points on $\mathbb{M}'_{K'}$ which is shown in [Car86], or can be seen by using Riemann–Roch to prove that $\mathbf{H}$ has zeros.

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Definition 9.3. Let $\widetilde{\mathbb{Y}}_r \otimes R_0$ denote the formal scheme over R_0 defined by the completion of $\mathbb{Y}_r \otimes R_0$ along the closed subscheme defined by $\pi = 0$. Denote the completion of $\mathbb{B}_r \otimes R_0$ and $\underline{\omega}$ on $\mathbb{Y}_r \otimes R_0$ by $\widetilde{\mathbb{B}}_r \otimes R_0$ and $\underline{\omega}_{\widetilde{\mathbb{B}}_r \otimes R_0/\widetilde{\mathbb{Y}}_r \otimes R_0}$ or simply $\underline{\omega}$.

Proposition 9.4. $S^D(R_0, r, K', k) = H^0(\widetilde{\mathbb{Y}}_r \otimes R_0, \underline{\omega}^{\otimes k})$.

Proof.

$$\begin{aligned} H^0(\widetilde{\mathbb{Y}}_r \otimes R_0, \underline{\omega}^{\otimes k}) &= \varprojlim_{n \geqslant 0} H^0(\mathbb{Y}_r \otimes R_0/\pi^n, \underline{\omega}^{\otimes k}) \\ &= \varprojlim_{n \geqslant 0} S^D(R_0/\pi^n, r, K', k) \\ &= S^D(R_0, r, K', k). \end{aligned}$$

Let $r', r'' \in R_0$ be such that $r' = rr''$. There is a morphism

$$\widetilde{\mathbb{Y}}_{r''} \otimes R_0 \to \widetilde{\mathbb{Y}}_{r'} \otimes R_0$$

which is defined by $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y) \mapsto (A, i, \theta, \bar{\alpha}^{\mathcal{P}}, rY)$ on the moduli level, and a morphism

$$\widetilde{\mathbb{Y}}_r \otimes R_0 \to \widetilde{\mathbb{M}'}_{K'} \otimes R_0$$

which is obtained by completion of the natural map $\mathbb{Y}_r \otimes R_0 \to \mathbb{M}'_{K'} \otimes R_0$ along $\pi = 0$. $\square$

Proposition 9.5. *The formal scheme $\widetilde{\mathbb{Y}}_r \otimes R_0$ is flat over R_0 and $\widetilde{\mathbb{Y}}_r \otimes R_0$ is 'integrally closed' in $\widetilde{\mathbb{Y}}_r \otimes L_0$.*

Proof. If $r = 1$ this is an immediate result of smoothness of $\mathbb{M}'_{K'} \otimes R_0$. Let $v(r) > 0$. Let $W = \mathrm{Spec}(R)$ be an open affine subset of $\mathbb{M}'_{K'} \otimes R_0$ over which there is a basis ω for $\underline{\omega}$. Assume $E_{q-1} = a\omega^{\otimes q-1}$ over W. Then the restriction of $\widetilde{\mathbb{Y}}_r \otimes R_0$ to W is given by $\mathrm{Spf}(\tilde{R}\langle x\rangle/(ax - r))$ where $\tilde{R}$ denotes the π-adic completion of R. First we show that $\tilde{R}\langle x\rangle/(ax-r)$ is flat over R_0 and hence $\tilde{R}\langle x\rangle/(ax-r) \subset (\tilde{R}\langle x\rangle/(ax-r)) \otimes L_0$. Let $f(x) + (ax - r) \in \tilde{R}\langle x\rangle/(ax-r)$ be π_0-torsion. Then $\pi_0 f(x) = (ax-r)h(x)$ for some $h(x) \in \tilde{R}\langle x\rangle$. Reducing modulo π_0, we get $0 = \bar{a}x\bar{h}(x)$. So for any coefficient b of $h(x)$ we know that $\bar{a}\bar{b} = 0$ in R. But $\bar{a}\bar{\omega}^{\otimes q-1}$ is the restriction of $\mathbf{H}$ to $W \otimes R_0/\pi_0$ and hence $\bar{a}$ has a finite number of zeros on $W \otimes R_0/\pi_0$. This implies $\bar{b} = 0$ and hence $h(x)$ is divisible by π_0. In other words $f(x)/\pi_0 \in (ax - r)$. This proves the flatness.

Next, we show that $\tilde{R}\langle x\rangle/(ax-r) \subset (\tilde{R}\langle x\rangle/(ax-r)) \otimes L_0$ is integrally closed. Assume $f(x)/\pi_0$ is integral over $\tilde{R}\langle x\rangle/(ax-r)$. We have an integral equation

$$(f(x)/\pi_0)^j + a_{j-1}(f(x)/\pi_0)^{j-1} + \cdots + a_0 = 0$$

in $(\tilde{R}\langle x\rangle/(ax-r)) \otimes L_0$. Multiplying by $(\pi_0)^j$ and reducing modulo π_0, we get $\bar{f}(x)^j = 0$ in $(\tilde{R}\langle x\rangle/(ax-r)) \otimes R/\pi_0 = (R/\pi_0)[x]/(\bar{a}x)$. However, by Lemma 8.2, we have $\bar{f}(x) = 0$, which means that $f(x)/\pi_0 \in \tilde{R}\langle x\rangle/(ax-r)$. $\square$

9.2 Rigid setting

We continue the assumptions on R_0 as in the beginning of this section. Let L_∞ denote the completion of an algebraic closure of L_0 and denote its ring of integers by R_∞.

Let X be a reduced proper flat scheme of finite type over R_0 and $\mathcal{L}$ a line bundle on X. Let l be a global section of $\mathcal{L}$. Coleman [Col96] explains how to associate to this data affinoid subdomains of $X \otimes L_0$. Let x be a closed point of $X \otimes L_0$ with residue field L_x which is a finite extension of L_0 and carries a unique extension of norm of L_0. Let R_x denote its ring of integers. Then since X is proper, the morphism $\mathrm{Spec}(L_x) \to X$ corresponding to x extends to a morphism $f_x : \mathrm{Spec}(R_x) \to X$. Now since R_x is a discrete valuation ring, $f_x^*(\mathcal{L})$ is a trivial line bundle generated by a section t.

Let $f_x^*(l) = at$ with $a \in R_x$. Define $|l(x)|_v = |a|_v$ which is independent of the choice of t. For any $s \in |L_\infty|_v$ there is a unique rigid subspace $X(s)$ of $(X \otimes L_0)^{\mathrm{an}}$ which is a finite union of affinoids and whose closed points are points x of $X \otimes L_0$ such that $|l(x)|_v \geqslant s$. Coleman shows that if X is an irreducible curve and $s \in |L_\infty|_v$, then $X(r)$ is an affinoid subdomain of $X \otimes L_0$ unless $\mathcal{L} = \mathcal{O}_X$ and l is nowhere vanishing in which case $X(r) = X$.

DEFINITION 9.6. Do the same construction as above with $X = \mathbb{M}'_{K'} \otimes R_0$, $\mathcal{L} = \underline{\omega}^{\otimes q-1}$, and $l = E_{q-1}$ to obtain affinoid subdomains $M'_{K'}(s)$ of $\mathbb{M}'_{K'} \otimes L_0$ for each $s \in |L_\infty|_v$. Denote the analytification of $\underline{\omega}$ on $\mathbb{M}'_{K'} \otimes L_0$ and its restrictions to any affinoid $M'_{K'}(s)$ again by $\underline{\omega}$.

When $s = 1$, $M'_{K'}(1)$ contains all points at which E_{q-1} has norm exactly 1. In other words, it contains all the points x such that $\mathbf{H}$ does not vanish at the reduction of x modulo the maximal ideal of R_0. So $M'_{K'}(1)$ (the ordinary part of $\mathbb{M}'_{K'} \otimes L_0$) is obtained by removing from the Shimura curve those $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ which have a supersingular reduction modulo the maximal ideal of R_0. The complement of $M'_{K'}(1)$ is the union of the finitely many supersingular discs (each disc consists of all points whose reduction is a fixed supersingular $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ over a finite extension of κ). Allowing s to be less than 1 corresponds to removing smaller supersingular discs from $\mathbb{M}'_{K'} \otimes L_0$. The ordinary part, $M'_{K'}(1)$, is an affinoid subdomain of $M'_{K'}(s)$ for each $s \leqslant 1$.

By rigid GAGA Theorem 9.1 modular forms of weight k and level K' with respect to D over L_0 are exactly the sections of $\underline{\omega}^{\otimes k}$ on $(\mathbb{M}'_{K'} \otimes L_0)^{\mathrm{an}}$. Following Coleman we will consider the sections of the same line bundle over the smaller affinoids $M'_{K'}(s)$. We define a *convergent modular form* of weight k and level K' over L_0 to be an element of $H^0(M'_{K'}(1), \underline{\omega}^{\otimes k})$. An *overconvergent modular form* of weight k, level K', and growth condition s over L_0 is an element of $H^0(M'_{K'}(s), \underline{\omega}^{\otimes k})$ for some $s < 1$, $s \in |L_\infty|_v$. These modular forms are called overconvergent since they can be partially extended to supersingular discs. We will see that these modular forms are related to $\mathcal{P}$-adic modular forms with respect to D.

PROPOSITION 9.7. *Let $r \in R_0$. Then $\widetilde{\mathbb{Y}}_r \otimes R_0$ is an admissible formal scheme and there is an isomorphism $(\widetilde{\mathbb{Y}}_r \otimes R_0)^{\mathrm{rig}} \xrightarrow{\sim} M'_{K'}(|r|_v)$. If $r' = rr''$, then the following diagram is commutative.*

$$\begin{tikzcd}
(\widetilde{\mathbb{Y}}_{r''} \otimes R_0)^{\mathrm{rig}} \arrow[r] \arrow[d] & {M'}_{K'}(|r''|_v) \arrow[d] \\
(\widetilde{\mathbb{Y}}_{r'} \otimes R_0)^{\mathrm{rig}} \arrow[r] \arrow[d] & {M'}_{K'}(|r'|_v) \arrow[d] \\
(\widetilde{\mathbb{M}'_{K'} \otimes R_0})^{\mathrm{rig}} \arrow[r] & (\mathbb{M}'_{K'} \otimes L_0)^{\mathrm{an}}
\end{tikzcd}$$

Proof. Let $\{\mathrm{Spec}(S_j)\}_{j \in I}$ be an open covering of $\mathbb{M}'_{K'} \otimes R_0$, each element of which intersects the special fibre and such that the restriction of $\underline{\omega}^{\otimes q-1}$ to $\mathrm{Spec}(S_j)$ is trivial and generated by the section t_j. Let $E_{q-1} = a_j t_j$ over $\mathrm{Spec}(S_j)$. Then $\mathbb{Y}_r \otimes R_0$ has an open covering, $\{\mathrm{Spec}(S_j[t_j]/(a_j t_j - r))\}_{j \in I}$, with gluing data induced by those of the above open covering of $\mathbb{M}'_{K'} \otimes R_0$. Completing along the $\pi = 0$ subscheme, we obtain $\widetilde{\mathbb{Y}}_r \otimes R_0$ in which the above covering becomes $\{\mathrm{Spf}(\tilde{S}_j\langle t_j\rangle/(a_j t_j - r))\}_{j \in I}$ where $\tilde{S}$ denotes the π-adic completion of S. This shows that $\widetilde{\mathbb{Y}}_r \otimes R_0$ is locally topologically finitely generated. Also by Proposition 9.5, $\widetilde{\mathbb{Y}}_r \otimes R_0$ is flat over R_0. Therefore, $\widetilde{\mathbb{Y}}_r \otimes R_0$ is admissible. Now the image of $\widetilde{\mathbb{Y}}_r \otimes R_0$ under the Raynaud's functor is given by a covering $\{\mathrm{Sp}(\tilde{S}_j \otimes L_0\langle t_j\rangle/(a_j t_j - r))\}_{j \in I}$ with the induced gluing data. On the other hand, $\{\mathrm{Sp}(\tilde{S}_j \otimes L_0)\}_{j \in I}$ with the induced gluing data gives us $(\mathbb{M}'_{K'} \otimes L_0)^{\mathrm{an}}$. But by Proposition 7.2.3.4 of [BGR84], $\mathrm{Sp}(\tilde{S}_j \otimes L_0\langle t_j\rangle/(a_j t_j - r))$ is the affinoid subdomain of $\mathrm{Sp}(\tilde{S}_j \otimes L_0)$ defined by $|a_j(x)|_v \geqslant |r|_v$. Finally to get the diagram first note that the lower isomorphism is given by Theorem 9.2, part iii as $\mathbb{M}'_{K'} \otimes R_0$ is proper over R_0. The commutativity of the diagram follows by looking at the image of t_j in each chart. □

We are now able to link the (over)convergent modular forms to $\mathcal{P}$-adic modular forms with respect to D.

Corollary 9.8. *If $r \in R_0$, we have $S^D(R_0, r, K', k) \otimes L_0 = H^0({M'}_{K'}(|r|_v), \underline{\omega}^{\otimes k})$.*

Proof. By Proposition 9.7, we have

$$\begin{aligned} H^0({M'}_{K'}(|r|_v), \underline{\omega}^{\otimes k}) &= H^0((\widetilde{\mathbb{Y}}_r \otimes R_0)^{\mathrm{rig}}, \underline{\omega}^{\otimes k}) \\ &= H^0(\widetilde{\mathbb{Y}}_r \otimes R_0, \underline{\omega}^{\otimes k}) \otimes L_0 \\ &= S^D(R_0, r, K', k) \otimes L_0. \end{aligned}$$

□

Corollary 9.9. *$S^D(R_0, r, K', 0) \otimes L_0$ is an L_0-affinoid algebra and*

$$M'_{K'}(|r|_v) = \mathrm{Sp}(S^D(R_0, r, K', 0) \otimes L_0).$$

Corollary 9.10. *Let $r'' = rr'$. The natural morphism*

$$S^D(R_0, r'', K', k) \to S^D(R_0, r', K', k)$$

defined from the transformation of functors $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y) \mapsto (A, i, \theta, \bar{\alpha}^{\mathcal{P}}, rY)$ is injective.

Proof. The map $S^D(R_0, r'', K', k) \otimes L_0 \to S^D(R_0, r', K', k) \otimes L_0$ is the restriction of sections of $\underline{\omega}^{\otimes k}$ from $M'_{K'}(r'')$ to $M'_{K'}(r')$ which is injective. Now Proposition 9.5 gives the result. □

We end this section by defining the universal family of test objects over $M'_{K'}(|r|_v)$. We define $A'_{K'}(|r|_v)$ to be the image of $\widetilde{\mathbb{B}}_r \otimes R_0$ under Raynaud's functor rig,

$$A'_{K'}(|r|_v) := (\widetilde{\mathbb{B}}_r \otimes R_0)^{\mathrm{rig}}.$$

This is a family of test objects $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ over $M'_{K'}(|r|_v)$.

10. The canonical subgroup

Throughout this section we assume R_0 to be an $\mathcal{O}_{\mathcal{P}}$-algebra which is a complete discrete valuation ring of characteristic 0 with a uniformizer π_0 and field of fractions L_0 such that $v(\pi) = 1$. (Recall that π is our choice of uniformizer in $\mathcal{O}_{\mathcal{P}}$.)

In analogy with the classical case, we will construct the canonical subgroup of a test object $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ which is not 'too supersingular'. Roughly speaking, the canonical subgroup of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ is a lifting of the kernel of Fr_q in the reduction of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ in characteristic p. It will be used in defining the Frobenius morphism of $\mathcal{P}$-adic modular forms.

Theorem 10.1 Canonical subgroups.

i) *Let $r \in R_0$ with $v(r) < q/(q+1)$. There is a canonical way to associate to every r-test object $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$, where*
   - *$(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ defined over an R_0-algebra R is as in § 4.2,*
   - *Y is a section of $\underline{\omega}_{A/R}^{\otimes 1-q}$ which satisfies $YE_{q-1} = r$,*

   a finite flat subgroup scheme C of A such that

   a) *C has rank q^{4d} and is stable under the action of $\mathcal{O}_D$,*
   b) *C depends only on the R-isomorphism class of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$,*

c) *the formation of* C *commutes with arbitrary base change of* π*-adically complete* R_0*-algebras,*
d) *if* $\pi/r = 0$ *in* R*, then* C *can be identified with the kernel of Frobenius morphism* $\mathrm{Fr}_q : A \to A^{(q)}$*,*
e) C *is of type* 1 *(as defined in § 4.4).*

ii) *Let* $r \in R_0$ *with* $v(r) < 1/(q+1)$*. There is a canonical way to associate to every* $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$ *as in part i, an* r^q*-test object* $(A', i', \theta', (\bar{\alpha}^{\mathcal{P}})', Y')$*, where*
 – $(A', i', \theta', (\bar{\alpha}^{\mathcal{P}})')$ *is the quotient of* $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ *by* C *(as defined in § 4.4),*
 – Y' *is a section of* $\underline{\omega}_{A'/R}^{\otimes 1-q}$ *which satisfies* $Y' E_{q-1} = r^q$*,*

such that

a) Y' *depends only on the* R*-isomorphism class of* $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$*,*
b) *the formation of* Y' *commutes with arbitrary base change of* π*-adically complete* R_0*-algebras,*
c) *if* $\pi/r^{q+1} = 0$ *in* R*, then* Y' *is equal to* $Y^{(q)}$ *on* $A^{(q)} = A/C$*.*

10.1 Proof of Theorem 10.1

The proof will be in several steps. Since C is supposed to be invariant under $\mathcal{O}_D$ it will decompose as $C_1^1 \oplus \cdots \oplus C_m^1 \oplus C_1^2 \oplus \cdots \oplus C_m^2$. The component C_1^2 will further decompose as $C_1^{2,1} \oplus C_1^{2,2}$. We first construct $C_1^{2,1}$.

10.1.1 *Constructing* $C_1^{2,1}$*.* The following lemma is due to Lubin and is recorded in [Kat73]. We will rewrite the proof in this context in order to fix the notation.

LEMMA 10.2. *Let* R *be a* π*-adically complete* R_0*-algebra which is flat over* $\mathcal{O}_{\mathcal{P}}$*, and* $r \in R_0$ *with* $v(r) < q/(q+1)$*, and* $r_1 = -\pi/r \in R_0$*. Let* G *be a formal* $\mathcal{O}_{\mathcal{P}}$*-module over* R*. By Proposition 6.2 there is a coordinate* x *on* G *for which* $[\zeta](x) = \zeta x$ *for* ζ *any* $(q-1)$*th root of unity in* $\mathcal{O}_{\mathcal{P}}$*, and we have* $[\pi](x) = \pi x + a x^q + \cdots$*. Assume that there are* $y, b \in R$ *such that* $(a + \pi b) y = r$*. Then, there is a canonical way to associate to this data a subscheme* C *of* G *of rank* q *killed by* π *such that* $C \otimes R/r_1$ *can be identified with the kernel of* $\mathrm{Fr}_q : G \otimes R/r_1 \to G^{(q)} \otimes R/r_1$ *and its formation commutes with arbitrary base change of* π*-adically complete* R_0*-algebras which are flat over* $\mathcal{O}_{\mathcal{P}}$*. Furthermore,* C *is independent of the choice of* x*.*

Proof. By Proposition 6.2 $[\pi](x) = \pi x + a x^q + \sum_{n \geqslant 2} c_n x^{n(q-1)+1}$. Define

$$f(T) := \pi + aT + \sum_{n \geqslant 2} c_n T^n.$$

It is clear that $[\pi](x) = x f(x^{q-1})$. The desired subscheme will be defined by means of a canonical zero t_{can} of the power series $f(T)$. It will consist of '0 and the $q-1$ solutions of $x^{q-1} = t_{\mathrm{can}}$'. Since $v(r_1) > 0$, we know that $1 + r_1 b y$ is invertible in R. Let $t_0 := r_1 y/(1 + r_1 b y)$ which satisfies $\pi + a t_0 = 0$. Now let $f_1(T) := f(t_0 T)$. Then $f_1(T) = \pi - \pi T + \sum_{n \geqslant 2} c_n t_0^n T^n$.

We study the coefficients of $f_1(T)$. By assumption $v(r_1^{q+1}/\pi) > 0$. Let r_2 be a generator of the ideal generated by r_1^{q+1}/π and r_1^2 in R_0. If $n \not\equiv 1 \bmod q$, then by Proposition 6.2 $c_n \in \pi R$ and we have $(c_n/\pi) t_0^n \in t_0^2 R \subset r_1^2 R \subset r_2 R$. If $n \equiv 1 \bmod q$, then $c_n t_0^n \in c_n t_0^{q+1} R \subset c_n r_1^{q+1} R$. On the other hand, $r_1^{q+1}/\pi \in R_0$. Thus, $c_n t_0^n/\pi \in (t_0^{q+1}/\pi) R \subset (r_1^{q+1}/\pi) R \subset r_2 R$. Clearly in each case $c_n t_0^n/\pi$ tends to zero as n tends to ∞. This shows that we can write $f_1(T) = \pi f_2(T)$, where $f_2(T) = 1 - T + \sum_{n \geqslant 2} d_n T^n$ with $d_n \in r_2 R$ and $d_n \to 0$ when $n \to \infty$.

Now let $f_3(T) = f_2(1-T)$. Then we can write $f_3(T) = e_0 + (1 + e_1) T + \sum_{n \geqslant 2} e_n T^n$ where $e_n \in r_2 R$ for all $n \geqslant 0$. Let $I = r_2 R$. We now show that there is a unique element $t_\infty \in I$ such that $f_3(t_\infty) = 0$.

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If $t \in I$, then $f_3(t) \in I$ and $f_3'(t) \in 1 + I$ and hence $f_3'(t)$ is invertible. The Newton process of successive approximation, $t_1 = 0, \ldots, t_{n+1} = t_n - f_3(t_n)/f_3'(t_n)$, converges to a zero of f_3 which lies in I. If t and $t' = t + \Delta$ are two zeros of f_3 in I, then we get $-f_3'(t)\Delta = f_3(t+\Delta) - f_3(t) - f_3'(t)\Delta \in \Delta^2 R$. Since $f_3'(t)$ is a unit in R, $\Delta \in \Delta^2 R$. However $\Delta \in I$ and R is I-adically separated. This proves $\Delta = 0$ and $t = t'$.

Going backwards we obtain a zero $t_{\mathrm{can}} = t_0(1 - t_\infty)$ of $f(T)$. Since $t_0 \in r_1 R$ is topologically nilpotent, we can expand $f(T)$ in terms of $(T - t_{\mathrm{can}})$ and deduce that $f(T)$ is divisible by $T - t_{\mathrm{can}}$ in $R[\![T]\!]$. Therefore, $[\pi](x)$ is divisible by $x^q - t_{\mathrm{can}}x$ in $R[\![x]\!]$. We define the canonical subscheme C of G to be the subscheme of $G[\pi]$ defined by $x^q - t_{\mathrm{can}}x$. Thus, $C = \mathrm{Spec}(R[\![x]\!]/(x^q - t_{\mathrm{can}}x))$ which is finite and flat of rank q over R. Furthermore, $C/r_1 = \mathrm{Spec}(R[\![x]\!]/(x^q, r_1)) = \mathrm{Spec}((R/r_1)[\![x]\!]/x^q)$ which is nothing but $\mathrm{Ker}(\mathrm{Fr}_q : G/r_1 \to (G/r_1)^q)$. It is an easy exercise to see that this is independent of choice of x. This ends the proof of the lemma. □

We now proceed with the construction of $C_1^{2,1}$. Basically, $C_1^{2,1}$ will be given as the canonical subscheme of $(G_A)_1^{2,1}$ as in the previous lemma. The only problem is that the base R_0 may not be flat over $\mathcal{O}_{\mathcal{P}}$. To get around this, we proceed as follows. It is enough to construct $C_1^{2,1}$ for $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$ defined over an R_0-algebra R in which π is nilpotent. There is a map $\nu : \mathrm{Spec}(R) \to \mathbb{Y}_r \otimes R_0$ such that $\nu^*(\mathbb{B}_r \otimes R_0, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y_r) = (A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$. We will construct $C_1^{2,1}$ locally Zariski on $\mathrm{Spec}(R)$. Working locally on $\mathrm{Spec}(R)$, we may assume that ν lands in an open affine $\mathrm{Spec}(S)$ of $\mathbb{Y}_r \otimes R_0$ so that $(G_{\mathbb{B}_r \otimes R_0})_1^{2,1}|_{\mathrm{Spec}(S)}$ is given by a formal $\mathcal{O}_{\mathcal{P}}$-module law. Note that still π is nilpotent in R which implies that R is π-adically complete. Therefore, the induced map $\nu^* : S \to R$ factors through the π-adic completion of S. In other words there is a map $\psi : \mathrm{Spec}(R) \to \mathrm{Spec}(\tilde{S})$ such that $\mathrm{Spec}(R) \xrightarrow{\psi} \mathrm{Spec}(\tilde{S}) \to \mathrm{Spec}(S)$ equals ν. We now apply the previous lemma to obtain a subscheme of $(G_{\mathbb{B}_r \otimes R_0})_1^{2,1}|_{\mathrm{Spec}(S)} \otimes \tilde{S}$ and use ψ to pull it back to $\mathrm{Spec}(R)$.

Let x be a coordinate on $(G_{\mathbb{B}_r \otimes R_0})_1^{2,1}|_{\mathrm{Spec}(S)}$ as in Proposition 6.2. Then we have $[\pi](x) = \pi x + a x^q + \cdots$. Let ω be a basis of $\underline{\omega}$ on $\mathrm{Spec}(S)$ which reduces to the dual differential with respect to x in $(G_{\mathbb{B}_r \otimes R_0})_1^{2,1}|_{\mathrm{Spec}(S)}$. Write $Y_r = y\omega^{\otimes 1-q}$. By Proposition 6.2 we have

$$E_{q-1}(\mathbb{B}_r \otimes R_0, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y_r, \omega) \equiv a \bmod \pi.$$

Thus, $E_{q-1}(\mathbb{B}_r \otimes R_0, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y_r, \omega) = a + \pi b$ for some $b \in S$ and $Y_r E_{q-1} = r$ implies $(a + \pi b)y = r$. Now $\tilde{S}$ is π-adically complete and flat over $\mathcal{O}_{\mathcal{P}}$ and therefore we can apply Lemma 10.2 to obtain a subscheme $(\mathbf{C}_r)_1^{2,1}$ of $(G_{\mathbb{B}_r \otimes R_0})_1^{2,1}|_{\mathrm{Spec}(S)} \otimes \tilde{S}$ which is a finite flat subscheme of rank q of $\mathbb{B}_r \otimes \tilde{S}$. Since $A = \psi^*(\mathbb{B}_r \otimes_S \tilde{S})$ we can define $C_1^{2,1}$ to be the pullback of $(\mathbf{C}_r)_1^{2,1}$ under ψ,

$$C_1^{2,1} = \psi^*((\mathbf{C}_r)_1^{2,1}).$$

Clearly $C_1^{2,1}$ is a finite flat subscheme of rank q of $(A[\pi])_1^{2,1} \subset A$ which reduces to the $(\mathrm{Ker}(\mathrm{Fr}_q))_1^{2,1}$ modulo r_1.

Proposition 10.3. *$C_1^{2,1}$ is a subgroup scheme of $(A[\pi])_1^{2,1}$.*

Proof. We argue as in § 3.8 of [Kat73]. First we consider the case when r is a unit in R_0. This implies that $E_{q-1}(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ is nowhere vanishing. Hence, in the notation of Proposition 6.2, we have

$$[\pi](x) = \pi x + a x^q + \cdots \quad a \in R \text{ unit}.$$

Factorize $[\pi](x) = (x^q - t_{\mathrm{can}}x)h(x)$. Writing $h(x) = \sum_{n \geqslant 0} h_i x^i$, we get $h_0 = t_{\mathrm{can}}h_{q-1} + a$. Since a is a unit in R and t_{can} is topologically nilpotent, we see that h_0 is a unit in R and therefore $h(x)$ is a unit in $R[\![x]\!]$. This implies that when r is a unit in R_0 the subscheme $C_1^{2,1}$ is equal to $\mathrm{Ker}([\pi] : (G_A)_1^{2,1} \to (G_A)_1^{2,1})$ which is a subgroup scheme of $(A[\pi])_1^{2,1}$.

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In the general case, let $G(x,y) \in R[\![x,y]\!]$ be the power series giving the addition in $(G_A)_1^{2,1}$. Then, $C_1^{2,1} = \mathrm{Spec}(R[\![x]\!]/(x^p - t_{\mathrm{can}}x))$ is a group scheme if

$$G(x,y)^q - t_{\mathrm{can}}G(x,y) = 0 \quad \text{in } R[\![x,y]\!]/(x^q - t_{\mathrm{can}}x, y^q - t_{\mathrm{can}}y).$$

Since t_{can} is topologically nilpotent in R, $R[\![x,y]\!]/(x^q - t_{\mathrm{can}}x, y^q - t_{\mathrm{can}}y)$ is free of rank q^2 with basis $\{x^i y^j\}_{0\leqslant i,j\leqslant q-1}$. Write

$$G(x,y)^q - t_{\mathrm{can}}G(x,y) = \sum_{0\leqslant i,j\leqslant q-1} g_{ij}x^i y^j$$

in $R[\![x,y]\!]/(x^q - t_{\mathrm{can}}x, y^q - t_{\mathrm{can}}y)$ with $g_{ij} \in R$. Note that the formation of g_{ij} is functorial. Now, $C_1^{2,1}$ is a subgroup scheme iff $g_{ij} = 0$ for all $0 \leqslant i,j \leqslant q-1$.

By construction, it suffices to show that $(\mathbf{C}_r)_1^{2,1} \subset (G_{\mathbb{B}_r\otimes R_0})_1^{2,1} \otimes \widetilde{\mathbb{Y}}_r$ is a subgroup scheme. We do that locally Zariski on $\widetilde{\mathbb{Y}}_r \otimes R_0$. Let $V = \mathrm{Spf}(S_r)$ be an open inside $\widetilde{\mathbb{Y}}_r \otimes R_0$. We need to show that all $g_{ij} \in S_r$ vanish. Let $W = \mathrm{Spf}(S_1)$ an open inside $\widetilde{\mathbb{Y}}_1 \otimes R_0$ such that $W \to V$ under the natural map, whose image under rig is an inclusion of affinoids in $M'_{K'}(|r|_v)$. This implies that $S_r \to S_1$ is injective. Since we have already seen that $(\mathbf{C}_{r=1})_1^{2,1}$ is a subgroup, all g_{ij} vanish in S_1 and hence in S_r as desired. □

Remark 10.4. In the above, we saw that if r is a unit in R_0, then we have $C_1^{2,1} = (G_A)_1^{2,1}[\pi]$.

10.1.2 *Constructing C.* For an r-test object $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$ defined over R we construct finite flat subgroup schemes C_j^i of $(A[q])_j^i$ such that

$$C_j^i \otimes R/r_1 = \mathrm{Ker}(\mathrm{Fr}_q|_{(A[q])_j^i\otimes R/r_1}) = (\mathrm{Ker}(\mathrm{Fr}_q))_j^i$$

for $i = 1,2$ and $1 \leqslant j \leqslant m$. First we construct C_1^2. As was explained before, choosing idempotents e and $f = 1-e$ in $M_2(\mathcal{O}_{\mathcal{P}})$ will give a decomposition $C_1^2 = C_1^{2,1} \oplus C_2^{2,2} \subset (A[\pi])_1^{2,1} \oplus (A[\pi])_1^{2,2} \subset (A[q])_1^{2,1} \oplus (A[q])_1^{2,2}$. If $g \in GL_2(\mathcal{O}_{\mathcal{P}})$ conjugates e and f, then it induces an isomorphism between $(A[q])_1^{2,1}$ and $(A[q])_1^{2,2}$. Define $C_1^{2,2}$ to be the image of $C_1^{2,1}$ under this isomorphism. We can then define

$$C_1^2 := C_1^{2,1} \oplus C_2^{2,2}.$$

By construction of $C_1^{2,1}$ and since the action of Fr_q commutes with that of $M_2(\mathcal{O}_{\mathcal{P}})$, we conclude that $C_1^2 \otimes R/r_1 = \mathrm{Ker}(\mathrm{Fr}_q|_{(A[q])_1^2\otimes R/r_1})$ as desired.

When $i = 2$ and $2 \leqslant j \leqslant m$, we know that $(A[q])_j^2$ is étale (see § 4.3). Therefore, Fr_q is injective on $(A[q])_j^2 \otimes R/r_1$ and we define

$$C_j^2 := \{0\}.$$

We use the Cartier duality between $(A[q])_j^1$ and $(A[q])_j^2$ (from § 4.3) to construct C_j^1 for $1 \leqslant j \leqslant m$. Define

$$C_j^1 := \big((A[q])_j^2/C_j^2\big)^\vee \subset \big((A[q])_j^2\big)^\vee \cong (A[q])_j^1.$$

In particular, $C_j^1 = (A[q])_j^1$ if $j \neq 1$. Let Ver_q denote the Cartier dual of Fr_q. Then $\big((A[q])_j^2/C_j^2\big)^\vee$ reduces modulo r_1 to $(((A[q])_j^2 \otimes R/r_1)/\mathrm{Ker}(\mathrm{Fr}_q))^\vee$. However, $\mathrm{Ker}(\mathrm{Fr}_q) = \mathrm{Im}(\mathrm{Ver}_q)$ and we have

$$\big(((A[q])_j^2 \otimes R/r_1)/\mathrm{Im}(\mathrm{Ver}_q)\big)^\vee = \mathrm{Ker}(\mathrm{Fr}_q|_{((A[q])_j^2\otimes R/r_1)^\vee}) \cong \mathrm{Ker}(\mathrm{Fr}_q|_{(A[q])_j^1\otimes R/r_1}),$$

which shows that C_j^1 reduces to $\mathrm{Ker}(\mathrm{Fr}_q|_{(A[q])_j^1\otimes R/r_1})$ modulo r_1. We now define the canonical subgroup of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$ to be

$$C := C_1^1 \oplus \cdots \oplus C_m^1 \oplus C_1^2 \oplus \cdots \oplus C_m^2.$$

Proposition 10.5. *The canonical subgroup C of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$ is a finite flat group scheme of rank q^{4d}, which is stable under the action of $\mathcal{O}_D$ and reduces to the kernel of Fr_q modulo $r_1 R$. Furthermore, it is of type 1 (see § 4.4).*

Proof. Since C_j^i reduces to $\mathrm{Ker}(\mathrm{Fr}_q)$ in $(A[q])_j^i \otimes R/r_1$, as we showed above, C will reduce to $\mathrm{Ker}(\mathrm{Fr}_q)$ in A. The rank of C over R equals its rank after reduction modulo r_1 which is $q^{\dim(A/r_1)} = q^{4d}$. To prove that C is stable under the action of $\mathcal{O}_D$, we show that each C_j^i is stable under the action of $\mathcal{O}_{D_j^i}$. This is clear for $j \neq 1$. We claim that when $j = 1$ we only need to prove this for $i = 2$. We have a commutative diagram (see §§ 4.1 and 4.3)

$$\begin{tikzcd}
(A[q])_1^1 \arrow[r, "\theta"] \arrow[d, "l"] & ((A[q])_1^2)^\vee \arrow[d, "(l^*)^\vee"] \\
(A[q])_1^1 \arrow[r, "\theta"] & ((A[q])_1^2)^\vee
\end{tikzcd}$$

By definition of C we know that θ takes C_1^1 isomorphically onto $(A[q]_1^2/C_1^2)^\vee$. To show that C_1^1 is invariant under $l \in \mathcal{O}_{D_1^1}$ amounts to proving that $(A[q]_1^2/C_1^2)^\vee$ is invariant under $(l^*)^\vee$. But $l^* \in \mathcal{O}_{D_1^2}$ and it is enough to show that $C_1^2 \subset A[q]_1^2$ is invariant under $\mathcal{O}_{D_1^2} = M_2(\mathcal{O}_{\mathcal{P}})$.

To show that C_1^2 is invariant under $M_2(\mathcal{O}_{\mathcal{P}})$, it is enough to show that $C_1^{2,1}$ is invariant under the action of $\mathcal{O}_{\mathcal{P}}$. We will prove this in the formal group. By construction, $C_1^{2,1} = \mathrm{Spec}(R[\![x]\!]/(x^q - t_{\mathrm{can}} x))$ and $[\pi]$ kills $C_1^{2,1}$. We therefore need to show that $\mathcal{O}_{\mathcal{P}}/\pi = \kappa$ keeps $C_1^{2,1}$ invariant. Let ζ be a primitive $(q-1)$th root of unity in $\mathcal{O}_{\mathcal{P}}$. Then $\kappa = \mathbb{F}_p(\bar{\zeta})$, where $\bar{\zeta}$ denotes the reduction modulo π of ζ. The action of $\mathbb{F}_p$ comes from that of $\mathbb{Z}$ and hence keeps $C_1^{2,1}$ stable. So we only need to show that $C_1^{2,1}$ is invariant under ζ. But by our choice of coordinate we have

$$\zeta[x]^q - t_{\mathrm{can}}\zeta[x] = \zeta^q x^q - t_{\mathrm{can}}\zeta x = \zeta(x^q - t_{\mathrm{can}} x)$$

which proves the claim. The fact that C is of type 1 is a consequence of the above discussions and the definition of C. $\square$

10.1.3 *Constructing Y'.* We will proceed as in § 3.9 of [Kat73]. Let R be an R_0-algebra in which π is nilpotent and $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$ be an r-test object with $v(r) < 1/(q+1)$. Let C be the canonical subgroup of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$. Let $(A', i', \theta', (\bar{\alpha}^{\mathcal{P}})')$ denote the quotient of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ by C as described in § 4.4. We can assume there is an open affine subset $\mathrm{Spec}(S)$ of $\mathbb{Y}_r \otimes R_0$, and a morphism $\psi : \mathrm{Spec}(R) \to \mathrm{Spec}(\tilde{S})$ such that the pullback of $(\mathbb{B}_r|_{\mathrm{Spec}(S)} \otimes \tilde{S}, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y_r)$ under ψ is equal to $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$. Hence, we only need to do the construction over $\mathbb{B}_r \otimes \tilde{S}$ for which the base scheme is flat over $\mathcal{O}_{\mathcal{P}}$ (see Proposition 9.5). For simplicity we denote $\mathbb{B}_r \otimes \tilde{S}$ by $\mathbf{B}$ and its canonical subgroup by $\mathbf{C}$. Let $(\mathbf{B}', i', \theta', (\bar{\alpha}^{\mathcal{P}})')$ be the quotient of $(\mathbf{B}, i, \theta, \bar{\alpha}^{\mathcal{P}})$ by $\mathbf{C}$ as in § 4.4. Let ω be a basis of $\underline{\omega}_{\mathbf{B}/\tilde{S}}$ on $\mathrm{Spec}(\tilde{S})$. Write $Y_r = y\omega^{\otimes 1-q}$ on $\mathrm{Spec}(\tilde{S})$. It follows from properties of $\mathbf{C}$ that $\mathbf{B}'$ reduces to $\mathbf{B}^{(q)}$ modulo $r_1\tilde{S}$. Let ω' be any basis of $\underline{\omega}_{\mathbf{B}'/\tilde{S}}$ which reduces to $\omega^{(q)}$ on $\mathbf{B}^{(q)}$ modulo $r_1\tilde{S}$. We have

$$E_{q-1}(\mathbf{B}', i', \theta', (\bar{\alpha}^{\mathcal{P}})', \omega') \equiv (E_{q-1}(\mathbf{B}, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega))^q \mod r_1\tilde{S},$$

which implies

$$E_{q-1}(\mathbf{B}', i', \theta', (\bar{\alpha}^{\mathcal{P}})', \omega') = (E_{q-1}(\mathbf{B}, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega))^q + r_1.c$$

for some $c \in \tilde{S}$. Since $v(r) < 1/(q+1)$ we know that r_1 is divisible by r^q in R_0 and hence $r_3 = r_1/r^q$ lies in R and is topologically nilpotent. Define

$$y' := y^q/(1 + r_3 c y^q).$$

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Since $yE_{q-1}(\mathbf{B}, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega) = r$ by calculation we deduce that $y'E_{q-1}(\mathbf{B}', i', \theta', (\bar{\alpha}^{\mathcal{P}})', \omega') = r^q$. So we can define

$$Y'_r = y'(\omega')^{\otimes 1-q}.$$

Clearly Y'_r reduces to $Y^{(q)}$ modulo r_3. The fact that $\tilde{S}$ is flat over $\mathcal{O}_{\mathcal{P}}$ uniquely determines Y'. This concludes the proof of Theorem 10.1. □

Remark 10.6. Let r', r'' and r be elements of R_0 such that $r'' = rr'$. It is easy to see that all the constructions in this section are compatible with the maps induced by $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y) \mapsto (A, i, \theta, \bar{\alpha}^{\mathcal{P}}, rY)$ from r'-test objects to r''-test objects.

11. The Frobenius morphism of $\mathcal{P}$-adic modular functions

Using canonical subgroups, we will define the Frobenius operator, Frob, which is the analogue of the V_p operator on the classical modular forms. The U operator will then essentially be defined as a trace of Frob.

Definition 11.1. Let $r \in R_0$ with $v(r) < 1/(q+1)$. The *Frobenius operator*

$$\begin{aligned} \mathrm{Frob} : S^D(R_0, r^q, K', 0) &\to S^D(R_0, r, K', 0) \\ f &\mapsto \mathrm{Frob}(f) \end{aligned}$$

is defined by

$$\mathrm{Frob}(f)(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y) = f(A', i', \theta', (\bar{\alpha}^{\mathcal{P}})', Y'),$$

where $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$ is an r-test object over an R_0-algebra R in which π is nilpotent, and the r^q-test object $(A', i', \theta', (\bar{\alpha}^{\mathcal{P}})', Y')$ is as in Theorem 10.1, part ii (obtained from dividing by the canonical subgroup).

From Remark 10.6, it is clear that if $r'' = rr'$ in R_0 and $v(r'') < 1/(q+1)$ (in particular, when $r' = 1$ and $r'' = r$), then the diagram

$$\begin{tikzcd} S^D(R_0, (r'')^q, K', 0) \arrow[r, "\mathrm{Frob}"] \arrow[d] & S^D(R_0, r'', K', 0) \arrow[d] \\ S^D(R_0, (r')^q, K', 0) \arrow[r, "\mathrm{Frob}"] & S^D(R_0, r', K', 0) \end{tikzcd}$$

is commutative, where the vertical arrows are the natural inclusions (see Corollary 9.10).

The Frobenius morphism of (convergent and) overconvergent modular functions

$$\mathrm{Frob}_{L_0} : S^D(R_0, r^q, K', 0) \otimes L_0 \to S^D(R_0, r, K', 0) \otimes L_0$$

is obtained by tensoring Frob with L_0.

11.1 Frob in the rigid setting

Define a morphism

$$\begin{aligned} \mathrm{Frob}_n : \mathbb{Y}_r \otimes R_0/\pi^n &\to \mathbb{Y}_{r^q} \otimes R_0/\pi^n \\ (A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y) &\mapsto (A', i', \theta', (\bar{\alpha}^{\mathcal{P}})', Y'), \end{aligned}$$

where $(A', i', \theta', (\bar{\alpha}^{\mathcal{P}})', Y')$ denotes the r^q-test object defined over an R_0/π^n-algebra which is obtained from $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$ by dividing by its canonical subgroup.

For any $n \geqslant 1$, let $\mathbf{C}_{r,n}$ denote the canonical subgroup of the r-test object $(\mathbb{B}_r \otimes R_0/\pi^n, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y_r)$ over $\mathbb{Y}_r \otimes R_0/\pi^n$. Let $\widetilde{\mathbf{C}}_r \subset \widetilde{\mathbb{B}}_r \otimes R_0$ be given by the inverse limit of the group schemes $\mathbf{C}_{r,n} \subset \mathbb{B}_r \otimes R_0/\pi^n$. From the description of Frob_n we have an isomorphism

$$\mathrm{Frob}_n^*(\mathbb{B}_{r^q} \otimes R_0/\pi^n) \xrightarrow{\sim} (\mathbb{B}_r \otimes R_0/\pi^n)/\mathbf{C}_{r,n}.$$

Let $\phi_n : \mathbb{B}_r \otimes R_0/\pi^n \to (\mathbb{B}_r \otimes R_0/\pi^n)/\mathbf{C}_{r,n}$ denote the canonical projection and let $\eta_n : (\mathbb{B}_r \otimes R_0/\pi^n)/\mathbf{C}_{r,n} \to \mathbb{B}_{r^q} \otimes R_0/\pi^n$ denote the base extension map via Frob_n. We have commutative diagrams

$$\begin{tikzcd}
\mathbb{B}_r \otimes R_0/\pi^n \arrow[r, "\mathrm{Frob}_n := \eta_n \circ \phi_n"] \arrow[d] & \mathbb{B}_{r^q} \otimes R_0/\pi^n \arrow[d] \\
\mathbb{Y}_r \otimes R_0/\pi^n \arrow[r, "\mathrm{Frob}_n"] & \mathbb{Y}_{r^q} \otimes R_0/\pi^n
\end{tikzcd}$$

which are compatible for varying n. By passing to completion along $\pi = 0$, we get the following commutative diagram:

$$\begin{tikzcd}
\widetilde{\mathbb{B}}_r \otimes R_0 \arrow[r, "\widetilde{\mathrm{Frob}}"] \arrow[d] & \widetilde{\mathbb{B}}_{r^q} \otimes R_0 \arrow[d] \\
\widetilde{\mathbb{Y}}_r \otimes R_0 \arrow[r, "\widetilde{\mathrm{Frob}}"] & \widetilde{\mathbb{Y}}_{r^q} \otimes R_0
\end{tikzcd}$$

Similarly, we have $\widetilde{\mathrm{Frob}}^*(\widetilde{\mathbb{B}}_{r^q} \otimes R_0) = (\widetilde{\mathbb{B}}_r \otimes R_0)/\widetilde{\mathbf{C}}_r$ and the map

$$\widetilde{\mathbb{B}}_r \otimes R_0 \to \widetilde{\mathrm{Frob}}^*(\widetilde{\mathbb{B}}_{r^q} \otimes R_0)$$

induced by $\widetilde{\mathrm{Frob}}$ is the canonical projection $\tilde{\phi} = \varprojlim_n \phi_n$. Let $\hat{\eta} = \varprojlim_n \eta_n$ be the base extension map $\widetilde{\mathbb{B}}_r \otimes R_0/\widetilde{\mathbf{C}}_r \to \widetilde{\mathbb{B}}_{r^q} \otimes R_0$. Then, we have

$$\widetilde{\mathrm{Frob}} = \tilde{\eta} \circ \tilde{\phi}.$$

It is clear from the definition of Frob_n and Frob and the above construction that the diagram

$$\begin{tikzcd}
H^0(\widetilde{\mathbb{Y}}_{r^q} \otimes R_0, \mathcal{O}_{\widetilde{\mathbb{Y}}_{r^q} \otimes R_0}) \arrow[r, "\widetilde{\mathrm{Frob}}^*"] \arrow[d, equal] & H^0(\widetilde{\mathbb{Y}}_r \otimes R_0, \mathcal{O}_{\widetilde{\mathbb{Y}}_r \otimes R_0}) \arrow[d, equal] \\
S^D(R_0, r^q, K', 0) \arrow[r, "\mathrm{Frob}"] & S^D(R_0, r, K', 0)
\end{tikzcd}$$

is commutative. We have so far described the Frobenius morphism in the formal setting. This can be used to derive a description of Frob in the rigid setting. The canonical subgroup of $A'_{K'}(|r|_v)$ is the image of $\widetilde{\mathbf{C}}_r$ under the functor rig and is denoted by $\mathbf{C}_r$.

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Proposition 11.2. *Let $r \in R_0$ with $v(r) < 1/(q+1)$. There exists a commutative diagram of rigid analytic spaces over L_0*

$$\begin{tikzcd} A'_{K'}(|r|_v) \arrow[r, "\mathrm{Frob}^{\mathrm{rig}}"] \arrow[d] & A'_{K'}(|r|_v^q) \arrow[d] \\ M'_{K'}(|r|_v) \arrow[r, "\mathrm{Frob}^{\mathrm{rig}}"] & M'_{K'}(|r|_v^q) \end{tikzcd}$$

in which the pullback of $A'_{K'}(|r|_v^q)$ under $\mathrm{Frob}^{\mathrm{rig}}$ is isomorphic to $A'_{K'}(|r|_v)/\mathbf{C}_r$ and the map induced by $\mathrm{Frob}^{\mathrm{rig}}$ and the natural map $A'_{K'}(|r|_v) \to M'_{K'}(|r|_v)$,

$$A'_{K'}(|r|_v) \xrightarrow{\phi} \mathrm{Frob}^{\mathrm{rig}*}(A'_{K'}(|r|_v^q)) \cong A'_{K'}(|r|_v)/\mathbf{C}_r,$$

is the natural projection ϕ^{rig} so that

$$\mathrm{Frob}^{\mathrm{rig}} = \eta^{\mathrm{rig}} \circ \phi^{\mathrm{rig}}.$$

Here, $\eta^{\mathrm{rig}} : A'_{K'}(|r|_v)/\mathbf{C}_r \to A'_{K'}(|r|_v^q)$ is the base extension map obtained by rigidification of η. Furthermore the Frobenius morphism of (convergent and) overconvergent modular functions can be described as the pullback of $\mathrm{Frob}^{\mathrm{rig}}$. In other words, the following diagram is commutative:

$$\begin{tikzcd} H^0(M'_{K'}(|r|_v^q), \mathcal{O}_{M'_{K'}(|r|_v^q)}) \arrow[r, "(\mathrm{Frob}^{\mathrm{rig}})^*"] \arrow[d, equal] & H^0(M'_{K'}(|r|_v), \mathcal{O}_{M'_{K'}(|r|_v)}) \arrow[d, equal] \\ S^D(R_0, r^q, K', 0) \otimes L_0 \arrow[r, "\mathrm{Frob}_{L_0}"] & S^D(R_0, r, K', 0) \otimes L_0 \end{tikzcd}$$

Proof. Apply Raynaud's functor rig to the construction we have done in the formal setting. □

11.2 Frob on points

We will study the action of $\mathrm{Frob}^{\mathrm{rig}}$ on points of $M'_{K'}(|r|_v)$. Let L_∞ denote the completion of an algebraic closure of L_0. The following lemma is a standard result of Raynaud's theory.

Lemma 11.3. *Let X be a scheme which is flat and of finite type over R_0. Let $\tilde{X}$ denote the completion of X along the subscheme $\pi = 0$. Let X^{rig} be the rigid analytic space over L_0 associated to $\tilde{X}$ under the Raynaud functor. For any L which is either L_∞ or a finite extension of L_0 with ring of integers R, we have a one-to-one correspondence*

$$\mathrm{Hom}_{R_0}(\mathrm{Spec}(R), X) \leftrightarrow \mathrm{Hom}_L(\mathrm{Sp}(L), X^{\mathrm{rig}} \hat{\otimes} L).$$

(Note that if $[L : L_0]$ is finite then $\mathrm{Hom}_L(\mathrm{Sp}(L), X^{\mathrm{rig}} \hat{\otimes} L) = \mathrm{Hom}_{L_0}(\mathrm{Sp}(L), X^{\mathrm{rig}})$.)

A closed point of $M'_{K'}(|r|_v)$ gives a map $\mathrm{Sp}(L) \to M'_{K'}(|r|_v)$ for some finite extension L of L_0 with ring of integers R. Thinking of $M'_{K'}(|r|_v)$ as an affinoid subdomain of $(\mathbb{M}'_{K'} \otimes L_0)^{\mathrm{an}}$ we get a map $\mathrm{Sp}(L) \to (\mathbb{M}'_{K'} \otimes L_0)^{\mathrm{an}}$ which, by rigid GAGA, corresponds to a map $\mathrm{Spec}(L) \to \mathbb{M}'_{K'} \otimes L_0$. This is nothing but (the analytification of) $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ over L. By Lemma 11.3 the map $\mathrm{Sp}(L) \to M'_{K'}(|r|_v)$ is obtained as the image of a map $\mathrm{Spec}(R) \to \mathbb{Y}_r$ under the Raynaud functor. This gives an r-test object $(\mathbb{A}, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$ over R. Now since the Raynaud functor agrees with an for proper schemes, we deduce that $(\mathbb{A}, i, \theta, \bar{\alpha}^{\mathcal{P}})$ is a model for $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ over R. The existence of Y is clearly equivalent to the condition $|E_{q-1}(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega)|_v \geqslant |r|_v$ and Y is uniquely determined from this inequality. Call a morphism $\mathrm{Sp}(L_\infty) \to M'_{K'}(|r|_v) \hat{\otimes} L_\infty$ an L_∞-point of $M'_{K'}(|r|_v)$. Similarly, by Lemma 11.3 giving an L_∞-point of $M'_{K'}(|r|_v)$ is equivalent to giving $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ over R_∞ (the ring of integers of L_∞) such that $|E_{q-1}(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega)|_v \geqslant |r|_v$.

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Now we investigate the canonical subgroup of a closed or L_∞-point of $M'_{K'}(|r|_v)$. Assume that $x : \mathrm{Sp}(L) \to M'_{K'}(|r|_v)\hat{\otimes}L$ is such a point giving $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ over R. The fibre of $\mathbf{C}_r\hat{\otimes}L$ (see § 11.1) over $\mathrm{Sp}(L)$ gives a finite flat subgroup scheme of $A \otimes L$. By construction of $A'_{K'}(|r|_v)$ and $\mathbf{C}_r$, this subgroup is the image of C, the canonical subgroup of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, Y)$, under rig. Since C is finite and flat over R, this equals the (analytification of the) generic fibre of C. Now the description of $\widetilde{\mathrm{Frob}}$ shows that the image of x under $\mathrm{Frob}^{\mathrm{rig}}$ is the point given by $(A', i', \theta', (\bar{\alpha}^{\mathcal{P}})')$ which is obtained from $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ by dividing by C.

We summarize the above observations in the following corollary.

Corollary 11.4. *Giving a closed point (respectively L_∞-point) of $M'_{K'}(|r|_v)$ is equivalent to giving $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ (as in § 4.2) over R, the ring of integers of a finite extension of L_0 (respectively over R_∞), which satisfies $|E_{q-1}(A, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega)|_v \geqslant |r|_v$ for a basis ω of $\underline{\omega}$. The fibre of $\mathbf{C}_r \subset A'_{K'}(|r|_v)$ over this point is the generic fibre of $C \subset A$ where C is the canonical subgroup of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$. The image of this point under* $\mathrm{Frob}^{\mathrm{rig}}$ *is determined by $(A', i', \theta', (\bar{\alpha}^{\mathcal{P}})')$ which is the quotient of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ by C as defined in § 4.4.*

11.3 Properties of Frob

We have seen that the Frobenius morphism of $\mathcal{P}$-adic modular functions can be described as the pullback of a morphism of rigid analytic varieties. This helps us in the study of the properties of Frob.

Proposition 11.5. *For $r \in R_0$ with $v(r) < 1/(q+1)$,*

$$\mathrm{Frob} : S^D(R_0, r^q, K', 0) \to S^D(R_0, r, K', 0)$$

is a finite morphism. If $r = 1$, it is finite and flat of rank q.

Proof. This can be proven in exactly the same way as Theorem 3.10.1 of [Kat73]. □

We have seen that when $r = 1$, the morphism Frob is finite and flat of rank q. This is not true for general r. However, the same result holds true after tensoring with L_0.

Proposition 11.6. *If $v(r) < 1/(q+1)$, then* $\mathrm{Frob}^{\mathrm{rig}} : M'_{K'}(|r|_v) \to M'_{K'}(|r|_v^q)$ *is a finite flat map of degree q between rigid analytic spaces over L_0.*

Proof. We have already seen that $\mathrm{Frob}^{\mathrm{rig}}$ is finite. First we prove that it is flat. Since $M'_{K'}(|r|_v)$ is an affinoid subdomain of $\mathbb{M}'_{K'} \otimes L_0$, the completion of the rigid local ring of $M'_{K'}(|r|_v)$ at any closed point of $M'_{K'}(|r|_v)$ equals the completion of the local ring of the corresponding closed point on $\mathbb{M}'_{K'} \otimes L_0$. But $\mathbb{M}'_{K'} \otimes L_0$ is smooth and hence the completion of the local ring of any of its closed points is regular. Therefore, the local rings of $S^D(R_0, r, K', 0) \otimes L_0$ at its maximal ideals are all regular. Hence, $S^D(R_0, r, K', 0) \otimes L_0$ and $S^D(R_0, r^q, K', 0) \otimes L_0$ are regular rings of dimension one and hence any finite morphism between them is flat. Therefore, Frob_{L_0} is flat. We have already seen that $\mathrm{Frob}^{\mathrm{rig}}$ has degree q over the affinoid $M'_{K'}(1)$. Therefore, $\mathrm{Frob}^{\mathrm{rig}}$ is finite flat of degree q. □

Proposition 11.7. *The morphism* $\mathrm{Frob}^{\mathrm{rig}} : M'_{K'}(1) \to M'_{K'}(1)$ *is étale of degree q.*

Proof. By Proposition 11.6, we only need to prove that the fibre of any L_∞-point of $M'_{K'}(1)$ consists of exactly q points. Let $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$ be an L_∞-point of $M'_{K'}(1)$ as described in § 11.2. The fibre of $\mathrm{Frob}^{\mathrm{rig}}$ over this point consists of all $(A', i', \theta', (\bar{\alpha}^{\mathcal{P}})')$ defined over R_∞ such that $E_{q-1}(A', i', \theta', (\bar{\alpha}^{\mathcal{P}})', \omega')$ is a unit in R_∞ for a choice of nowhere vanishing section ω' of $\underline{\omega}$ and

$$(A', i', \theta', (\bar{\alpha}^{\mathcal{P}})')/C' = (A, i, \theta, \bar{\alpha}^{\mathcal{P}}),$$

where C' is the canonical subgroup of $(A', i', \theta', (\bar{\alpha}^{\mathcal{P}})')$. Let $g : A' \to A$ be the projection and let $f : A \to A'$ be the morphism such that $g \circ f = [q]_A$. Then $\mathrm{Ker}(f) \xrightarrow{\sim} A'[q]/C' \subset A'/C' \xrightarrow{\sim} A$.

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First we study $\mathrm{Ker}(f)$. Since C' is of type 1 (defined in § 4.4), $\mathrm{Ker}(f) \subset A$ is of type 2 and is uniquely determined by the component $(\mathrm{Ker}(f))_1^{2,1}$. As a consequence of the properties of canonical subgroups, the subgroup $(\mathrm{Ker}(f))_1^{2,1}$ of A has the following properties:

i) $A([q/\pi])_1^{2,1} \subset (\mathrm{Ker}(f))_1^{2,1} \subset (A[q])_1^{2,1}$;

ii) $(\mathrm{Ker}(f))_1^{2,1}$ has index q in $(A[q])_1^{2,1}$;

iii) $(\mathrm{Ker}(f))_1^{2,1}$ is $\mathcal{O}_{\mathcal{P}}$-stable.

Subgroups with the above properties are in bijection with $\mathcal{O}_{\mathcal{P}}$-invariant subgroups $\bar{C}_j$ of rank q in $A[\pi]_1^{2,1}$ and hence there are exactly $q+1$ subgroups of this form. For $0 \leqslant j \leqslant q$ let $(C_j)_1^{2,1}$ denote all such subgroups. The correspondence is via an exact sequence

$$0 \longrightarrow (A[q/\pi])_1^{2,1} \longrightarrow (C_j)_1^{2,1} \xrightarrow{\times q/\pi} \bar{C}_j \longrightarrow 0.$$

Furthermore, assume $\bar{C}_0 = C_1^{2,1}$ where C is the canonical subgroup of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$. Denote by C_j the subgroup of type 2 determined uniquely by $(C_j)_1^{2,1}$. So $\mathrm{Ker}(f) = C_j$ for some $0 \leqslant j \leqslant q$ and we have

$$(A', i', \theta') = (A, i, \theta)/C_j,$$
$$(\bar{\alpha}^{\mathcal{P}})' = (\alpha^p)' \oplus (\alpha_p^{\mathcal{P}})' = (1/q)(\alpha_j)^p \oplus (\alpha_j)_p^{\mathcal{P}},$$

where $\bar{\alpha_j}^{\mathcal{P}} = (\alpha_j)^p \oplus (\alpha_j)_p^{\mathcal{P}}$ denotes the induced level structure on (A', i', θ') form $f : A \to A'$. Now $(A/C_j, i', \theta', (\bar{\alpha}^{\mathcal{P}})')$ is in the fibre if and only if $A[q]/C_j$ is the canonical subgroup of $A/C_j \xrightarrow{\sim} A'$. Since $A[q]/C_j$ is of type 1, by definition of the canonical subgroup this is to say that $(A[q]/C_j)_1^{2,1}$ is the canonical subgroup of $(G_{A/C_j})_1^{2,1}$. Since $r = 1$, Remark 10.4 implies that the canonical subgroup of $(G_{A/C_j})_1^{2,1}$ equals $(G_{A/C_j})_1^{2,1}[\pi]$. We claim that $(A[q]/C_j)_1^{2,1} = (G_{A/C_j})_1^{2,1}[\pi]$, if and only if $j \neq 0$. This will show that the fibre has q points and prove the proposition. The image of $(A[q]/C_j)_1^{2,1}$ in $(G_{A/C_j})_1^{2,1}$ is $\mathcal{O}_{\mathcal{P}}$-invariant and hence of rank 0 or q, and A/C_j is not in the fibre if and only if this image is zero. This happens exactly when the image of $(C_j)_1^{2,1}$ in $(G_A)_1^{2,1}$ equals $(G_A)_1^{2,1}[q]$. But then, the above exact sequence shows that this occurs if and only if the image of $\bar{C}_j$ in $(G_A)_1^{2,1}$ is $(G_A)_1^{2,1}[\pi]$ as we have the following exact sequence:

$$0 \longrightarrow (G_A)_1^{2,1}[q/\pi] \longrightarrow (G_A)_1^{2,1}[q] \xrightarrow{\times q/\pi} (G_A)_1^{2,1}[\pi] \longrightarrow 0.$$

Since $r = 1$, Remark 10.4 implies that the image of $\bar{C}_j$ in $(G_A)_1^{2,1}$ is $(G_A)_1^{2,1}[\pi]$ if and only if $\bar{C}_j = C_1^{2,1}$, where C is the canonical subgroup of $(A, i, \theta, \bar{\alpha}^{\mathcal{P}})$. That is true only for $j = 0$. $\square$

Remark 11.8. A lengthy but straightforward argument similar to Katz's proof of Theorem 3.10.7 of [Kat73] shows that $\mathrm{Frob}^{\mathrm{rig}}$ is also étale for any r with $0 < v(r) < 1/(q+1)$.

12. The U operator

We have seen that if $v(r) < 1/(q+1)$, then Frob_{L_0} is finite flat of rank q. We can therefore define the trace of Frob_{L_0} as

$$\mathrm{Tr}_{\mathrm{Frob}_{L_0}} : S^D(R_0, r, K', 0) \otimes L_0 \to S^D(R_0, r^q, K', 0) \otimes L_0.$$

This morphism is compatible with the injection of $S^D(R_0, r'', K', 0)$ in $S^D(R_0, r', K', 0)$ if $r'' = rr'$. If $r = 1$, then Frob is already finite and flat of rank q before tensoring with L_0 and thus, in this case, $\mathrm{Tr}_{\mathrm{Frob}} : S^D(R_0, 1, K', 0) \to S^D(R_0, 1, K', 0)$ can be defined and we have the equality $\mathrm{Tr}_{\mathrm{Frob}} \otimes L_0 = \mathrm{Tr}_{\mathrm{Frob}_{L_0}}$ for $r = 1$.

Lemma 12.1. $(\mathrm{Frob}^{\mathrm{rig}})^*\underline{\omega}_{A'_{K'}(|r|_v^q)/M'_{K'}(|r|_v^q)} \xrightarrow{\sim} \underline{\omega}_{(A'_{K'}(|r|_v)/\mathbf{C}_r)/M'_{K'}(|r|_v)}$.

Proof. Consider the following diagram:

$$\begin{tikzcd}
(\mathrm{Frob}^{\mathrm{rig}})^* A'_{K'}(|r|_v^q) \arrow[r, "\eta^{\mathrm{rig}}"] \arrow[d] & A'_{K'}(|r|_v^q) \arrow[d] \\
M'_{K'}(|r|_v) \arrow[r, "\mathrm{Frob}^{\mathrm{rig}}"] & M'_{K'}(|r|_v^q)
\end{tikzcd}$$

From § 11.1 recall that $\eta^{\mathrm{rig}} : (\mathrm{Frob}^{\mathrm{rig}})^* A'_{K'}(|r|_v^q) \to A'_{K'}(|r|_v^q)$ is the base extension of $\mathrm{Frob}^{\mathrm{rig}}$. So, there is an isomorphism $(\eta^{\mathrm{rig}})^*\Omega^1_{A'_{K'}(|r|_v^q)/M'_{K'}(|r|_v^q)} \xrightarrow{\sim} \Omega^1_{(\mathrm{Frob}^{\mathrm{rig}})^* A'_{K'}(|r|_v^q)/M'_{K'}(|r|_v)}$. On the other hand, from Proposition 11.2 we have $(\mathrm{Frob}^{\mathrm{rig}})^* A'_{K'}(|r|_v^q) \xrightarrow{\sim} A'_{K'}(|r|_v)/\mathbf{C}_r$. Taking the component of Ω^1 corresponding to $\mathcal{O}_{D_1^{2,1}}$, we get the desired natural isomorphism. $\square$

We define $\mathrm{Tr}_{\mathrm{Frob}_{L_0}}$ for overconvergent modular forms of arbitrary weight. It is defined as the trace function on the sections of the sheaf $\underline{\omega}^{\otimes k}$. When $k \geqslant 0$ we define $\mathrm{Tr}_{\mathrm{Frob}_{L_0}}$ by the following diagram

$$\begin{tikzcd}
H^0(M'_{K'}(|r|_v), \underline{\omega}^{\otimes k}_{(A'_{K'}(|r|_v)/\mathbf{C}_r)/M'_{K'}(|r|_v)}) \arrow[r, "\mathrm{Tr}_{\mathrm{Frob}^{\mathrm{rig}}}"] & H^0(M'_{K'}(|r|_v^q), \underline{\omega}^{\otimes k}_{A'_{K'}(|r|_v^q)/M'_{K'}(|r|_v^q)}) \arrow[dd, equal] \\
H^0(M'_{K'}(|r|_v), \underline{\omega}^{\otimes k}_{A'_{K'}(|r|_v)/M'_{K'}(|r|_v)}) \arrow[u, "(\phi')^*"] \arrow[d, equal] & \\
S^D(R_0, r, K', k) \otimes L_0 \arrow[r, "\mathrm{Tr}_{\mathrm{Frob}_{L_0}}"] & S^D(R_0, r^q, K', k) \otimes L_0
\end{tikzcd}$$

Here $\phi = \phi^{\mathrm{rig}} : A'_{K'}(|r|_v) \to A'_{K'}(|r|_v)/\mathbf{C}_r$ is the natural projection and the morphism $(\phi')^* : \underline{\omega}_{A'_{K'}(|r|_v)} \to \underline{\omega}_{A'_{K'}(|r|_v)/\mathbf{C}_r}$ is our refined pullback defined in Proposition 4.5. The upper horizontal morphism, $\mathrm{Tr}_{\mathrm{Frob}^{\mathrm{rig}}}$, is the trace from the global sections of the pullback of $\underline{\omega}^{\otimes k}_{A'_{K'}(|r|_v^q)/M'_{K'}(|r|_v^q)}$ on $M'_{K'}(|r|_v)$ to the global sections of $\underline{\omega}^{\otimes k}_{A'_{K'}(|r|_v^q)/M'_{K'}(|r|_v^q)}$ on $M'_{K'}(|r|_v^q)$. For $k = 0$ we get the same definition as before. For $k < 0$ we replace $(\phi')^*$ with $\pi^k(\phi^*)^\vee$.

We now define the U operator.

Definition 12.2. Let $r \in R_0$ be such that $v(r) < 1/(q+1)$. We define the $\mathrm{U}_{(k)}$ operator of $S^D(R_0, r^q, K', k) \otimes L_0$,

$$\mathrm{U}_{(k)} : S^D(R_0, r^q, K', k) \otimes L_0 \to S^D(R_0, r^q, K', k) \otimes L_0,$$

to be the following composite:

$$S^D(R_0, r^q, K', k) \otimes L_0 \hookrightarrow S^D(R_0, r, K', k) \otimes L_0 \xrightarrow{(1/q)\,\mathrm{Tr}_{\mathrm{Frob}_{L_0}}} S^D(R_0, r^q, K', k) \otimes L_0.$$

Here the first arrow is the natural inclusion. Again $\mathrm{U}_{(k)}$ is compatible with the natural injection of $S^D(R_0, (r'')^q, K', 0)$ in $S^D(R_0, (r')^q, K', 0))$, if $r'' = rr'$. In particular, when $r' = 1$, and $r'' = r$,

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we have

$$\begin{tikzcd}
S^D(R_0, r^q, K', k) \otimes L_0 \arrow[r, "\mathrm{U}_{(k)}"] \arrow[d] & S^D(R_0, r^q, K', k) \otimes L_0 \arrow[d] \\
S^D(R_0, 1, K', k) \otimes L_0 \arrow[r, "\mathrm{U}_{(k)}"] & S^D(R_0, 1, K', k) \otimes L_0
\end{tikzcd}$$

Therefore, we can think of $\mathrm{U}_{(k)}$ as an operator on $S^D(R_0, 1, K', k) \otimes L_0$, the space of convergent modular forms, which preserves the subspace of overconvergent modular forms. We will usually drop the subscript k when the weight is understood and simply refer to this operator as the U operator. We now define the Frobenius morphism for overconvergent modular forms of general weight.

DEFINITION 12.3. The Frobenius morphism for overconvergent modular forms,

$$\mathrm{Frob}_{L_0} : S^D(R_0, r^q, K', k) \otimes L_0 \to S^D(R_0, r, K', k) \otimes L_0,$$

is defined as follows. For $k \geqslant 0$ we have

$$\begin{tikzcd}
H^0(M'_{K'}(|r|_v), \underline{\omega}^{\otimes k}_{(A'_{K'}(|r|_v)/\mathbf{C}_r)/M'_{K'}(|r|^q_v)}) \arrow[r, "(\phi^*)/\pi^k"] & H^0(M'_{K'}(|r|_v), \underline{\omega}^{\otimes k}_{A'_{K'}(|r|_v)/M'_{K'}(|r|_v)}) \arrow[dd, equal] \\
H^0(M'_{K'}(|r|^q_v), \underline{\omega}^{\otimes k}_{A'_{K'}(|r|^q_v)/M'_{K'}(|r|^q_v)}) \arrow[u, "(\mathrm{Frob}^{\mathrm{rig}})^*"] \arrow[d, equal] & \\
S^D(R_0, r^q, K', k) \otimes L_0 \arrow[r, "\mathrm{Frob}_{L_0}"] & S^D(R_0, r, K', k) \otimes L_0
\end{tikzcd}$$

and for $k < 0$ we replace $(\phi^*)/\pi^k$ with $((\phi')^*)^\vee$ in the above diagram, where $(\phi')^*$ (in weight $-k$) is defined as in Proposition 4.5. Note that when $k = 0$ we get the same definition as before.

To conclude this section, we prove a projection formula.

LEMMA 12.4. *Let $r \in R_0$ with $v(r) < 1/(q+1)$. Assume that $f \in S^D(R_0, r^q, K', k) \otimes L_0$ and $g \in S^D(R_0, r, K', k') \otimes L_0$. Then, we have*

$$\mathrm{U}(\mathrm{Frob}_{L_0}(f)g) = f\mathrm{U}(g).$$

Proof. For $k, k' \geqslant 0$ we write

$$\begin{aligned}
\mathrm{U}(\mathrm{Frob}_{L_0}(f)g) &= (1/q)\,\mathrm{Tr}_{\mathrm{Frob}^{\mathrm{rig}}}\,\phi'^*(((\phi^*/\pi^k)(\mathrm{Frob}^{\mathrm{rig}})^*f)g) \\
&= (1/q)\,\mathrm{Tr}_{\mathrm{Frob}^{\mathrm{rig}}}(((\mathrm{Frob}^{\mathrm{rig}})^*f)(\phi'^*g)) \\
&= (1/q)f\,\mathrm{Tr}_{\mathrm{Frob}^{\mathrm{rig}}}(\phi'^*g) \\
&= f\mathrm{U}(g).
\end{aligned}$$

Here we use Proposition 4.5 which states that $\phi'^*\phi^* = \pi^k$ on $\underline{\omega}^{\otimes k}_{(A'_{K'}(|r|_v)/\mathbf{C}_r)/M'_{K'}(|r|^q_v)}$. A similar argument works for other cases. $\square$

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13. Continuity properties of U

In this section we study continuity properties of U. Since $S^D(R_0, r, K', 0) \otimes L_0$ is a reduced affinoid L_0-algebra, it carries a canonical L_0-Banach space topology which is induced by the supremum norm $|\ |_{\sup}$. We also describe $|\ |_{\sup}$ defined on $S^D(R_0, r, K', k) \otimes L_0$ for general k. Let $f \in S^D(R_0, r, K', k) \otimes L_0$. Cover $\mathbb{M}'_{K'} \otimes R_0$ with finitely many open affines V_i such that on each V_i, ω_i gives a basis for $\underline{\omega}^{\otimes k}$. Write $f = a_i \omega_i$ on the intersection of V_i and $M'_{K'}(|r|_v)$. Define $|f|_{\sup} = \sup\{|a_i|_{\sup}\}$. This definition is clearly independent of choice of the affine covering. It is easy to see that $S^D(R_0, r, K', k) \otimes L_0$ is complete and separated with respect to this norm (as a result of the same fact for $k = 0$). One can also define $|f(x)|_v$ for any closed point of $M'_{K'}(|r|_v)$ applying Lemma 11.3 and using a construction similar to the one described in § 9.2. We have $|f|_{\sup} = \sup\{|f(x)|_v\}$ where x varies over all closed points of $M'_{K'}(|r|_v)$.

Proposition 13.1. $S^D(R_0, r, K', k) = \{f \in S^D(R_0, r, K', k) \otimes L_0 \mid |f|_{\sup} \leqslant 1\}$.

Proof. Let $f \in S^D(R_0, r, K', k) \otimes L_0 = H^0(M'_{K'}(|r|_v), \underline{\omega}^{\otimes k})$ with $|f|_{\sup} \leqslant 1$. We only need to show that $f \in H^0(\widetilde{\mathbb{Y}}_r \otimes R_0, \underline{\omega}^{\otimes k}) \subset H^0(M'_{K'}(|r|_v), \underline{\omega}^{\otimes k})$. It is enough to prove this locally. Let $W = \mathrm{Spec}(R)$ be an open affine of $\mathbb{M}'_{K'} \otimes R_0$ on which $\underline{\omega}$ has a nowhere vanishing section ω and write $E_{q-1}|_W = a\omega^{\otimes q-1}$. The restriction of $\widetilde{\mathbb{Y}}_r \otimes R_0$ to W is given by $\mathrm{Spf}(\tilde{R}\langle x\rangle/(ax - r))$ where $\tilde{R}$ as usual denotes the π-adic completion of R. The corresponding affinoid in $M'_{K'}(|r|_v)$ is given by $\mathrm{Sp}((\tilde{R}\langle x\rangle/(ax - r)) \otimes L_0)$. Let $f = b\omega^{\otimes k}$ with $b \in (\tilde{R}\langle x\rangle/(ax - r)) \otimes L_0$. By our definition of the supremum norm for general k, the fact that $|f|_{\sup} \leqslant 1$ implies $|b|_{\sup} \leqslant 1$. By Theorem 5.2 of [Tat71] b has to be integral over $\tilde{R}\langle x\rangle/(ax - r)$. Thus, Proposition 9.5 implies that $b \in \tilde{R}\langle x\rangle/(ax - r)$ and hence $f \in H^0(\widetilde{\mathbb{Y}}_r \otimes R_0, \underline{\omega}^{\otimes k})$ as desired. □

This allows us to prove the independence of our theory of the noncanonical choice of the lifting of **H**.

Corollary 13.2. *The theory is independent of choice of E_{q-1} as long as $v(r) < q/(q + 1)$.*

Proof. Let E'_{q-1} be another lifting. Since $E_{q-1} \equiv E'_{q-1} \pmod{\pi}$, for any closed point x of $\mathbb{M}'_{K'}$ we have $|E(x) - E'(x)|_v \leqslant |\pi|_v = 1/q$. On the other hand, $v(r) < q/(q+1)$ implies $|r|_v > 1/q$ and hence $|E(x)|_v \geqslant |r|_v$ if and only if $|E(x)|_v \geqslant |r|_v$. Therefore, the definition of $M'_{K'}(|r|_v)$ is independent of choice of E_{q-1} and hence so is the definition of $S^D(R_0, r, K', k) \otimes L_0$. Now by Proposition 13.1 $S^D(R_0, r, K', k)$ can be recovered from $S^D(R_0, r, K', k) \otimes L_0$ independently of choice of E_{q-1}. It is straightforward to check that the canonical subgroup of a closed point of $M'_{K'}(|r|_v)$ depends only on the reduction modulo π of E_{q-1}. This shows that the definition of Frob and U are also independent of the choice of E_{q-1}. □

We can now study the continuity of U.

Proposition 13.3. *Let $r \in R_0$ with $v(r) < 1/(q + 1)$. For any $k \in \mathbb{Z}$ the operator* U *is continuous on $S^D(R_0, r^q, K', k) \otimes L_0$.*

Proof. First assume $k = 0$. For any finite flat morphism $f : S_1 \to S_2$ of affinoid L_0-algebras, Tr_f takes power bounded elements of S_1 to power bounded elements of S_2. So if $f \in S^D(R_0, r, K', 0)$, then $|\,\mathrm{Tr}_{\mathrm{Frob}_{L_0}}(f)|_{\sup} \leqslant 1$. Therefore, we have $\mathrm{Tr}_{\mathrm{Frob}_{L_0}}(S^D(R_0, r, K', 0)) \subset S^D(R_0, r^q, K', 0)$ by Proposition 13.1. The definition of U now implies the desired result.

Now assume $k > 0$. Let Ω_r denote $S^D(R_0, r, K', k)$ and let S_r denote $S^D(R_0, r, K', 0)$ for simplicity. Let Ω denote $H^0(\widetilde{\mathbb{Y}}_r \otimes R_0, \underline{\omega}^{\otimes k}_{(\widetilde{\mathbb{B}}_r \otimes R_0/\widetilde{\mathbf{C}}_r)/\widetilde{\mathbb{Y}}_r \otimes R_0})$. By definition qU is obtained as the following composite:

$$\Omega_{r^q} \otimes L_0 \longrightarrow \Omega_r \otimes L_0 \xrightarrow{(\phi')^*} \Omega \otimes L_0 \xrightarrow{\mathrm{Tr}_{\mathrm{Frob}^{\mathrm{rig}}}} \Omega_{r^q} \otimes L_0.$$

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Since $(\phi')^*(\Omega_r) \subset \Omega$ and $\Omega_{r^q} \hookrightarrow \Omega_r$, to prove the desired result it suffices to show that

$$\mathrm{Tr}_{\mathrm{Frob}^{\mathrm{rig}}}(\Omega) \subset (1/\pi^j)\Omega_{r^q}$$

for some integer $j \geqslant 0$. From the case $k = 0$, we know that $\mathrm{Tr}_{\mathrm{Frob}^{\mathrm{rig}}}(\Omega_{r^q} \hat{\otimes}_{S_{r^q}} S_r) \subset \Omega_{r^q}$. Now the result follows, since $\Omega \otimes L_0 = (\Omega_{r^q} \hat{\otimes}_{S_{r^q}} S_r) \otimes L_0$, and Ω is finitely generated S_r-module. A similar argument works in the case $k < 0$. $\square$

The operator theory of U heavily depends on whether it acts on the full space of convergent modular forms or just on the subspace of overconvergent ones. It turns out that in the latter case, apart from the kernel of U, the eigenspaces are all finite dimensional, whereas this does not hold in the first case. The reason is, as we will see, that when $v(r) > 0$ the U operator is a completely continuous operator of orthonormizable L_0-Banach spaces.

An operator of L_0-Banach spaces is called completely continuous if it is a limit of operators whose images are finite dimensional over L_0. Serre [Ser62] defines a Fredholm determinant $P_u(T) = \det(1 - Tu)$ for a completely continuous operator $u : \Omega \to \Omega$, where Ω is an L_0-Banach space. This is a generalization of the usual $\det(1-Tu)$ when u is of finite rank and enjoys similar properties. Let β be a rational number. An element f of Ω is said to be a generalized eigenform of slope β of u if there exists a polynomial $h(T)$ in $L_0[T]$ such that $h(u)(f) = 0$ and all the roots of $h(T)$ have valuation β. If $h(T) = (T - \lambda)^n$ we call f a generalized eigenform with eigenvalue λ. A $\lambda \neq 0$ is an eigenvalue of u if and only if $P_u(\lambda^{-1}) = 0$ and the dimension of the generalized eigenspace corresponding to λ is the multiplicity of λ^{-1} as a root of $P_u(T)$.

Coleman [Col97] has generalized Serre's construction for general Banach algebras. Let S be a Banach algebra and Ω be a Banach module over S. We say that Ω is orthonormizable if it has a Banach basis over S. In other words if there is a set $\{f_i : i \in I\}$ of elements of Ω, for some index set I, such that every element $f \in \Omega$ can be uniquely written as $\sum_{i \in I} a_i f_i$ with $a_i \in S$ such that $\lim |a_i| = 0$ and $|f| = \sup\{|a_i|\}_{i \in I}$.

Let $u : \Omega_1 \to \Omega_2$ be a continuous operator between Banach modules Ω_1 and Ω_2 over a Banach algebra S. A completely continuous operator is one which is a limit of operators whose images are finitely generated over S. Let $u : \Omega \to \Omega$ be completely continuous and assume Ω is orthonormizable. Let $a \in S$ be a multiplicative element (i.e. $|aa'| = |a|.|a'|$ for all $a' \in S$). Assume that $|u|$ is at most $|a|$. Let S^0 denote the set of all elements of S of norm at most 1. Then Coleman shows that there is a power series $P_u(T) \in S^0[\![aT]\!]$ which is called the Fredholm determinant of u. It is entire in T (i.e. if $P_u(T) = \sum_{m \geqslant 0} a_m T^m$, $|a_m| M^m \to 0$ for any real number M). This Fredholm determinant has similar properties as the one defined by Serre and coincides with it when $S = L_0$. Its formation commutes with contractive base changes. If $u : \Omega_1 \to \Omega_2$ is a completely continuous operator of L_0-Banach spaces and $u_1 : \Omega_1' \to \Omega_1$ and $u_2 : \Omega_2 \to \Omega_2'$ are continuous operators then $u_2 \circ u \circ u_1$ is also completely continuous. Also if $u : \Omega_1 \to \Omega_2$ is completely continuous and $v : \Omega_2 \to \Omega_1$ is continuous, then both $u \circ v$ and $v \circ u$ are completely continuous and $P_{u \circ v}(T) = P_{v \circ u}(T)$. We will use these results in the next section.

PROPOSITION 13.4. *Let R_0 be discretely valued. Then $S^D(R_0, r, K', k) \otimes L_0$ is an orthonormizable Banach module over L_0.*

Proof. We use Lemma A1.2. of [Col97]. First notice that $S^D(R_0, r, K', 0) \otimes R_0/\pi_0$ is reduced since it injects into $H^0(\mathbb{Y}_r \otimes R_0/\pi_0, \mathcal{O})$ which is itself reduced by Lemma 8.2. Thus,

$$|S^D(R_0, r, K', k) \otimes L_0|_{\sup} = |S^D(R_0, r, K', 0) \otimes L_0|_{\sup} = |L_0|_v$$

(see § A5 of [Col97] for example). By Proposition 13.1 we only need to show that $S^D(R_0, r, K', k) \otimes R_0/\pi_0$ is free over R_0/π_0 which is clear. $\square$

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Proposition 13.5. *Assume* $0 < v(r) < 1/(q+1)$. *Then* U *is a completely continuous operator of* $S^D(R_0, r^q, K', k) \otimes L_0$.

Proof. By definition U is the following composite:

$$S^D(R_0, r^q, K', k) \otimes L_0 \hookrightarrow S^D(R_0, r, K', k) \otimes L_0 \xrightarrow{(1/q)\,\mathrm{Tr}_{\mathrm{Frob}_{L_0}}} S^D(R_0, r^q, K', k) \otimes L_0.$$

We have seen that the second arrow is continuous. Therefore, it is enough to show that

$$S^D(R_0, r^q, K', k) \otimes L_0 \hookrightarrow S^D(R_0, r, K', k) \otimes L_0$$

is a completely continuous homomorphism of L_0-Banach spaces.

For the case $k = 0$, we use Proposition A5.2. of [Col97]. We showed that $S^D(R_0, r, K', 0) \otimes R_0/\pi_0$ is reduced in Proposition 13.4. Therefore, we only need to show that the image of the above inclusion is finite over R_0/π_0. It is enough to prove the same statement for the map

$$H^0(\mathbb{Y}_{r^q} \otimes R_0/\pi_0, \mathcal{O}) \to H^0(\mathbb{Y}_r \otimes R_0/\pi_0, \mathcal{O}).$$

We will prove this by showing that the map

$$\mathbb{Y}_r \otimes R_0/\pi_0 \to \mathbb{Y}_{r^q} \otimes R_0/\pi_0$$

factors through $\mathbb{M}'_{K'} \otimes R_0/\pi_0$.

To see this, we use our usual local Zariski picture (for example as in Lemma 8.2) to show that for $f : \mathbb{Y}_r \otimes R_0 \to \mathbb{Y}_{r^q} \otimes R_0$, locally Zariski f^* is the map $R[x]/(ax - r^q) \to R[x]/(ax - r)$ which sends x to $r^{q-1}x$ and hence the reduction of f^* mod π_0 sends x to 0 and factors through R.

For general k, we argue as follows. For simplicity let Ω_r denote $S^D(R_0, r, K', k)$ and let S_r denote $S^D(R_0, r, K', 0)$. Therefore, the inclusion $\Omega_{r^q} \otimes L_0 \hookrightarrow \Omega_r \otimes L_0$ is indeed the natural map

$$\Omega_{r^q} \otimes L_0 \to (\Omega_{r^q} \otimes L_0) \hat{\otimes}_{S_{r^q} \otimes L_0} S_r \otimes L_0.$$

Now, we only need notice that $\Omega_{r^q} \otimes L_0$ is a finitely generated module over $S_{r^q} \otimes L_0$ and therefore this map can be written as a limit of maps with finite dimensional image, by the case $k = 0$. $\square$

As a corollary of the above, we see that if $0 < v(r) < 1/(q+1)$, then there is a Fredholm determinant $P_{\mathrm{U}}(T) \in L_0[\![T]\!]$ for U, in any weight k. The next proposition shows that this power series is indeed independent of r.

Proposition 13.6. *Assume that* R_0 *is discretely valued. The Fredholm determinant of* U *is independent of* r, *such that* $0 < v(r) < 1/(q+1)$.

Proof. Let us denote $S^D(R_0, r, K', k)$ by Ω_r for simplicity. We will also denote the natural map $\Omega_{rr'} \otimes L_0 \to \Omega_r \otimes L_0$ by $R_r^{rr'}$. We will include r in the notation for U and write U_r in this proof. Let r' and r'' be elements of R_0 such that $0 < v(r'), v(r'') < 1/(q+1)$. Without loss of generality, we can assume $v(r'') \leqslant v(r') \leqslant qv(r'')$. Hence, we can write $r' = ar''$, and $(r'')^q = br'$ for $a, b \in R_0$. This implies that we have natural inclusions $R_{(r'')^q}^{(r')^q}$ and $R_{r'}^{(r'')^q}$. Denote $\mathrm{Tr}_{\mathrm{Frob}_{L_0}}$ by Tr for simplicity. Let $T : \Omega_{(r'')^q} \otimes L_0 \to \Omega_{(r')^q} \otimes L_0$ be given by $\mathrm{Tr} \circ R_{r'}^{(r'')^q}$. We have

$$T \circ R_{(r'')^q}^{(r')^q} = \mathrm{Tr} \circ R_{r'}^{(r'')^q} \circ R_{(r'')^q}^{(r')^q} = \mathrm{Tr} \circ R_{r'}^{(r')^q} = q\mathrm{U}_{(r')^q}.$$

On the other hand

$$R_{(r'')^q}^{(r')^q} \circ T = R_{(r'')^q}^{(r')^q} \circ \mathrm{Tr} \circ R_{r'}^{(r'')^q} = \mathrm{Tr} \circ R_{r''}^{r'} \circ R_{r'}^{(r'')^q} = \mathrm{Tr} \circ R_{r''}^{(r'')^q} = q\mathrm{U}_{(r'')^q}.$$

Furthermore, $R_{(r'')^q}^{(r')^q}$ is completely continuous from Proposition 13.5. Therefore, we have

$$P_{\mathrm{U}_{(r')^q}} = P_{T/q \circ R_{(r'')^q}^{(r')^q}} = P_{R_{(r'')^q}^{(r')^q} \circ T/q} = P_{\mathrm{U}_{(r'')^q}}. \qquad \square$$

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DEFINITION 13.7. Let β be a rational number. Define $d_r(K', k, \beta)$ to be the dimension of generalized eigenforms of slope β of U acting on $S^D(R_0, r, K', k) \otimes L_0$ over any L_0 in which r is a qth power. By Proposition 13.5, $d_r(K', k, \beta)$ is finite. On the other hand, because of Proposition 13.6 we know that $d_r(K', k, \beta)$ does not depend on r, as long as $0 < v(r) < q/(q+1)$. We define $d(K', k, \beta)$ to be this common value.

14. Eigenforms of U

Coleman uses rigid geometry to study the overconvergent eigenforms of U over modular curves. We will use this approach to obtain results about the eigenforms of U in this setting. As was explained earlier, the fact that U is a completely continuous operator of the space of overconvergent modular forms allows us to use the Fredholm theory of the U operator.

We let $\mathbb{C}_p$ denote the completion of an algebraic closure of $F_{\mathcal{P}}$ with normalized valuation such that $v(\pi) = 1$, and the corresponding norm $|\ |_v = (1/q)^v$. We assume that L_0 is a finite extension of $F_{\mathcal{P}}$ (embedded in $\mathbb{C}_p$), with ring of integers R_0.

We will need the following result for a construction. However this result is our first example of a π-adic congruence between $\mathcal{P}$-adic modular forms (of the same weight). In a future article we will define and study a notion of congruence between $\mathcal{P}$-adic modular forms of possibly different weights. One has to remember that these modular forms lack q-expansions. In the case of modular curves, the following result (for the classical E_{p-1}) is proven essentially as a result of the q-expansion principle.

First we remark that $\mathrm{Frob}(E_{q-1}) \in H^0(M'_{K'}(|r|_v), \underline{\omega}^{\otimes q-1})$ is a nowhere vanishing section for any $r \in R_0$ with $v(r) < 1/(q+1)$. The reason is that by § 11.2 we have

$$|(\mathrm{Frob}^{\mathrm{rig}})^*(E_{q-1})(x)|_v = |E_{q-1}(\mathrm{Frob}^{\mathrm{rig}}(x))|_v \geqslant |r|_v^q$$

at any closed point x of $M'_{K'}(|r|_v)$, and $\pi^{q-1}\mathrm{Frob}(E_{q-1})$ is obtained as a pullback of the nowhere vanishing differential form $(\mathrm{Frob}^{\mathrm{rig}})^*(E_{q-1})$ on $A'_{K'}(|r|_v)/\mathbf{C}_r$, under the étale map

$$A'_{K'}(|r|_v) \xrightarrow{\phi} A'_{K'}(|r|_v)/\mathbf{C}_r.$$

Define an element $\mathbf{e} \in H^0(M'_{K'}(|r|_v), \mathcal{O}_{M'_{K'}(|r|_v)}) = S^D(R_0, r, K', 0) \otimes L_0$ by

$$\mathbf{e} := E_{q-1}/\mathrm{Frob}(E_{q-1}).$$

THEOREM 14.1. *Let $|\ |_{M'_{K'}(1)}$ denote the supremum norm on $M'_{K'}(1)$. We have*

$$|\mathbf{e} - 1|_{M'_{K'}(1)} \leqslant |\pi|_v.$$

Proof. We may assume $L_0 = F_{\mathcal{P}}$ throughout this proof. For notation see § 9. First we show that $\mathrm{Frob}(E_{q-1}) \in H^0(\widetilde{\mathbb{Y}}_1, \underline{\omega}^{\otimes q-1}) \subset H^0(M'_{K'}(1), \underline{\omega}^{\otimes q-1})$. By definition of $\mathrm{Frob}(E_{q-1})$ we know that

$$\pi^{q-1}\mathrm{Frob}(E_{q-1}) = (\tilde{\phi}^*)(\widetilde{\mathrm{Frob}}^*(E_{q-1})) \in H^0(\widetilde{\mathbb{Y}}_1, \underline{\omega}^{\otimes q-1}).$$

So we have to show that $(\tilde{\phi}^*)(\widetilde{\mathrm{Frob}}^*(E_{q-1}))$ is divisible by π^{q-1} in $H^0(\widetilde{\mathbb{Y}}_1, \underline{\omega}^{\otimes q-1})$. Since locally we can write $\widetilde{\mathrm{Frob}}^*(E_{q-1})$ as a tensor product of $q-1$ differential forms on $\widetilde{\mathbb{B}}_1/\widetilde{\mathbf{C}}_1$, it suffices to show that locally Zariski over $\widetilde{\mathbb{Y}}_1$, for any $\gamma \in H^0(\widetilde{\mathbb{B}}_1/\widetilde{\mathbf{C}}_1, \Omega^1_{(\widetilde{\mathbb{B}}_1/\widetilde{\mathbf{C}}_1)/\widetilde{\mathbb{Y}}_1})$, the pullback $\tilde{\phi}^*(\gamma)$ is divisible by π in $H^0(\widetilde{\mathbb{B}}_1, \Omega^1_{\widetilde{\mathbb{B}}_1/\widetilde{\mathbb{Y}}_1})$ noting that $\widetilde{\mathbb{Y}}_1$ is flat over $\mathcal{O}_{\mathcal{P}}$.

Since $\widetilde{\mathbf{C}}_1$ (the canonical subgroup of $\widetilde{\mathbb{B}}_1$) modulo π is the kernel of the (qth) Frobenius morphism of $\mathbb{B}_1 \otimes \kappa$, by reduction modulo π we get the following diagram from discussions of § 9:

$$\begin{tikzcd}
\mathbb{B}_1 \otimes \kappa \arrow[r, "\mathrm{Fr}_q"] \arrow[dr] & (\mathbb{B}_1 \otimes \kappa)/\mathbf{C}_{1,1} = (\mathbb{B}_1 \otimes \kappa)^{(q)} \arrow[r, "\eta_1"] \arrow[d] & \mathbb{B}_1 \arrow[d] \\
& \mathbb{Y}_1 \otimes \kappa \arrow[r, "\mathrm{Frob}_1"] & \mathbb{Y}_1 \otimes \kappa
\end{tikzcd}$$

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in which Frob_1 is the qth power morphism of $\mathbb{Y}_1$ over κ, $\eta_1 : (\mathbb{B}_1 \otimes \kappa)^{(q)} \to \mathbb{B}_1 \otimes \kappa$ is the base extension morphism, and Fr_q is the relative Frobenius morphism of $\mathbb{B}_1 \otimes \kappa$ (which is the reduction of $\tilde{\phi}$ modulo π). Denote the reduction modulo π of γ by $\bar{\gamma}$. We know that

$$\mathrm{Fr}_q^*(\bar{\gamma}) = 0,$$

which shows that $\tilde{\phi}^*(\gamma)$ is divisible by π in $H^0(\widetilde{\mathbb{B}}_1, \Omega^1_{\widetilde{\mathbb{B}}_1/\widetilde{\mathbf{C}}_1})$ as desired.

By Proposition 4.5, $\phi'^*\phi^* = \pi^{q-1}$ on sections of $\underline{\omega}^{\otimes q-1}$and therefore we know that

$$\phi'^*(\mathrm{Frob}(E_{q-1})) = (\mathrm{Frob}^{\mathrm{rig}})^*(E_{q-1}) = \widetilde{\mathrm{Frob}}^*(E_{q-1}).$$

On the other hand, ϕ'^* is $H^0(M'_{K'}(1), \mathcal{O}_{M'_{K'}(1)})$-invariant and hence

$$\phi'^*(E_{q-1}) = \phi'^*(\mathbf{e}\,\mathrm{Frob}(E_{q-1})) = \mathbf{e}\phi'^*(\mathrm{Frob}(E_{q-1})).$$

Therefore, we have

$$\phi'^*(E_{q-1}) = \mathbf{e}(\widetilde{\mathrm{Frob}})^*(E_{q-1}).$$

We will prove the theorem essentially by reducing this equality modulo π. Since the desired result is local on $M'_{K'}(1)$ we will henceforth work locally on $\widetilde{\mathbb{Y}}_1$ so that we can assume $\underline{\omega}$ to be a trivial line bundle. Fix generators ω (respectively ω') of $\underline{\omega}_{\widetilde{\mathbb{B}}_1/\widetilde{\mathbb{Y}}_1}$ (respectively $\underline{\omega}_{(\widetilde{\mathrm{Frob}})^*\widetilde{\mathbb{B}}_1/\widetilde{\mathbb{Y}}_1}$) such that ω' reduces to $\omega^{(q)}$ modulo π. Notice that this is possible as the reduction of $(\widetilde{\mathrm{Frob}})^*\widetilde{\mathbb{B}}_1/\widetilde{\mathbb{Y}}_1$ modulo π is $(\mathbb{B}_1 \otimes \kappa)^{(q)}$.

Let us denote $(\mathbb{B}_1, i, \theta, \bar{\alpha}^{\mathcal{P}}, \omega)$ by $\underline{\mathbb{B}_1}$ for simplicity. Write

$$E_{q-1} = E_{q-1}(\underline{\mathbb{B}_1})\omega^{\otimes q-1},$$
$$(\widetilde{\mathrm{Frob}})^*(E_{q-1}) = \lambda.(\omega')^{\otimes q-1}.$$

We use $\bar{}$ to denote reduction modulo π. Since $\widetilde{\mathrm{Frob}}$ reduces to the qth power morphism, we have

$$\lambda \equiv E_{q-1}(\underline{\mathbb{B}_1})^q \equiv \mathbf{H}(\underline{\mathbb{B}_1} \otimes \kappa)^q \mod \pi.$$

On the other hand, by Proposition 4.5 ϕ'^* reduces to V^* modulo π. Hence, $\phi'^*(E_{q-1}) = \phi'^*(E_{q-1}(\underline{\mathbb{B}_1})\omega^{\otimes q-1})$ reduces to $\mathrm{V}^*(\mathbf{H}(\underline{\mathbb{B}_1} \otimes \kappa)\bar{\omega}^{\otimes q-1})$, which is equal to

$$\mathbf{H}(\underline{\mathbb{B}_1} \otimes \kappa)(\mathbf{H}(\underline{\mathbb{B}_1} \otimes \kappa)\bar{\omega}^{(q)})^{\otimes q-1}$$

by the definition of the Hasse invariant (see § 6). Combining the above congruences we get

$$\mathbf{H}(\underline{\mathbb{B}_1} \otimes \kappa)^q = \bar{\mathbf{e}}\mathbf{H}(\underline{\mathbb{B}_1} \otimes \kappa)^q$$

in $H^0(\widetilde{\mathbb{Y}}_1, \mathcal{O}_{\widetilde{\mathbb{Y}}_1}) \otimes \kappa$ and hence

$$\mathbf{e} \equiv 1 \mod \pi.$$

This proves that $|e-1|_{M'_{K'}(1)} \leqslant |\pi|_v$. ∎

Let $r \in L_0$ be a qth power such that $0 < v(r) < 1/(q+1)$. Consider the isomorphism of L_0-Banach spaces defined by multiplication by E_{q-1},

$$S^D(R_0, r, K', k) \otimes L_0 \xrightarrow{\sim} S^D(R_0, r, K', k+q-1) \otimes L_0$$
$$h \mapsto hE_{q-1}.$$

This is an isomorphism because E_{q-1} is nowhere vanishing on $M'_{K'}(|r|_v)$. The pullback of the operator $\mathrm{U}_{(k+q-1)}$ via this isomorphism is an operator on $S^D(R_0, r, K', k)$ given by

$$h \mapsto E_{q-1}^{-1}\mathrm{U}_{(k+q-1)}(hE_{q-1}).$$

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From Lemma 12.4 we have

$$E_{q-1}^{-1}\mathrm{U}_{(k+q-1)}(hE_{q-1}) = E_{q-1}^{-1}\mathrm{U}_{(k+q-1)}(\mathrm{Frob}(E_{q-1})\mathbf{e}h) = \mathrm{U}_{(k)}(\mathbf{e}h).$$

In other words the pullback of $\mathrm{U}_{(k+q-1)}$ via the above isomorphism can be written as

$$\mathrm{U}_{(k)} \circ m_{\mathbf{e}},$$

where $m_{\mathbf{e}}$ denotes multiplication by $\mathbf{e}$. This proves the following lemma.

LEMMA 14.2. *Let the notation be as above. For each $n \geqslant 0$, we have*

$$P_{\mathrm{U}_{(k+n(q-1))}} = P_{\mathrm{U}_{(k)}\circ m_{\mathbf{e}^n}}.$$

So studying the operator theory of U in weight $k+n(q-1)$ is equivalent to studying the operator theory of $\mathrm{U}\circ m_{\mathbf{e}^n}$ in weight k. Coleman's idea is to put all the $\mathrm{U}\circ m_{\mathbf{e}^n}$ for varying n together in a rigid analytic family of operators, producing a completely continuous operator of a Banach module. The study of the Fredholm theory of the latter will give us information about that of each of the original operators. We will use this idea to study $d(K', k, \beta)$. By Definition 13.7 this equals $d_r(K', k, \beta)$ for any r such that $0 < v(r) < 1/(q+1)$.

It is easy to check that there are $\epsilon > 0$ and $\delta \leqslant 1$ such that $(1+x)^t = \sum_{n\geqslant 0}\binom{t}{n}x^n$ is convergent for all $x, t \in \mathbb{C}_p$ with $|x|_v \leqslant |\pi|_v + \epsilon$, and $|t|_v \leqslant \delta$.

Since $|\mathbf{e}-1|_{M'_{K'}(1)} = \lim_{s\to 1^-}|\mathbf{e}-1|_{M'_{K'}\hat{\otimes}\mathbb{C}_p(s)}$, there exists an $s_0 \in |\mathbb{C}_p|_v$ such that $|\pi|_v^{1/(q+1)} < s_0 < 1$ and $|\mathbf{e}-1|_{M'_{K'}\hat{\otimes}\mathbb{C}_p(s_0)} \leqslant |\pi|_v + \epsilon$. We can enlarge L_0 to a finite extension of $F_{\mathcal{P}}$ such that there is an element $r \in R_0$ which is a qth power with $|r|_v = s_0$ (which also implies that $0 < v(r) < 1/(q+1)$). Therefore, for this choice of r, we know that $\mathbf{e}^t$ is convergent on $M'_{K'}(|r|_v)$ for any t with $|t|_v \leqslant \delta$.

Let us fix an integer k_0 throughout the discussion. For any $t \in \mathbb{C}_p$ with $|t|_v \leqslant \delta$ define

$$\begin{aligned} u_t : H^0(M'_{K'}(|r|_v), \underline{\omega}^{\otimes k_0}) &\to H^0(M'_{K'}(|r|_v), \underline{\omega}^{\otimes k_0}) \\ h &\mapsto \mathrm{U}_{(k_0)}(h\cdot \mathbf{e}^t). \end{aligned}$$

Let $B = B_{L_0}[0,\delta]$ denote the affinoid subset of the rigid space $\mathbb{A}^1_{L_0}$ given by $|x|_v \leqslant \delta$. Define the affinoid rigid space $Z := B \times M'_{K'}(|r|_v)$. Let us denote the pullback of $\underline{\omega} = \underline{\omega}_{A'_{K'}(|r|_v)/M'_{K'}(|r|_v)}$ to Z under the second projection again by $\underline{\omega}$. Then we have

$$H^0(Z, \underline{\omega}^{\otimes n}) = H^0(B, \mathcal{O}_B)\hat{\otimes}_{L_0} H^0(M'_{K'}(|r|_v), \underline{\omega}^{\otimes n}).$$

Now consider the operator

$$\mathrm{id}\hat{\otimes}\mathrm{U}_{(k_0)} : H^0(Z, \underline{\omega}^{\otimes k_0}) \to H^0(Z, \underline{\omega}^{\otimes k_0}).$$

This continuous $H^0(B, \mathcal{O}_B)$-linear operator is obtained by base extension of $\mathrm{U}_{(k_0)}$ under the map $L_0 \to H^0(B, \mathcal{O}_B)$. Since we can think of $\mathbf{e}^t$ as a rigid function on Z, there is a continuous operator

$$m_{\mathbf{e}^t} : H^0(Z, \underline{\omega}^{\otimes k_0}) \to H^0(Z, \underline{\omega}^{\otimes k_0})$$

which is given by multiplication by $\mathbf{e}^t$. Define

$$\mathbb{U}_{(k_0)} := \mathrm{id}\hat{\otimes}\mathrm{U}_{(k_0)} \circ m_{\mathbf{e}^t}.$$

This is a continuous operator of the $H^0(B, \mathcal{O}_B)$-Banach space $H^0(Z, \underline{\omega}^{\otimes k_0})$. It is clear that for any $t_0 \in \mathbb{C}_p$ with $|t_0|_v \leqslant \delta$ the restriction of $\mathbb{U}_{(k_0)}$ to the fibre of Z over t_0 is the already defined operator u_{t_0}.

We show that $\mathbb{U}_{(k_0)}$ is completely continuous. We have seen in Proposition 13.5 that $\mathrm{U}_{(k_0)}$ is completely continuous. This shows that $\mathrm{id}\hat{\otimes}\mathrm{U}_{(k_0)}$ is completely continuous as it is the base extension of $\mathrm{U}_{(k_0)}$ under a contractive map of Banach algebras. Now since $m_{\mathbf{e}^s}$ is continuous, we deduce that

$\mathbb{U}_{(k_0)}$ is completely continuous. We also note that $H^0(Z, \underline{\omega}^{\otimes k_0})$ is orthonormizable over $H^0(B, \mathcal{O}_B)$. This is because by Proposition 13.4 we know that $H^0(M'_{K'}(|r|_v), \underline{\omega}^{\otimes k_0})$ is orthonormizable and hence so is its base extension $H^0(Z, \underline{\omega}^{\otimes k_0})$ (see Proposition A.1.3 of [Col97]). Let $P_{\mathbb{U}_{(k_0)}}(s,T) \in L_0[\![s,T]\!]$ denote the Fredholm determinant of $\mathbb{U}_{(k_0)}$. Let n be an integer such that $|n|_v \leqslant \delta$. Letting $s = n$ corresponds to a base extension $H^0(B, \mathcal{O}_B) \to L_0$. Under this base extension $\mathbb{U}_{(k_0)}$ becomes

$$u_n : H^0(M'_{K'}(|r|_v), \underline{\omega}^{\otimes k_0}) \to H^0(M'_{K'}(|r|_v), \underline{\omega}^{\otimes k_0}).$$

Since the formation of Fredholm determinant commutes with contractive base change (see Lemma A.2.5 of [Col97]) and by Lemma 14.2, we deduce that

$$P_{\mathbb{U}_{(k_0)}}(n,T) = P_{u_n}(T) = P_{\mathrm{U}_{(k_0+n(q-1))}}(T).$$

This shows that the number of zeros of $P_{\mathbb{U}_{(k_0)}}(n,T)$ over $\mathbb{C}_p$ (counting multiplicities) which have valuation $-\beta$ is the same as $d(K', k_0 + n(q-1), \beta)$.

We study the zero locus of the entire series $P = P_{\mathbb{U}_{(k_0)}}(s,T)$. Enlarge L_0 so that $\beta \in |L_0|_v$. Let A_β denote the affinoid subdomain of $B \times \mathbb{A}^1_{L_0}$ determined by $|T|_v = |\pi|_v^{-\beta}$. The subspace of this affinoid determined by $P(T) = 0$ is an affinoid over B which we call Z_β. The projection map $f : Z_\beta \to B$ is quasi-finite as U is completely continuous. For any closed point x of B, $f^{-1}(x)$ is a scheme of dimension 0 over the residue field of x. Denote the dimension of its ring of functions over the residue field of x by $\deg(f^{-1}(x))$. By Proposition A.5.5 of [Col97] for each integer $i \geqslant 0$ the set of closed points x of B such that $\deg(f^{-1}(x)) \geqslant i$ is the set of closed points of an affinoid subdomain B_i of B. Furthermore, B_i is empty for large i. A standard argument shows that for any $x \in B \cap \mathbb{Z}_p$ there is a δ_x such that if $x' \in B \cap \mathbb{Z}_p$ and $|x - x'|_v \leqslant \delta_x$, then $\deg(f^{-1}(x')) = \deg(f^{-1}(x))$. Since $\mathbb{Z}_p$ is compact we can find a uniform δ' which works for any pair of elements of $\mathbb{Z}_p$ which lie in B. Now for any integer n such that $|n|_v \leqslant \delta$ the degree of $f^{-1}(n)$ is the number of zeros of $P_{\mathbb{U}_{(k_0)}}(n,T)$ over $\mathbb{C}_p$ which is itself equal to $d(K', k_0 + n(q-1), \beta)$. Choose N such that $|p|_v^N \leqslant \min\{\delta, \delta'\}$. Varying k_0 modulo $p^N(q-1)$ implies the following.

Theorem 14.3. *Assume K' is small enough, and $q > 3$. There exists an $N > 0$, depending only on β, K', and D, such that if*

$$k \equiv k' \mod p^N(q-1)$$

then

$$d(K', k, \beta) = d(K', k', \beta).$$

Moreover, $d(K', k, \beta)$ is uniformly bounded for all $k \in \mathbb{Z}$.

As we explained in § 1, the key step to passage from this kind of result to results about (certain) Hilbert modular forms is proving a criterion for classicality of overconvergent $\mathcal{P}$-adic modular forms. In presence of such a criterion in terms of slopes, one can prove a similar statement for dimension of spaces of classical modular forms on $M'_{K'}$. That kind of statement could be translated for quaternionic modular forms over F using Theorem 4.2. The Jacquet–Langlands correspondence then will establish the connection with Hilbert modular forms.

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