

ANALYSIS OF CELL POPULATION PDE MODELS WITH GENERAL MATURATION RATES

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(Received 2 June, 1998; revised 16 September, 1998)

Abstract

This paper considers a cell population model with a general maturation rate. This model is described by a nonlinear PDE. We use the theory of operator semigroups to study the problem under simple hypotheses on the growth function and the nonlinear term. By showing that a related operator generates a strongly continuous semigroup, we prove the existence of a classical solution of the nonlinear problem and its positivity. It is also proved that under simple hypotheses, the problem generates a semiflow. The invariance of the semiflow is studied as well.

1. Introduction

The study of cell population growth models has greatly contributed to the development of mathematical biology. See Metz and Diekmann [14] and Webb [18] for excellent accounts of this subject. Recently, a maturity structured model of a blood cell production system has been studied by Rey and Mackey [16]. The governing equation is

$$\begin{cases} \frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} (x u(x, t)) \\ \quad = \mu u(\alpha x, t - \tau)(1 - u(\alpha x, t - \tau)), & t > 0; \\ u(x, t) = \phi(x, t), & -\tau \leq t \leq 0, x \in [0, 1]. \end{cases} \quad (1.1)$$

Here $u(x, t)$ is the population density of cells with respect to maturity x at time t and μ, α, τ are parameters satisfying $\mu \geq 0, 0 < \alpha < 1, \tau > 0$. If we assume that

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the maturity transport term in the model is of the form $(\partial/\partial x)[g(x)u(x, t)]$ under the assumption that all cells have general maturation rate $g(x)$, this gives rise to the following model:

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + g(x) \frac{\partial u(x, t)}{\partial x} = F(u_t), \\ u(x, t) = \phi(x, t), \quad -\tau \leq t \leq 0, \quad x \in [0, 1], \end{cases} \tag{1.2}$$

where u_t denotes the derivative of u with respect to t , $g(x)$ is nonnegative and satisfies $g(0) = 0$ and $\int_0^1 dx/g(x) = \infty$.

The maturity structured model of a cell population system given by

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + \frac{\partial(a(x)u(x, t))}{\partial x} = \beta u(x, t), \\ u(x, 0) = \psi(x), \quad 0 \leq x \leq 1, \end{cases} \tag{1.3}$$

where $\beta > 1$ is a constant, was studied by Webb [19] for the special case $a(x) = x$. For a general maturation velocity $a(x)$, system (1.3) may be rewritten as

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + a(x) \frac{\partial u(x, t)}{\partial x} = f(x, u(x, t)), \\ u(x, 0) = \psi(x), \quad 0 \leq x \leq 1, \end{cases} \tag{1.4}$$

where $a(x)$ is nonnegative, $a(0) = 0$ and $\int_0^1 dx/a(x) = \infty$.

The model studied by Gyllenberg and Heijmans [9] is described by

$$\begin{aligned} \frac{\partial n(x, t)}{\partial t} + \frac{\partial(g(x)n(x, t))}{\partial x} &= -\mu(x)n(x, t) - b(x)n(x, t) \\ &+ \frac{2p(y^{-1}(x))b(y^{-1}(x))}{y'(y^{-1}(x))n(t - \tau, y^{-1}(x))}, \end{aligned} \tag{1.5}$$

where $n(x, t)$ is the size distribution of cells in the first phase at time t and size x . The functions g , μ and b are the rates at which cells of size x grow, die and transit to the second phase respectively. Here $\tau > 0$ is the constant duration of the second phase, $y(x)$ is the size of a new born cell whose mother entered the second phase with size x , and $p(x)$ is the fraction of cells who survive the second phase given that they entered it with size x . Under an appropriate substitution (see [9] and [20]) (1.5) can be transformed into

$$\frac{\partial u(x, t)}{\partial t} + g(x) \frac{\partial u(x, t)}{\partial x} = F(u_t), \tag{1.6}$$

where $g(x)$ is nonnegative and the same as in (1.5) but should also satisfy $\int_0^1 \frac{dx}{g(x)} = \infty$.

The model studied by Greiner and Nagel [8], and Metz and Diekmann [14], is described by

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + \frac{\partial g(x)u(x, t)}{\partial x} = -\mu(x)u(x, t) - b(x)u(x, t) + 4b(2x)u(2x, t), \\ u(\alpha/2, t) = 0, \quad u(x, 0) = u_0(x). \end{cases} \quad (1.7)$$

The growth function $g(x)$ was taken to be $0 < \epsilon \leq g(x) \leq \delta$ for all $x \geq 0$. It is more realistic in this case that $g(x)$ be taken to be $g(x) \geq 0$ for $x \geq 0$.

From the foregoing review, we see that it is of interest to study Problem (1.4) in the case where $f : [0, 1] \times R \mapsto R$ is continuous and $a(x)$ satisfies the following conditions:

- (i) $a \in C[0, 1]$ and $a(x) \geq 0$ for $0 \leq x \leq 1$;
- (ii) $a(0) = 0$, $\text{measure}\{x \mid a(x) = 0\} = 0$;
- (iii) $\int_x^1 ds/a(s) < \infty$, if $0 < x \leq 1$;
- (iv) $\int_0^1 ds/a(s) = \infty$.

A similar problem was studied in [4, 5, 10, 11], under stronger conditions on $a(x)$ and $f(x, v)$ (that is, $a(x) > 0$ ($x > 0$), a and f are continuously differentiable), using the classical method of characteristics.

It is well known that issues concerning the smoothness of various coefficients and terms appearing in differential equations greatly affect the existence, uniqueness and regularity of solutions and this is especially so in the theory of partial differential equations (see [17]). We will use the theory of operator semigroups to study problem (1.4) under simple hypotheses on $a(x)$ (conditions (i)–(iv)) and $f(x, u)$ given above.

The only assumption made about the nonlinear term is that $f(x, v)$ and $f_v(x, v)$ are continuous. In fact, when a and f are continuously differentiable and $a(x) > 0$ for $x > 0$, as mentioned earlier, the existence of the problem can be studied by the classical theory of characteristics (see [10]). In our case, however, the classical theory of characteristics cannot be applied. Thus the approach taken in this paper is totally different from that in [10] and we not only obtain strong results on the existence and positivity of a classical solution, but also establish a basis for analysing time-delay cell populations with general maturation velocity, such as models (1.2) and (1.6). In this paper, we will show that the related operator generates a strongly continuous semigroup. By studying the properties of the semigroup, we prove the existence of a classical solution of the nonlinear problem and the positivity of the solution. It is also proved that the problem generates a semiflow. We will prove that the phase space, that is, all nonnegatively continuous functions on $[0, 1]$, of the semiflow splits into two disjoint invariant sets.

The paper is organized as follows: in Section 2, we prove that the operator $a(x)(d/dx)$ generates a strongly continuous semigroup of operators on $C[0, 1]$ and

show that the semigroup is positive and contractive. Although an analytic semigroup has many nice properties, it will be shown that our semigroup cannot be extended to being an analytic semigroup. Next, in Section 3, we consider the existence of a classical solution of problem (1.4) and the positivity of the solution. We will show that the mild solution of the problem generates a semiflow. We finally present one of the properties of the semiflow—invariance (Section 4).

2. Related semigroup of operators

Let $Y = C[0, 1]$ be the Banach space with norm $\|\phi\| = \max_{0 \leq x \leq 1} |\phi(x)|$ for $\phi \in Y$. Let $a(x)$ be a given continuous function on $[0, 1]$. Similar to [12], we define an operator H on Y . Define the domain of H to be:

$$D(H) = \{ \phi \in Y \mid \begin{aligned} &\phi'(x) \text{ exists and is continuous at } x \text{ when } a(x) \neq 0; \\ &\lim_{x \rightarrow x_0} a(x)\phi'(x) \text{ exists when } a(x_0) = 0 \text{ and } x_0 \neq 0; \\ &\lim_{x \rightarrow 0} \tilde{\phi}(x) \text{ exists} \end{aligned} \} \tag{2.1}$$

and for $\phi \in D(H)$

$$(H\phi)(x) = \begin{cases} a(x) \frac{d\phi(x)}{dx}, & \text{if } a(x) \neq 0, \\ \lim_{x \rightarrow x_0} a(x) \frac{d\phi(x)}{dx}, & \text{if } a(x_0) = 0 \text{ and } x_0 \neq 0, \\ \lim_{x \rightarrow 0} \tilde{\phi}(x), & \text{if } x = 0, \end{cases} \tag{2.2}$$

where

$$\tilde{\phi}(x) = \begin{cases} a(x) \frac{d\phi(x)}{dx}, & \text{if } a(x) \neq 0, \\ \lim_{y \rightarrow x} a(y) \frac{d\phi(y)}{dy}, & \text{if } a(x) = 0 \text{ and } x \neq 0. \end{cases}$$

REMARK 2.1. The operator H has also been studied in many other places (see [1, 2, 6, 21]). It is also related to the problem of characterizing all flows on $[0, 1]$ (see, for example, [1]).

Using an idea developed in [12] and [21], we can prove the following theorem.

THEOREM 2.1. *If $a(x)$ satisfies conditions (i)–(iii) in Section 1, then the operator $-H$ defined by (2.1) and (2.2) generates a strongly continuous contraction semigroup if and only if $\int_0^1 dx/a(x) = \infty$.*

PROOF. Let $a(x)$ satisfy conditions (i)–(iv) in Section 1. We will show that $-H$ is an infinitesimal generator of a strongly continuous contraction semigroup. First,

we note that $D(-H)$ is a dense subset of Y since $C^1[0, 1] \subset D(-H)$. We note also that $-H$ is a closed operator. In fact, if $\phi_n \rightarrow \phi$ and $-H\phi_n \rightarrow \psi$ in Y as $n \rightarrow \infty$, then $\lim_{n \rightarrow \infty} \phi'_n(x) = -\psi(x)/a(x)$ a.e. on $[0, 1]$ and $|\phi'_n(x)| \leq (|\psi(x)| + 1)/a(x)$ a.e. on $[0, 1]$ for sufficiently large n . Therefore $\phi(x) = \phi(1) - \int_x^1 \psi(s)/a(s) ds$ for $0 < x \leq 1$, and we see that $\phi \in D(-H)$ and $-H\phi = \psi$.

We need to show that the image $R(\lambda I + H)$ of operator $\lambda I + H$ is dense in Y for $\lambda > 0$. For a given $\psi \in Y$, let

$$\int_0^x \frac{\psi(t)}{a(t)} e^{-\int_t^x \lambda/a(s) ds} dt = \lim_{\epsilon \rightarrow 0^+} \int_\epsilon^x \frac{\psi(t)}{a(t)} e^{-\int_t^x \lambda/a(s) ds} dt$$

for $0 < x \leq 1$. It is clear that the limit exists. Define

$$\begin{cases} \phi(x) = e^{\int_x^1 \lambda/a(s) ds} \int_0^x \frac{\psi(t)}{a(t)} e^{-\int_t^1 \lambda/a(s) ds} dt, & 0 < x \leq 1; \\ \phi(0) = \psi(0)/\lambda, & x = 0. \end{cases} \tag{2.3}$$

Since there exist $\xi_1, \xi_2 \in [0, x]$ such that

$$\psi(\xi_1)/\lambda \leq \phi(x) \leq \psi(\xi_2)/\lambda, \quad \text{for } t \in [0, x],$$

we see that $\lim_{x \rightarrow 0^+} \phi(x) = \psi(0)/\lambda$. Therefore $\phi \in Y$ and $(\lambda I + H)\phi = \psi$. It follows that $R(\lambda I + H) = Y$ for $\lambda > 0$. From (2.3) we also have

$$\|\phi\| \leq \|\psi\|/\lambda, \quad \text{for } \lambda > 0. \tag{2.4}$$

For a given $\lambda > 0$ if $(\lambda I + H)\phi = 0$ then $(e^{\lambda \int_0^x ds/a(s)} \phi(x))' = 0$ a.e. on $[0, 1]$. In fact, according to condition (iv), there is an at most countable subset Γ of $[0, 1]$ such that $(e^{\lambda \int_0^x ds/a(s)} \phi(x))' = 0$ for all $x \in [0, 1] - \Gamma$ (see [6]) and $0 \in \Gamma$ is the only accumulation point of Γ .

We must have $\phi \equiv 0$, since $\lim_{x \rightarrow 0^+} e^{\lambda \int_0^x ds/a(s)} = 0$ and ϕ is a continuous function. So, $(\lambda I + H)$ is injective for $\lambda > 0$ and we have that $(\lambda I + H)^{-1} \in L(Y, Y)$ with $\|(\lambda I + H)^{-1}\| \leq 1/\lambda$ for $\lambda > 0$.

By the Hille-Yosida theorem (see [7, 15]), $-H$ generates a strongly continuous contraction semigroup on Y . Conversely if $-H$ generates a C_0 contraction semigroup on Y , then there is a $\lambda > 0$ with $\lambda \in \rho(-H)$ such that for $\psi \equiv 1$, there exists a unique $\phi \in D(-H)$ satisfying

$$(\lambda I + H)\phi = \psi \equiv 1. \tag{2.5}$$

If $\int_0^1 dx/a(x) < \infty$, there are two different functions ϕ_1 and ϕ_2 that satisfy (2.5) and are defined as follows:

$$\phi_1(x) = \frac{1}{\lambda} \left(1 - e^{-\int_0^x \lambda/a(s) ds} \right), \quad 0 \leq x \leq 1,$$

and

$$\phi_2(x) = e^{\int_x^1 \lambda/a(s) ds} \int_0^x \frac{1}{a(t)} e^{-\int_t^1 \lambda/a(s) ds} dt, \quad 0 \leq x \leq 1.$$

This contradiction shows that we must have $\int_0^1 dx/a(x) = \infty$ and the theorem is proved.

We will show that the strongly continuous contraction semigroup $\{T(t)\}_{t \geq 0}$ generated by $-H$ is positive. In fact, the positivity of the resolvent operator of $-H$ implies the positivity of the semigroup. In order to explore further properties of the operator $-H$, we give the following proof of positivity.

Let $Y_+ = \{\phi \in Y \mid \phi(x) \geq 0 \text{ for } x \in [0, 1]\}$ be a positive cone. Y can be ordered by setting $f \geq g$ whenever the function $f - g \in Y_+$. In such a setting, we see that $(Y, Y_+, \|\cdot\|)$ is a Banach lattice. Recall that $\{T(t)\}_{t \geq 0}$ is positive if $T(t)Y_+ \subset Y_+$ for $t > 0$. For the reader's convenience, we cite the following result.

LEMMA 2.1 (see [3]). *Let $(Y, Y_+, \|\cdot\|)$ be an ordered Banach space for which the norm is monotone and the operator norm on $L(Y, Y)$ is positively attained. If $-H$ is a closed densely defined, N -dissipative operator, and $R(I + \alpha H) = Y$ for some $\alpha > 0$, then $-H$ generates a positive C_0 semigroup of contraction.*

We now prove the following result.

THEOREM 2.2. *Let $a(x)$ satisfy (i)–(iv) in Section 1, then the operator $-H$ defined by (2.1) and (2.2) generates a positive C_0 contraction semigroup.*

PROOF. It follows from Theorem 2.1 that $-H$ generates a C_0 contraction semigroup and $R(I + \alpha H) = Y$ for $\alpha > 0$.

It is easy to see that Y_+ is generating, that is, $Y = Y_+ - Y_+$ and the norm is monotone, that is, $\|\phi\| \leq \|\psi\|$ if $0 \leq \phi \leq \psi$.

We will show that $L(Y, Y)$ is positively attained. The norm on Y is a Riesz norm. In fact, the $\|\cdot\|_Y$ is absolutely monotone ($-\psi \leq \phi \leq \psi$ always implies $\|\phi\| \leq \|\psi\|$) and Y_+ is approximately absolutely dominating (for each $\phi \in Y$ there is a $\psi \geq 0$ such that $-\psi \leq \phi \leq \psi$ and $\|\psi\| \leq \alpha \|\phi\|$ for all $\alpha > 1$). According to [3, Corollary 1.7.5], the operator norm on $L(Y, Y)$ is absolutely monotone. Since $\text{int } Y_+ \neq \emptyset$ ($\text{int } Y_+$ denotes the set of interior points of Y_+), we see that the operator norm is positively attained (by [3, Theorem 1.7.9]).

It now follows from the arguments above and Lemma 2.1 that $-H$ generates a positive semigroup if we can show that H is N -dissipative, that is,

$$N((I + \alpha H)\phi) \geq H(\phi),$$

for all (small) $\alpha > 0$, and $\phi \in D(H)$ (see [3, Theorem 2.1.1]), where $N(\psi) = \|\psi_+\|$ for $\psi \in Y$, $\psi_+ = (|\psi| + \psi)/2$. From the proof of Theorem 2.1, if $(I + \alpha H)\phi = \psi$, then

$$\phi(x) = \frac{1}{\alpha} e^{\int_x^t ds/\alpha a(s)} \int_0^x \frac{\psi(t)}{a(t)} e^{-\int_t^s ds/\alpha a(s)} dt,$$

therefore

$$\phi_+ \leq (I + \alpha H)^{-1}\psi_+ \quad \text{and} \quad N(\phi) = \|\phi_+\| \leq \|\psi_+\| = N(\psi).$$

So, H is N -dissipative and we conclude the proof of the theorem.

It is interesting to note that positive C_0 -semigroups automatically satisfy a stronger positivity condition.

COROLLARY 2.1. *Suppose $a(x)$ satisfies conditions (i)–(iv) in Section 1, then the positive C_0 semigroup $\{T(t)\}_{t \geq 0}$ generated by the operator $-H$ satisfies*

$$T(t)(\text{int } Y_+) \subset \text{int } Y_+ \tag{2.6}$$

for all $t \geq 0$.

PROOF. This is an immediate consequence of [3, Proposition 2.2.9].

The following result is obvious.

COROLLARY 2.2. *If $a \leq 0 \leq b$ and*

$$\Gamma = \{\phi \in Y \mid a \leq \phi(x) \leq b, \text{ for all } x \in [0, 1]\},$$

then $T(t)\Gamma \subset \Gamma$, for $t \geq 0$.

We are interested in the possibility of extending the C_0 semigroup $\{T(t)\}_{t \geq 0}$ to an analytic semigroup. We first recall the following result.

LEMMA 2.2 (see [15]). *Let $\{S(t)\}_{t \geq 0}$ be a uniformly bounded C_0 semigroup. Let A be the infinitesimal generator of $\{S(t)\}_{t \geq 0}$ and assume $0 \in \rho(A)$. The following statements are equivalent:*

- (a) $\{S(t)\}_{t \geq 0}$ can be extended to an analytic semigroup in a sector $\Delta_\delta = \{z \mid |\arg z| < \delta\}$ and $\|S(z)\|$ is uniformly bounded in every closed subsector $\bar{\Delta}_{\delta'}$, $\delta' < \delta$, of Δ_δ .
- (b) There exists a constant C such that for every $\sigma > 0$ and $\tau \neq 0$

$$\|R(((\sigma + i \tau)I - A)^{-1})\| \leq C/|\tau|.$$

It is well known that a semigroup which can be extended to an analytic semigroup has many nice properties, but we have the following result for the semigroup $\{T(t)\}_{t \geq 0}$.

THEOREM 2.3. *Suppose $a(x)$ satisfies (i)–(iv) in Section 1. Let $\{T(t)\}_{t \geq 0}$ be the C_0 semigroup generated by the operator $-H$, then $\{T(t)\}_{t \geq 0}$ cannot be extended to an analytic semigroup.*

PROOF. Since multiplication of a C_0 semigroup by $e^{i\omega t}$ does not affect the possibility or impossibility of extending it to an analytic semigroup, we will consider the uniformly bounded semigroup $T(t)e^{-\epsilon t}$ for some $\epsilon > 0$. Then $-H - \epsilon I$ is the infinitesimal generator of $T(t)e^{-\epsilon t}$. According to the proof of Theorem 2.1 we see $0 \in \rho(-H - \epsilon I)$.

Take $\sigma > 0$, $\tau \neq 0$ and $\tau \neq (\sigma + \epsilon)^2$. We also take $\psi_1, \psi_2 \in C^1[0, 1]$ such that $\psi_1^2(0) + \psi_2^2(0) \neq 0$ and define

$$\phi(x) = \begin{cases} e^{\int_x^1 (\sigma + \epsilon + i\tau)/a(s) ds} \int_0^x \frac{\psi_1(t) + i\psi_2(t)}{a(t)} e^{-\int_t^1 (\sigma + \epsilon + i\tau)/a(s) ds} dt, & 0 < x \leq 1, \\ \frac{(\sigma + \epsilon)\psi_1(0) + \tau\psi_2(0) - i\tau\psi_1(0) + i(\sigma + \epsilon)\psi_2(0)}{(\sigma + \epsilon)^2 + \tau}, & x = 0. \end{cases} \tag{2.7}$$

When $0 < x \leq 1$, we have

$$\begin{aligned} \phi(x) &= \int_0^x \frac{\psi_1(t)}{a(t)} e^{-\int_t^x (\sigma + \epsilon)/a(s) ds} \cos\left(\int_t^x \frac{\tau}{a(s)} ds\right) dt \\ &\quad + \int_0^x \frac{\psi_2(t)}{a(t)} e^{-\int_t^x (\sigma + \epsilon)/a(s) ds} \sin\left(\int_t^x \frac{\tau}{a(s)} ds\right) dt \\ &\quad - i \int_0^x \frac{\psi_1(t)}{a(t)} e^{-\int_t^x (\sigma + \epsilon)/a(s) ds} \sin\left(\int_t^x \frac{\tau}{a(s)} ds\right) dt \\ &\quad + i \int_0^x \frac{\psi_2(t)}{a(t)} e^{-\int_t^x (\sigma + \epsilon)/a(s) ds} \cos\left(\int_t^x \frac{\tau}{a(s)} ds\right) dt. \end{aligned} \tag{2.8}$$

Since

$$\int_0^x \frac{1}{a(t)} e^{-\int_t^x (\sigma + \epsilon)/a(s) ds} \cos\left(\int_t^x \frac{\tau}{a(s)} ds\right) dt = \frac{\sigma + \epsilon}{(\sigma + \epsilon)^2 + \tau}$$

and

$$\int_0^x \frac{1}{a(t)} e^{-\int_t^x (\sigma + \epsilon)/a(s) ds} \sin\left(\int_t^x \frac{\tau}{a(s)} ds\right) dt = \frac{\tau}{(\sigma + \epsilon)^2 + \tau},$$

we easily see that the function ϕ defined by (2.7) is continuous at $x = 0$, $\phi \in Y$ and

$$((\sigma + \epsilon + i\tau)I + H)\phi = \psi.$$

We also get

$$|\phi(0)| = \left(\frac{(\sigma + \epsilon)^2 + \tau^2}{[(\sigma + \epsilon)^2 + \tau]^2} \right)^{1/2} (\psi_1^2(0) + \psi_2^2(0))^{1/2}.$$

So

$$\lim_{\tau \rightarrow +\infty} |\phi(0)| = (\psi_1^2(0) + \psi_2^2(0))^{1/2} \neq 0. \tag{2.9}$$

If $\{T(t)\}_{t \geq 0}$ can be extended to an analytic semigroup, there exists a constant C such that for every $\tau > (\sigma + \epsilon)^2$, the following inequality is valid:

$$|\phi(0)| \leq \|\phi\| \leq C/\tau,$$

and $\lim_{\tau \rightarrow +\infty} \phi(0) = 0$. This contradicts (2.9) and shows that $\{T(t)\}_{t \geq 0}$ cannot be extended to an analytic semigroup. This completes the proof.

3. Existence

We should notice that the problem (1.7) can be written as an abstract ODE in the Banach space Y :

$$\begin{cases} du(t)/dt + Hu(t) = g(u(t)), & t \geq 0; \\ u(0) = \phi, \end{cases} \tag{3.1}$$

where $u(t) = u(\cdot, t)$, $\phi \in Y$, H is defined by (2.1)–(2.2) and

$$g(u(t))(x) = f(x, u(x, t)). \tag{3.2}$$

In this section, we return to the consideration of (3.1) with the aim of establishing sufficient conditions for solutions to exist and remain in certain closed convex subsets of Y . The first result establishes that mild solutions exist and are, in fact, classical solutions.

More precisely, $u : [0, \tau) \rightarrow Y$ is a mild solution of (3.1) if it is continuous and satisfies the following equation on $[0, \tau)$:

$$u(t) = T(t)\phi + \int_0^t T(t-s)g(u(s))ds. \tag{3.3}$$

Let Λ be a nonempty closed convex subset of R and Y_Λ be the subset of Y consisting of functions which take all their values in Λ :

$$Y_\Lambda = \{\phi \in Y \mid \phi(x) \in \Lambda, x \in [0, 1]\}. \tag{3.4}$$

Sufficient conditions for Y_Λ to be positively invariant with respect to the semiflow generated by (3.1) are given below. The first is the well-known condition:

$$\lim_{h \rightarrow 0^+} h^{-1} \text{dist}(\Lambda, v + hf(x, v)) = 0, \quad \text{for all } (x, v) \in [0, 1] \times \Lambda, \quad (3.5)$$

or

$$\lim_{h \rightarrow 0^+} \inf h^{-1} \text{dist}(Y_\Lambda, \phi + hg(\phi)) = 0, \quad \text{for all } \phi \in Y_\Lambda, \quad (3.6)$$

where $(g(\phi))(x) = f(x, \phi(x))$.

The second condition requires that the C_0 semigroup $\{T(t)\}_{t \geq 0}$ generated by $-H$ leaves Y_Λ positively invariant:

$$T(t)Y_\Lambda \subset Y_\Lambda, \quad t \geq 0. \quad (3.7)$$

The third condition requires smoothness of f :

$$f(x, v) \text{ and } \partial f(x, v)/\partial v : [0, 1] \times R \rightarrow R \text{ are continuous.} \quad (F)$$

Together these conditions imply the following existence of a solution of (3.1) and positive invariance.

THEOREM 3.1. *Let $a(x)$ satisfy conditions (i)–(iv) in Section 1 and $f(x, v)$ satisfy condition (F). Suppose that (3.5) and (3.7) hold. Then for each $\phi \in Y_\Lambda$, (3.1) has a unique noncontinuable mild solution $u(t) = u(t, \phi) \in Y_\Lambda$ defined on $[0, \sigma)$, where $\sigma = \sigma(\phi) \leq \infty$. Furthermore, the flowing properties hold:*

- (a) *if $\sigma < \infty$, then $\|u(t)\| \rightarrow \infty$, as $t \rightarrow \sigma$;*
- (b) *if $\phi \in D(-H)$, then $u(t)$ is a classical solution of the initial value problem, that is, $u(t)$ is continuous on $[0, \sigma)$ and $u(t)$ is continuously differentiable on $(0, \sigma)$ and $u(t)$ satisfies (3.1).*

Before we prove the theorem, we need the following fact.

PROPOSITION 3.1. *If (3.5) holds, so does (3.6).*

PROOF. If (3.5) holds, we will show that

$$\lim_{h \rightarrow 0^+} h^{-1} \text{dist}(Y_\Lambda, \phi + hg(\phi)) = 0 \quad \text{for all } \phi \in Y_\Lambda, \quad (3.8)$$

where $(g(\phi))(x) = f(x, \phi(x))$.

If (3.8) doesn't hold, then there exists $\phi_0 \in Y_\Lambda$, $\epsilon_0 > 0$ and $h_k > 0$ ($k = 1, 2, \dots$), such that $0 < h_k < h_{k+1}$, $\lim_{k \rightarrow \infty} h_k = 0$, and

$$h_k^{-1} \text{dist}(Y_\Lambda, \phi_0 + h_k g(\phi_0)) \geq \epsilon_0, \quad k = 1, 2, \dots$$

Therefore, there exist $\phi_k \in Y_\Lambda$ and $x_k \in [0, 1]$ for $k = 1, 2, \dots$, such that

$$h_k^{-1} |\phi_k(x_k) - [\phi_0(x_k) + h_k f(x_k, \phi_0(x_k))]| \geq \epsilon/2, \quad k = 1, 2, \dots \tag{3.9}$$

Define

$$F(h, x, v) = \begin{cases} h^{-1} \text{dist}(\Lambda, v + hf(x, v)), & \text{for } h > 0, (x, v) \in [0, 1] \times (\Lambda \cap [-\|\phi_0\|, \|\phi_0\|]), \\ 0, & \text{for } h = 0. \end{cases}$$

Since (3.5) holds, $F(h, x, v)$ is a continuous function in (h, x, v) and there exists $0 < \delta < 1/2$ such that

$$F(h, x, v) < \epsilon/3, \text{ for } 0 \leq h < \delta \text{ and } (x, v) \in [0, 1] \times (\Lambda \cap [-\|\phi_0\|, \|\phi_0\|]).$$

For sufficiently large k , there are

$$h_k^{-1} |\phi_k(x_k) - [\phi_0(x_k) + h_k f(x_k, \phi_0(x_k))]| < \epsilon/2.$$

This contradicts (3.9) and shows that (3.8) holds and the proof of the proposition is completed.

PROOF OF THEOREM 3.1. The existence of a unique local mild solution of (3.1) and the fact that the solution can be continued to a maximal interval of existence $[0, \tau)$, such that (a) holds, are a consequence of [15, Chapter 6, Theorem 1.4], since the hypothesis (F) on f implies that $g : Y \rightarrow Y$ is locally Lipschitz continuous. Theorem 1.5, of the same reference, shows that the mild solution $u(t, \phi)$ of (3.1) with $\phi \in D(-H)$ is a classical solution of the initial value problem (3.1) and thus (b) holds. This result requires that $g : Y \rightarrow Y$ is continuously differentiable, which can be guaranteed by (F).

The fact that the mild solution belongs to Y_Λ is a consequence of [13, Chapter 8, Theorem 2.1] and the proof of Theorem 3.1 is completed.

The nonnegative functions on Y are just Y_Λ where $\Lambda = R^+$ and R^+ is the set of all nonnegative real numbers.

THEOREM 3.2. *Suppose that $a(x)$ satisfies conditions (i)–(iv) in Section 1. Let $\Lambda = R^+$ and suppose that $f : [0, 1] \times R \rightarrow R$ satisfies (F), $f(x, 0) \geq 0$ for $x \in [0, 1]$ and*

$$f(x, v) \leq k_1 + k_2v, \quad \text{for } v \geq 0, \tag{3.10}$$

where k_1 and k_2 are positive constants. Then (3.5) and (3.7) hold for $Y_\Lambda = Y_+$, so that the conclusion of Theorem 3.1 holds. Moreover, the maximum interval of existence of the solution is $[0, \infty)$; the mild solution $\Psi_t(\phi) = u(t, \phi)$ is a semiflow on Y_+ .

PROOF. Equation (3.7) follows by Theorem 2.2. For $(x, v) \in [0, 1] \times R^+$,

$$v + hf(x, v) \geq 0,$$

for all small $h > 0$. So $\text{dist}(R^+, v + hf(x, v)) = 0$ for all small $h > 0$. This obviously implies that (3.5) holds. It follows that the conclusion of Theorem 3.1 holds.

Moreover, the maximum interval of existence of the solution is $[0, \infty)$. If the maximum interval of existence of the solution is $[0, t_0]$ for some $\phi \in Y_+$ and we have $0 < t_0 < \infty$, then $\lim_{t \rightarrow t_0} \|u(t, \phi)\| = \infty$. On the other hand

$$\|u(t, \phi)\| \leq \|\phi\| + \int_0^t (k_1 + k_2 \|u(s, \phi)\|) ds, \quad \text{for } 0 \leq t < t_0,$$

and it follows from Gronwall’s inequality that $\lim_{t \rightarrow t_0} \|u(t, \phi)\| < \infty$. This contradiction shows $t_0 = \infty$.

In order to prove that $\Psi_t(\phi)$ is a semiflow on Y_Λ , take $\phi_0, \phi \in Y_\Lambda$ and fix t_0 and ϕ_0 . From (3.10) for a given $\delta > 0$, there exists $M > 0$ such that if $\|\phi - \phi_0\| < \delta$, then

$$\|g(u(s, \phi))\| \leq M \quad \text{for } s \in [0, t_0 + 1].$$

Therefore

$$\begin{aligned} & \lim_{\substack{t \rightarrow t_0 \\ \phi \rightarrow \phi_0}} (u(t, \phi) - u(t_0, \phi)) \\ &= \lim_{\substack{t \rightarrow t_0 \\ \phi \rightarrow \phi_0}} \left([T(t) - T(t_0)]\phi + \int_{t_0}^t T(t-s)g(u(s, \phi)) ds \right) = 0. \end{aligned} \tag{3.11}$$

Noting that $g : Y \rightarrow Y$ is locally Lipschitz continuous, we see that if $\|\phi - \phi_0\| < \delta$ and $\phi \in Y_\Lambda$, then

$$\begin{aligned} \|u(t_0, \phi) - u(t_0, \phi_0)\| &= \|\phi - \phi_0\| + \int_0^{t_0} \|g(u(s, \phi)) - g(u(s, \phi_0))\| ds \\ &\leq \|\phi - \phi_0\| + L \int_0^{t_0} \|u(s, \phi) - u(s, \phi_0)\| ds, \end{aligned} \tag{3.12}$$

where L is a constant related to t_0 and δ . This implies, by Gronwall’s inequality, that

$$\lim_{\phi \rightarrow \phi_0} \|u(t_0, \phi) - u(t_0, \phi_0)\| = 0. \tag{3.13}$$

It follows from (3.11) and (3.13) that

$$\Phi : R^+ \times Y_\Lambda \rightarrow Y_\Lambda \tag{3.14}$$

is a continuous mapping. It is easy to show that $\Phi_0 = id$ and that $\Phi_t \circ \Phi_s = \Phi_{t+s}$ for $t, s \geq 0$. This completes the proof of Theorem 3.1.

REMARK 3.1. It follows from Theorem 3.2 that the solution $u(t, x) = u(t, \phi)(x)$ is nonnegative if $\phi(x) \geq 0$.

Let $C_+[0, 1] = \{\phi \in Y \mid \phi(x) \geq 0, x \in [0, 1]\}$. From Theorems 3.1 and 3.2, we have the following result:

COROLLARY 3.1. *Let $a(x)$ and $f(x, v)$ satisfy the assumptions of Theorem 3.2. For each $\phi \in C_+[0, 1]$, Problem (3.1) has a unique mild solution on $[0, \infty)$, $u(t, \phi) \geq 0$. For each $\phi \in (C_+[0, 1] \cap D(-H))$, the problem (3.1) has a unique nonnegative classical solution $u(t, \phi)$ on $[0, \infty)$.*

4. Invariance

Define $\Psi_t(\phi) = u(t, \phi)$ and we see $\Psi : [0, \infty) \times C_+[0, 1] \rightarrow C_+[0, 1]$ is a semiflow on $C_+[0, 1]$ by Theorem 3.1 and Corollary 3.1. Let

$$U_+ = \{\phi \in C_+[0, 1] \mid \phi(0) > 0\} \tag{4.1}$$

and

$$U_0 = \{\phi \in C_+[0, 1] \mid \phi(0) = 0\}. \tag{4.2}$$

The following result shows that the semiflow $\{\Psi_t\}_{t \geq 0}$ splits the phase space $C_+[0, 1]$ into two disjoint invariant sets U_+ and U_0 .

THEOREM 4.1. *Suppose that $a(x)$ and $f(x, v)$ satisfy the assumptions of Theorem 3.2 and there exists $v_0 > 0$ such that $f(0, v) \geq 0$ for $0 \leq v \leq v_0$. Then the set U_+ is positively invariant under $\{\Psi_t\}_{t \geq 0}$. The same property exists for set U_0 if $f(0, 0) = 0$.*

PROOF. We first prove that $\Psi_t U_0 \subset U_0$ for $t \geq 0$. Given $\phi_0 \in U_0$, for every $\epsilon > 0$, there exists $\phi_h \geq 0$ such that

$$\|\phi_h - [\phi_0 + hg(\phi_0)]\| \leq h\epsilon/2, \quad h > 0.$$

Notice that $(\phi_0 + hg(\phi_0))(0) = 0$ since $f(0, 0) = 0$. We can choose a $\psi_h \in U_0$, such that

$$\|\phi_h - \psi_h\| < h\epsilon/2.$$

Hence $\|\psi_h - [\phi_0 + hg(\phi_0)]\| < h\epsilon, h > 0$. It follows that

$$\lim_{h \rightarrow 0^+} h^{-1} \text{dist}(U_0, \phi + hg(\phi)) = 0, \quad \text{for } \phi \in U_0. \tag{4.3}$$

Next, we show that

$$T(t)U_0 \subset U_0 \quad \text{for } t \geq 0. \tag{4.4}$$

Let $\phi_0 \in U_0$ be fixed. For every $\epsilon > 0$, it is easy to show that there exists $1 > \delta_\epsilon > 0$ and $\phi_\epsilon \in C_+[0, 1]$ such that

$$\begin{cases} 0 \leq \phi_\epsilon(x) - \phi_0(x) \leq \epsilon, & \text{for } x \in [0, 1], \\ \phi'_\epsilon(x) \text{ exists} & \text{on } [0, \delta_\epsilon], \\ \phi'_\epsilon(x) \geq 0, & \text{on } [0, \delta_\epsilon]. \end{cases} \tag{4.5}$$

For $0 \leq x \leq \delta_\epsilon$, we have

$$\begin{aligned} ((I + \alpha H)^{-1}\phi_\epsilon)(x) - \phi_\epsilon(x) &= \frac{1}{\alpha} e^{\int_x^1 ds/\alpha a(s)} \int_0^x \frac{\phi_\epsilon(t)}{a(t)} e^{-\int_t^1 ds/\alpha a(s)} dt - \phi_\epsilon(x) \\ &\leq \frac{1}{\alpha} e^{\int_x^1 ds/\alpha a(s)} \int_0^x \frac{\phi_\epsilon(t) - \phi_\epsilon(x)}{a(t)} e^{-\int_t^1 ds/\alpha a(s)} dt \\ &= \frac{1}{\alpha} e^{\int_x^1 ds/\alpha a(s)} \int_0^x \frac{\phi'_\epsilon(\theta)(t-x)}{a(t)} e^{-\int_t^1 ds/\alpha a(s)} dt \leq 0, \end{aligned}$$

that is, $((I + \alpha H)^{-1}\phi_\epsilon)(x) \leq \phi_\epsilon(x)$ for $0 \leq x \leq \delta_\epsilon$.

It is well known that $T(t)\phi = \lim_{n \rightarrow \infty} (I + (t/n)H)^{-n}\phi$, for $t \geq 0$. Also when $0 \leq x \leq \delta_\epsilon$, we see that $(T(t)\phi_\epsilon)(x) \leq \phi_\epsilon(x)$, $t \geq 0$, in particular,

$$(T(t)\phi_\epsilon)(0) \leq \phi_\epsilon(0) \leq \epsilon$$

by (4.5). It follows that $(T(t)\phi_0)(0) \leq (T(t)\phi_\epsilon)(0) \leq \epsilon$, for $t \geq 0$. There must be $(T(t)\phi_0)(0) = 0$ for $t \geq 0$, so (4.4) holds.

Equations (4.3) and (4.4) imply that $\Psi_t(U_0) \subset U_0$ by Theorem 3.1.

We will prove that

$$\Psi_t(U_+) \subset U_+. \tag{4.6}$$

If $\phi \in Y$ and $\phi(x) > 0$ for $x \in [0, 1]$, then there exists $0 < t_0$ such that $\Psi_{t_0}(\phi)(x) > 0$ for all $x \in [0, 1]$. Otherwise, there exist $0 < t_n$ and $x_n \in [0, 1]$ such that $t_n \rightarrow 0$, $x_n \rightarrow x_0$ and

$$\Psi_{t_n}(\phi)(x_n) = 0, \quad n = 1, 2, \dots,$$

thus it follows that $\Psi_0(\phi)(x_0) = \phi(x_0) = 0$. It is a contradiction.

Pick $\phi_+ \in U_+$, then there exists $\psi \in C_+[0, 1]$ such that $\psi(0) = \phi_+(0)$, $\psi(x) > 0$ and $\psi(x) \geq \phi_+(x)$ for $x \in [0, 1]$, and by (2.6) and (4.4) we have

$$(T(t)\phi_+)(0) = (T(t)\psi)(0) > 0, \quad \text{for } t \geq 0. \tag{4.7}$$

Let $t_0 = \sup\{s \mid \Psi_r(\phi_+)(0) > 0, 0 \leq t \leq s\}$. It is clear that $t_0 > 0$. If $t_0 < \infty$ then

$$\int_0^{t_0} (T(t_0 - s)g(u(s)))(0) ds < 0$$

therefore there must be s^* such that $0 < s^* < t_0$ and $g(u(s))(0) = f(0, u(s, 0)) < 0$. We see that $u(s^*, 0) > v_0$. Since $u(t_0, 0) \equiv \Psi_{v_0}(\phi_+)(0) = 0$ and $f(0, v) \geq 0$ for $0 \leq v \leq v_0$, there exists t^* such that $0 < t^* < t_0$ and

$$g(u(s))(0) = f(0, u(s, 0)) \geq 0.$$

We have

$$\int_0^{t_0} (T(t_0 - s)g(u(s)))(0) ds \geq \int_0^{t^*} (T(t_0 - s)g(u(s)))(0) ds,$$

since $\int_0^{t^*} (T(t_0 - s)g(u(s)))(0) ds \geq 0$. We get

$$\begin{aligned} u(t_0, 0) &\geq T(t_0)\phi_+(0) + \int_0^{t^*} (T(t_0 - s)g(u(s)))(0) ds \\ &= (T(t_0 - t^*)[\Psi_r(\phi)])(0). \end{aligned} \quad (4.8)$$

It follows from $u(t_0, 0) = 0$, (4.7) and (4.8) that

$$[\Psi_r(\phi_+)](0) = 0.$$

On the other hand, $\Psi_r(\phi_+)(0) > 0$ since $t^* < t_0$. This contradiction shows that (4.6) is correct and completes the proof of Theorem 4.1.

Acknowledgements

This research was supported by NSERC–Canada. The third author was also supported by The China Scholarship Council. This paper was written while the third author was visiting the Department of Applied Mathematics of The University of Waterloo. It is his pleasure to express deep gratitude to all staff of the Department for their warm hospitality.

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