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IDEALS IN THE WIENER ALGEBRA W^+

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Abstract

Let W^+ denote the Banach algebra of all absolutely convergent Taylor series in the open unit disc. We characterize the finitely generated closed and prime ideals in W^+. Finally, we solve a problem of Rubel and McVoy by showing that W^+ is not coherent.

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1. Introduction

One of the earliest examples in Gelfand's theory of Banach algebras is the Wiener algebra W^+ (sometimes denoted by A^+ or l_1^+) of functions f analytic in the open unit disc $\mathbf{D}$ whose Taylor series converges absolutely, that is,

$$W^+ := \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n| < \infty \right\}.$$

W^+ is a commutative complex Banach algebra with identity element under the usual pointwise algebraic operations and the norm $\|f\| = \sum_{n=0}^{\infty} |a_n|$. Obviously, each element f of W^+ can be continuously extended to the closed unit disc $\overline{\mathbf{D}}$. Therefore W^+ is a subalgebra of the disc algebra $A(\mathbf{D})$, but only in the algebraic sense, since the topology induced by the W^+-norm is stronger than the topology induced by the sup-norm $\|f\|_\infty$. In particular, W^+ is not a function algebra (uniform algebra), and therefore the situation is more complicated. For example, the structure of the closed ideals in W^+ is not known and seems extremely difficult. But W^+ has also other unusual properties which cause serious difficulties. We list the following three.

1. W^+ has not the F-property (Gurarii 1972, see [13, page 416]). We say, that a subset S of the Hardy space H^1 has the F-property (factorization property), if for any function $f \in S$ and for any inner function ϕ, $f/\phi \in H^1$ implies $f/\phi \in S$. The F-property is an essential tool in many proofs in connection with uniqueness theorems, description of ideals in algebras of analytic functions etc. Most of the commonly considered spaces of analytic functions in the unit disc have the F-property, e.g. the Hardy spaces H^p $(1 \le p \le \infty)$, the disc algebra $A(\mathbf{D})$, the algebras $A^n(\mathbf{D})$ and many more (see Shirokov [13, Theorem 1]).

2. The zero sets of the algebra W^+ can be very complicated. A subset E of the unit circle T is called a zero set for W^+ (ZW^+-set, for short) if there exists a non-identically zero function f in W^+ which vanishes on E. If f vanishes precisely on E then E is called an exact ZW^+-set. Any Carleson set is a ZW^+-set [5, page 146], but there is no characterization known for ZW^+-sets. That such a characterization would probably be difficult might be seen from the following example of R. Kaufman [7]. There exists a ZW^+-set E such that $g(E)$ is not a ZW^+-set for any Möbius transformation

$$g(z) = e^{i\phi}\frac{z-a}{1-\overline{a}z} \qquad (a \in \mathbf{D})$$

other than a rotation, that is, $a \ne 0$. It follows that there are functions $f \in W^+$ such that the composition $f \circ g$ does not belong to W^+ for any $a \ne 0$. Even more is true, the set of all such functions is dense in W^+ [4, page 307, Satz 2].

3. W^+ has not the bounded inverse property (H. S. Shapiro [12]). Let B be a commutative Banach algebra with identity element. If the Gelfand transform $\hat{f}$ of an element $f \in B$ does not vanish then the inverse f^{-1} exists. We say that B has the bounded inverse property if for any $\delta > 0$ there exists a constant $C(B,\delta)$ depending only on B and δ such that

$$\|f^{-1}\| \le C(B,\delta)$$

holds for all $f \in B$ with $\|f\| = 1$ and $|\hat{f}(m)| \ge \delta > 0$ on the maximal ideal space of B.

After this brief view of unusual properties of the algebra W^+ we shall now turn to some classical results concerning the structure of maximal ideals. The maximal ideals of W^+ are exactly the ideals of the form

$$M_a := \{f \in W^+ : f(a) = 0\} \qquad (a \in \overline{\mathbf{D}}).$$

An immediate consequence of this fact is Wieners theorem: A function f is invertible in the algebra W^+ if and only if f has no zero in $\overline{\mathbf{D}}$. We list two other consequences. To this end let $Z(f)$ denote the zero set of a function f in W^+, that is,

$$Z(f) := \{z \in \overline{\mathbf{D}}: f(z) = 0\}$$

and let $Z(S)$ denote the zero set of a subset S of W^+, that is,

$$Z(S) := \bigcap\{Z(f): f \in S\}.$$

PROPOSITION 1. *If I is an ideal in W^+ with $Z(I) = \varnothing$, then $1 \in I$, that is $I = W^+$.*

In particular, if I is finitely generated in W^+ then Proposition 1 implies the following corona type theorem.

PROPOSITION 2. *If the functions $f_1, \dots, f_N \in W^+$ satisfy in $\mathbf{D}$ the condition*

$$|f_1| + \cdots + |f_N| \geq \delta > 0,$$

then there exist functions $g_1, \dots, g_N \in W^+$ so that 1 is a linear combination, that is,

$$1 = g_1 f_1 + \cdots + g_N f_N.$$

At this stage the finitely generated ideals in W^+ come into consideration. Since the structure of the closed ideals seems very complicated it is therefore natural to ask for the structure of finitely generated closed ideals. One purpose of this paper is to give a complete characterization of those ideals. Furthermore, by the same method we determine the structure of the finitely generated prime ideals. Finally, we give an answer to a question of Rubel and Mc Voy [11, page 77] by showing that the algebra W^+ is not coherent. We present even two principal ideals whose intersection is not finitely generated.

2. Auxiliary results

In our further investigations the maximal ideals M_a which correspond to a boundary point $a \in \partial\mathbf{D}$ are of great importance for us. Each M_a may be viewed as a subalgebra of W^+. Since every maximal ideal is closed, M_a is again a commutative complex Banach algebra but obviously without identity element. But there is a substitute, namely the notion of the approximate identity, which turns out to be useful.

DEFINITION. Let A be a commutative Banach algebra (without identity element). We say that A has a (strong) approximate identity if there exists a bounded (sequence) net (e_α) of elements e_α in A such that

$$\lim_\alpha \|e_\alpha f - f\| = 0$$

for any $f \in A$.

The following result shows that every maximal ideal M_a in W^+ which corresponds to a boundary point has an approximate identity. To simplify notations we take $a = 1$ without loss of generality.

PROPOSITION 3 (FAIVYŠEVSKIJ [3, PROPOSITION 1]). *Let* $n \in \mathbb{N}$, $r_n > 1$, $r_n \searrow 1$ *and*

$$g_n(z) = \frac{z-1}{z-r_n}.$$

Then (g_n) *is a strong approximate identity for* M_1.

The existence of an approximate identity for the maximal ideal M_1 in W^+ is not obvious (since W^+ and therefore M_1 is not a function algebra) and this fact was reproved in the last years, see for example [1, page 49] and [8, page 71]. There are other approximate identities for M_1 which are of interest, for example (g_n) with

$$g_n(z) = \sqrt[n]{1-z} \quad \text{(see [1, page 49])}.$$

These examples of (g_n) show very well the property, that (g_n) "approximates pointwise" the identity element $e(z) \equiv 1$ in W^+, that is, $\lim_{n\to\infty} g_n(z) = 1$ for any $z \in \mathbb{D}$.

The great importance of an approximate identity lies in the factorization theorem of P. J. Cohen. We formulate this theorem in the special case of a maximal ideal in a Banach algebra.

THEOREM (SEE [2, PAGE 74]). *Let A be a commutative Banach algebra with identity element, M a maximal ideal of A and suppose M has an approximate identity. Then for every $f \in M$, there exist $g, h \in M$ such that $f = gh$. Moreover, h can be chosen in the closed ideal generated by f.*

In our proofs the following lemma plays an essential role.

LEMMA 1. *Let $I \neq (0)$ be an ideal in W^+ contained in the maximal ideal M. If $I = IM$, that is if every function $f \in I$ can be factorized in a product $f = gh$ of two functions $g \in I$ and $h \in M$, then I cannot be finitely generated.*

PROOF. Suppose that $I = (f_1, \dots, f_N) \neq (0)$ is a finitely generated ideal in W^+ contained in a maximal ideal M_a, $a \in \overline{\mathbb{D}}$. By our assumption there are functions $g_i \in I$, $h_i \in M_a$ with $f_i = g_i h_i$ $(i = 1, \dots, N)$. Since $g_i \in I$, there exist functions $q_k^{(i)} \in W^+$ $(i = 1, \dots, N, k = 1, \dots, N)$ with $g_i = \sum_{k=1}^N q_k^{(i)} f_k$. From this it follows that

$$\sum_{i=1}^N |g_i| \le NC \sum_{i=1}^N |f_i| = NC \sum_{i=1}^N |g_i h_i|,$$

where C is a constant chosen so that $\|q_k^{(i)}\|_\infty \le C$ for all k and i. This implies together with the Cauchy-Schwarz inequality that

$$\sum_{i=1}^N |g_i|^2 \le \left(\sum_{i=1}^N |g_i|\right)^2 \le N^2C^2\left(\sum_{i=1}^N |g_ih_i|\right)^2$$
$$\le N^2C^2\sum_{i=1}^N |g_i|^2 \sum_{i=1}^N |h_i|^2.$$

This inequality holds for all z in the closed unit disc. Now we divide. With $\delta = 1/N^2C^2$ we obtain the inequality $\delta \le \sum_{i=1}^N |h_i(z)|^2$ for all points $z \in E$, where

$$E = \left\{ z \in \overline{\mathbf{D}} \colon \sum_{i=1}^N |g_i(z)|^2 > 0 \right\}.$$

Since $I \ne (0)$, E is a dense subset of $\overline{\mathbf{D}}$. Therefore, by continuity, this inequality holds on the closed unit disc. But this is a contradiction to the fact that each function h_i vanishes at the point a.

REMARK. Lemma 1 says that if I is a finitely generated ideal and if $I = IM$ for a maximal ideal M containing I, then $I = (0)$. This result can also be derived from Nakayamas lemma (see for example, [6, Theorem 76]) by noting that $1 + f \ne 0$ for any $f \in I$ and that W^+ has no zero divisors.

The assumption that I is finitely generated is essential in Nakayamas lemma. The advantage of our proof is that we can derive the conclusion $I = (0)$ also for ideals which are not finitely generated, for example, for ideals of the form

$$W = W(f_1, \dots, f_N) := \Big\{ f \in W^+ \colon |f| \le C \sum |f_i| \text{ on } \overline{\mathbf{D}} \quad \text{for some constant } C = C(f) \Big\}$$

and by a slight modification of the proof for countably generated closed ideals.

3. Finitely generated closed ideals

THEOREM 1. *Every finitely generated closed ideal $I \ne (0)$ in the algebra W^+ is a principal ideal generated by a finite Blaschke product.*

REMARK. We remark that in general a finitely generated ideal is not principal, for example the ideal $I = (f_1, f_2)$ with

$$f_1(z) = (1-z)^3 \quad \text{and} \quad f_2(z) = (1-z)^3 \exp\left(-\frac{1+z}{1-z}\right)$$

is not principal since f_1, f_2 have no greatest common divisor in W^+.

PROOF. Let $I \neq (0)$ be a closed finitely generated ideal in W^+. We may assume $I \neq W^+$. We show that $Z(I) \cap \partial\mathbf{D} = \emptyset$.

Suppose not, then there exists a point on the boundary, take without loss of generality $z = 1$, such that all functions in I vanish at $z = 1$. In particular, I is contained in the maximal ideal M_1. By Proposition 3, M_1 has an approximate identity, and therefore we can apply the factorization theorem of P. J. Cohen. Thus for every $f \in I$ there exist functions $g, h \in M_1$ with $f = gh$. Moreover, the functions h may be chosen in the closure of the principle ideal (f).

Since I is closed we have $h \in I$, and hence $I = IM_1$. By Lemma 1, I cannot be finitely generated, which contradicts our assumption. Therefore we have established $Z(I) \cap \partial\mathbf{D} = \emptyset$. By the identity theorem for analytic functions we conclude that $Z(I)$ is a finite subset of the open disc $\mathbf{D}$. Let B be the Blaschke product associated to the sequence of the common zeros of the functions in I including multiplicities. Then B is a finite Blaschke product and therefore belongs to W^+.

By construction, any zero of B is also a zero of any f_i $(i = 1, \dots, N)$, including multiplicities. Since B is finite we have $f_i/B \in W^+$ $(i = 1, \dots, N)$. It follows that $I \subset (B)$. To obtain the reverse we define the following subset of W^+:

$$I/B := \{f/B \colon f \in I\}.$$

I/B is obviously an ideal in W^+ and has the property $Z(I/B) = \emptyset$. By Proposition 1 we have $1 \in I/B$, i.e. $B \in I$ or $(B) \subset I$. Altogether we have established $I = (B)$.

REMARK. An inspection shows that the proof goes through in more general cases.

(1) Lemma 1 and Theorem 1 hold for all Banach algebras A of holomorphic functions in the open unit disc $\mathbf{D}$ which are continuous on the closed disc $\overline{\mathbf{D}}$, containing the polynomials and satisfying

(i) A is stable, that is, $f \in A$ and $f(a) = 0$ for $a \in \mathbf{D}$ implies $f/(z-a) \in A$,

(ii) every maximal ideal of A has the form M_a with $a \in \overline{\mathbf{D}}$,

(iii) every maximal ideal M_a which corresponds to a point a of the boundary, that is, $a \in \partial\mathbf{D}$, has an approximate identity.

Examples of such algebras are the disc algebra $A(\mathbf{D})$ and the Lipschitz algebras λ_α [9, Lemma 4].

(2) On the other hand one can generalize Theorem 1 to countably generated closed ideals. To do so, one has to use Cohen's factorization theorem with norm control and the fact that each function of a countably generated closed ideal

$$I = (f_1, f_2, \dots), \quad \|f_i\| \le 2^{-i} \qquad (i = 1, 2, \dots),$$

can be represented in the form

$$f = \sum_{i=1}^{n} h_i f_i, \quad \text{where } \|h_i\| \le C\|f\|, \; n = n(f) \in \mathbf{N},$$

and C is a positive constant independent of f.

4. Prime ideals

THEOREM 2. *The finitely generated prime ideals $P \neq (0)$ in W^+ are exactly the maximal ideals M_a corresponding to a point a of the open unit disc $\mathbf{D}$.*

PROOF. Since $P \neq (0)$ is a proper ideal there exists a maximal ideal M_a of W^+ with $P \subset M_a$, $a \in \overline{\mathbf{D}}$. We show that $a \in \mathbf{D}$. Assume $a \in \partial\mathbf{D}$. Since M_a has by Proposition 3 an approximate identity we can apply Cohen's factorization theorem. Therefore there exist for every $f \in P$ functions g and h in M_a with $f = gh$. Since P is prime, g or h belongs to P. Thus $P = PM_a$. By Lemma 1 this contradicts the fact that P is finitely generated. Therefore $a \in \partial\mathbf{D}$ is impossible and we conclude that $P \subset M_a$ with $a \in \mathbf{D}$. We claim $P = M_a$. Let $f \in P$, $f \neq 0$. Since W^+ is stable f has a factorization of the form $f(z) = (z-a)^n g(z)$ with $g \in W^+$, $g(a) \neq 0$. This means $g \notin M_a$, in particular $g \notin P$. P prime now implies $(z-a) \subset P$. Hence $M_a = (z-a) = P$.

REMARK. Our proof shows that this result holds also for the disc algebra. This was proven earlier by the first author [10, page 302]. We note that the proof given there cannot be used here since the algebra W^+ does not have the F-property.

5. W^+ is not coherent

In [11] Rubel and Mc Voy proved that the disc algebra $A(\mathbf{D})$ is not coherent by showing that the intersection of two finitely generated ideals can fail to be finitely generated. On the other hand, they showed that the algebra H^∞ is coherent and asked the question for other algebras of analytic functions, for example, for the algebra W^+. We prove that for this algebra the answer is negative.

THEOREM 3. *The algebra W^+ is not coherent.*

PROOF. We present two finitely generated ideals I and J such that $I \cap J$ is not finitely generated. In fact, in our example I and J are even principal ideals. Let

$$p(z) = (1-z)^3 \quad \text{and} \quad S(z) = \exp\left(-\frac{1+z}{1-z}\right).$$

We define the ideals $I = (p)$and $J = (pS)$. Note that p and pS belong actually to the algebra $A^1(\mathbf{D}) := \{f \in A(\mathbf{D})\colon f' \in A(\mathbf{D})\}$ and that $A^1(\mathbf{D}) \subset W^+$ [5, page 56]. Let

$$K := \{pSF\colon f \in W^+ \text{ and } Sf \in W^+\}.$$

We claim that $K = I \cap J$. Trivially $K \subset I \cap J$. To prove the reverse let $g \in I \cap J$. Then there exist two functions f and h in W^+ such that $g = ph = pSf$. Hence $Sf = h \in W^+$. So $g \in K$. Let L denote the ideal

$$L := \{f \in W^+\colon Sf \in W^+\}.$$

Then $K = pSL$. Clearly, $L \subset M_1$, since S has a singularity in $z = 1$. Let $f \in L$. We shall show that $L = LM_1$. Since L is not closed we cannot derive this result directly from Cohen's factorization theorem. Therefore we repeat the main steps in Browder's proof [2, page 76] of Cohen's factorization theorem simultaneously for the function f and Sf, that is for any $\delta > 0$ we construct inductively functions $g_n \in W^+$, invertible in W^+, converging in W^+ to a function $g \in M_1$, such that the following two estimates hold for every $n \in \mathbf{N}$:

$$\|g_n^{-1}f - g_{n+1}^{-1}f\| \leq \frac{\delta}{2^n}, \qquad \|g_n^{-1}Sf - g_{n+1}^{-1}Sf\| \leq \frac{\delta}{2^n}.$$

It is important that we can choose in both estimates the same functions g_n. This is possible because M_1 has an approximate identity and f, Sf belong to M_1. Put $h_n := g_n^{-1}f$ and $H_n := g_n^{-1}Sf$. Then h_n and H_n belong to M_1. The estimates above imply that (h_n) resp. (H_n) are Cauchy sequences in W^+. Hence, since M_1 is closed, they converge to elements h resp. H in M_1, that is, $h_n = f_n^{-1}f \to h$ and $H_n = g_n^{-1}Sf = Sh_n \to H$. Since the W^+-convergence implies the uniform convergence we obtain $H = Sh$.
Also in the W^+-norm we have

$$f = \lim_{n\to\infty} h_n g_n = hg.$$

This yields the desired factorization $L = LM_1$ since $h \in L$ (because $h, Sh \in M_1 \subset W^+$) and $g \in M_1$.

Since $L \neq (0)$, L cannot be finitely generated by Lemma 1. Therefore $K = pSL = I \cap J$ is not finitely generated.

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