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HECKE ALGEBRAS AND CLASS-GROUP INVARIANTS

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ABSTRACT. Let G be a finite group. To a set of subgroups of order two we associate a mod 2 Hecke algebra and construct a homomorphism, ψ, from its units to the class-group of $\mathbf{Z}[G]$. We show that this homomorphism takes values in the subgroup, $D(\mathbf{Z}[G])$. Alternative constructions of Chinburg invariants arising from the Galois module structure of higher-dimensional algebraic K-groups of rings of algebraic integers often differ by elements in the image of ψ. As an application we show that two such constructions coincide.

1. **Introduction.** Let G be a finite group. In the study of Chinburg invariants arising from the Galois module structure of higher-dimensional algebraic K-groups of rings of algebraic integers ([1], [2], [3], [9], [10], [11]) the following situation frequently arises. Suppose given an element of $\mathrm{Ext}^2_{\mathbf{Z}[G]}(Y, X)$ which is represented by a 2-extension of finitely generated $\mathbf{Z}[G]$-modules of the form

$$X \longrightarrow A \longrightarrow B \longrightarrow Y$$

in which A and B are cohomologically trivial (*cf.* [8] Chapter 7). In this case the Euler characteristic

$$[A] - [B] \in CL(\mathbf{Z}[G])$$

gives a well-defined element of the class-group of $\mathbf{Z}[G]$, depending only on the quasi-isomorphism class of the 2-extension. The following sort of commutative diagram, whose rows are such 2-extensions, arose in [3] during the comparison of two such Euler characteristics originating from the 2- and 3-dimensional K-groups of rings of integers in number fields.

$$\begin{array}{ccccccc}
E_+ & & & & & & \\
\downarrow & & & & & & \\
X_1 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & Y_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_2 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & Y_2 \\
 & & & & & & \downarrow \\
 & & & & & & E_-
\end{array}$$

In this diagram the right-hand and left-hand columns are short exact sequences and

$$E_\pm = \bigoplus_{i=1}^{r} \mathrm{Ind}_{H_i}^{G}(\mathbf{Z}_\pm)$$

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where $H_1, \ldots, H_r$ are distinct subgroups of order two, each of whose generators acts by ± 1 on a copy of the integers denoted by $\mathbf{Z}_{\pm}$.

The exact Mayer-Vietoris sequence resulting from this diagram defines a 3-extension, in $\mathrm{Ext}^3_{\mathbf{Z}[G]}(E_-, E_+)$, whose Euler characteristic is equal to

$$(1) \qquad [A_1] - [B_1] - [A_2] + [B_2] \in CL(\mathbf{Z}[G]).$$

In this paper we study the equivalent process of producing Euler characteristics in the class-group from elements of

$$\mathrm{Ext}^1_{\mathbf{Z}[G]}(E_-, E_+) \cong \mathrm{Ext}^3_{\mathbf{Z}[G]}(E_-, E_+)$$

with the objective of proving that they often vanish. This isomorphism is induced by a "change of groups" isomorphism together with the periodicity in the cohomology of H_i (*cf.* Section 3.1). Our main result (Theorem 4.4) shows that, if $N_G H_i$ is the normaliser of H_i in G, the Euler characteristics which arise lie in the subgroup generated by the images of compositions of the form

$$K_1(\mathbf{Z}[N_G H_i]) \xrightarrow{\delta} CL(\mathbf{Z}[N_G H_i]) \xrightarrow{\mathrm{Ind}^G_{N_G H_i}} CL(\mathbf{Z}[G]).$$

As explained in Section 4.6, this suffices to show that the difference of Euler characteristics in equation (1) vanishes in the arithmetical setting (see Example 2.4) in which $G = G(E/\mathbf{Q})$ is the Galois group of a number field extension in which E is totally complex. Heuristically, this is because ("in the limit") $N_G H_i = H_i$ in this case and $CL(\mathbf{Z}[H]) = 0$ when $|H| = 2$. Therefore Theorem 4.4 affords an alternative proof of the 2-primary part of the comparison results of [3], as explained in detail in Section 4.6.

The paper is arranged in the following manner. In Section 1 we introduce the subalgebras, $S^{\pm}_{R,T}(G)$, of the Hecke algebras $\mathrm{End}_{R[G]}(E_{\pm} \otimes R)$ and relate them to $\mathrm{Ext}^1_{\mathbf{Z}[G]}(E_-, E_+)$. In Section 2 we construct the homomorphism

$$\psi\colon \frac{S_{\mathbf{Z}/2,T}(G)^*}{S^+_{\mathbf{Z},T}(G)^*} \longrightarrow CL(\mathbf{Z}[G])$$

which arises from the Euler characteristics constructed from $\mathrm{Ext}^1_{\mathbf{Z}[G]}(E_-, E_+)$. In Section 3 we give some examples and relate the Hecke subalgebras to matrix rings with entries in $\mathbf{Z}[N_g H_i]$ in order to identify the image of ψ.

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2. The Hecke algebras.

2.1. Let R be a commutative ring with identity and let G be a finite group. Suppose that

$$T = \{\tau_i \in G\,;\, 1 \le i \le r\}$$

is a set of distinct elements of order two and set $H_i = \langle \tau_i \rangle$, the subgroup of order two generated by τ_i. Let $R_\pm$ be a copy of R on which τ_i acts by ± 1, respectively. Then there are isomorphisms, for $1 \le i, j \le r$,

$$\begin{aligned}\mathrm{Hom}_G\big(\mathrm{Ind}_{H_i}^G(R_\pm), \mathrm{Ind}_{H_j}^G(R_\pm)\big) &\cong \mathrm{Hom}_{R[H_i]}\big(R_\pm, \mathrm{Ind}_{H_j}^G(R_\pm)\big)\\ &\cong \mathrm{Hom}_{R[H_i]}\Big(R_\pm, \bigoplus_{z\in H_i\backslash G/H_j} \mathrm{Ind}_{H_i\cap zH_jz^{-1}}^{H_i}(R_\pm)\Big)\end{aligned}$$

where $\mathrm{Ind}_H^G(R_\pm)$ is the induced $R[G]$-module, $R[G] \otimes_H R_\pm$. The second isomorphism is induced by the Double Coset isomorphism

$$Res_{H_i}^G \mathrm{Ind}_{Hj}^G(R_\pm) \cong \bigoplus_{z\in H_i\backslash G/H_j} \mathrm{Ind}_{H_i\cap zH_jz^{-1}}^{H_i}(R_\pm)$$

under which $a \otimes_{H_i\cap zH_jz^{-1}} b$ on the right corresponds to $az \otimes_{H_j} b$ on the left.

Inside the Hecke algebra

$$\begin{aligned}\mathrm{End}_{R[G]}\Big(\bigoplus_{i=1}^r \mathrm{Ind}_{H_i}^G(R_\pm)\Big) &\cong \mathrm{Hom}_{R[G]}\Big(\bigoplus_{i=1}^r \mathrm{Ind}_{H_i}^G(R_\pm), \bigoplus_{j=1}^r \mathrm{Ind}_{H_j}^G(R_\pm)\Big)\\ &\cong \bigoplus_{i=1}^r\bigoplus_{j=1}^r \bigoplus_{z\in H_i\backslash G/H_j} \mathrm{Hom}_{R[H_i]}\big(R_\pm, \mathrm{Ind}_{H_i\cap zH_jz^{-1}}^{H_i}(R_\pm)\big)\end{aligned}$$

let $S_{R,T}^\pm(G)$ correspond to the subset of summands for which $H_i = zH_jz^{-1}$. Hence, additively, if we write $J \sim K$ to indicate conjugacy,

$$S_{R,T}^\pm(G) \cong \bigoplus_{H_i\sim H_j} \bigoplus_{z\in H_i\backslash G/H_j, H_i=zH_jz^{-1}} \mathrm{Hom}_R(R_\pm, R_\pm).$$

Henceforth we shall identify $\mathrm{Hom}_R(R_\pm, R_\pm)$ with R by the isomorphism which sends f to $f(1)$. Write $\underline{\mu} = \{\mu(i,j,z) \in R\} \in S_{R,T}^\pm(G)$ for the element whose (i,j,z)-component is equal to $\mu(i,j,z) \in R$. Therefore we may consider $\mu(i,j,-)$ as a function from G to R with the property that, in the case of $S_{R,T}^\pm(G)$,

$$\mu(i,j,\tau_i z) = \pm\mu(i,j,z) = \mu(i,j,z\tau_j),$$

respectively. Hence $\underline{\mu}$ is characterised, as a homomorphism, by the fact that it sends $g \otimes_{H_i} v \in \mathrm{Ind}_{H_i}^G(R_\pm)$ $(g \in G, v \in R)$ to

$$\bigoplus_{H_i\sim H_j} \bigoplus_{z\in H_i\backslash G/H_j, H_i=zH_jz^{-1}} gz \bigotimes_{H_j} \mu(i,j,z)v \in \bigoplus_{j=1}^r \mathrm{Ind}_{H_j}^G(R_\pm).$$

PROPOSITION 2.2. *The subgroup, $S_{R,T}^\pm(G)$, is a subring of* $\mathrm{End}_{R[G]}\big(\oplus_{i=1}^r \mathrm{Ind}_{H_i}^G(R_\pm)\big)$. *In fact, the product of $\underline{\mu} = \{\mu(i,j,z)\}$ and $\underline{\lambda} = \{\lambda(s,t,w)\}$ is given by ($g \in G, v \in R$)*

$$(\underline{\lambda}\cdot\underline{\mu})\Big(g \bigotimes_{H_i} v\Big) = \sum_{\substack{H_i\sim H_j, z\in H_i\backslash G/H_j\\ H_i=zH_jz^{-1}}} \sum_{\substack{H_j\sim H_k, w\in H_j\backslash G/H_k\\ H_j=wH_kw^{-1}}} gzw \bigotimes_{H_k} \mu(i,j,z)\lambda(j,k,w)v.$$

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Therefore

$$\underline{\lambda} \cdot \underline{\mu} = \underline{\nu} = \{\nu(a, b, y)\}$$

where

$$\nu(i, k, y) = \sum_{\substack{H_i \sim H_j, z \in H_i \backslash G / H_j \\ H_i = z H_j z^{-1}}} \sum_{\substack{H_j \sim H_k, w \in H_j \backslash G / H_k \\ H_j = w H_k w^{-1}}} \mu(i, j, z) \lambda(j, k, w)$$

the sum being taken only over pairs of double cosets, $H_i z H_j$ *and* $H_j w H_k$ *such that* $H_i y H_k = H_i z w H_k$*.*

PROOF. The homomorphism corresponding to $\underline{\mu}$ sends $g \otimes_{H_i} v$ to

$$\bigoplus_{H_i \sim H_j} \bigoplus_{z \in H_i \backslash G / H_j, H_i = z H_j z^{-1}} gz \bigotimes_{H_j} \mu(i, j, z) v$$

which is sent by $\underline{\lambda}$ to

$$\sum_{\substack{H_i \sim H_j, z \in H_i \backslash G / H_j \\ H_i = z H_j z^{-1}}} \sum_{\substack{H_j \sim H_k, w \in H_j \backslash G / H_k \\ H_j = w H_k w^{-1}}} gzw \bigotimes_{H_k} \mu(i, j, z) \lambda(j, k, w) v,$$

as required. ∎

EXAMPLE 2.3. Let $G = Q_8 = \{x, y \mid x^2 = y^2, x^4 = 1, xyx^{-1} = y^{-1}\}$ denote the quaternion group of order eight. Then $H_1 = \langle x^2 \rangle$ is the unique subgroup of order two, which is central. Let $V = Q_8 / \langle x^2 \rangle \cong \mathbf{Z}/2 \times \mathbf{Z}/2$ consisting of the images of $1, x, y, xy$. Then $S^+_{R, \langle x^2 \rangle}(Q_8) \cong R[V]$, the isomorphism being to send $\underline{\mu}$ to $\sum_{z \in V} \mu(1, 1, z) z$, where we have identified $\langle x^2 \rangle \backslash Q_8 / \langle x^2 \rangle = Q_8 / \langle x^2 \rangle$ with V.

Similarly, $S^-_{R, \langle x^2 \rangle}(Q_8) \cong \mathbf{H}_{\mathbf{Z}}$, the integral quaternions.

EXAMPLE 2.4. Let L/K be a Galois extension of number fields and let $E/\mathbf{Q}$ be a large Galois extension of number fields such that $L \subset E$ and E is totally complex. Let c denote complex conjugation in $\mathbf{Q}^{\mathrm{sep}}$. Let Ω_L denote the absolute Galois group, $\Omega_L = G(\mathbf{Q}^{\mathrm{sep}}/L)$, where $\mathbf{Q}^{\mathrm{sep}}$ is a separable closure of $\mathbf{Q}$, the rationals. Let $v_\infty : L \longrightarrow E \longrightarrow \mathbf{Q}^{\mathrm{sep}}$ be a fixed embedding which restricts to a real embedding, $w_\infty : K \longrightarrow E^{\langle c \rangle} \longrightarrow (\mathbf{Q}^{\mathrm{sep}})^{\langle c \rangle}$. Assigning to $g \in \Omega_{\mathbf{Q}}$ the embedding $(v_\infty)g : L \xrightarrow{v_\infty} \mathbf{Q}^{\mathrm{sep}} \xrightarrow{g} \mathbf{Q}^{\mathrm{sep}}$ defines a bijection between embeddings of L and $\Omega_L \backslash \Omega_{\mathbf{Q}}$. I must apologise for my notation, which the reader may find rather awkward; it is chosen to make the double cosets emerge the same way round here as they appear later in the homological algebra of Section 3.1. The set of embeddings, $\{(v_\infty)g, (v_\infty)gc\}$, corresponds to an Archimedean place of L, since the completions of of $(v_\infty)g$ and $(v_\infty)gc$ coincide. Hence assigning the double coset $\Omega_L g \langle c \rangle$ to this Archimedean place defines a bijection between $\Omega_L \backslash \Omega_{\mathbf{Q}} / \langle c \rangle$ and $\Sigma_\infty(L)$, the set of Archimedean places of L.

If $(v_\infty)g$ is a complex place then gcg^{-1} does not belong to Ω_L and $\Omega_L \cap \langle gcg^{-1} \rangle = \{1\}$. If $(v_\infty)g$ is real then $\Omega_L \cap \langle gcg^{-1} \rangle = \langle gcg^{-1} \rangle$ is of order two. In the first case, if $(w_\infty)g$ is a real place of K then $gcg^{-1} \in \Omega_K$ and its image in $G(L/K) \cong \Omega_K / \Omega_L$ is the decomposition group, $H_g = G(L_{(v_\infty)g} / K_{(w_\infty)g})$.

Taking $G = G(L/K)$ and $\mathcal{T} = \{H_i = \langle g_i c g_i^{-1}\rangle \ ; \ 1 \leq i \leq r\}$, the set of non-trivial decomposition groups at infinity, one copy for each Archimedean prime, gives the example which was the original motivation for the study of the class-group invariant which is introduced in Section 3.2.

3. Some homological algebra.

3.1. Let R, G and H_i $(1 \leq i \leq r)$ be as in Section 2.1 and let $R_{\pm}$ denote the $R[H_i]$-module upon which τ_i acts as multiplication by ± 1. Let $E_{R,\pm}$ denote the $R[G]$ module given by $\oplus_{i=1}^{r} \mathrm{Ind}_{H_i}^{G}(R_{\pm})$.

Taking $R = \mathbf{Z}$ or $\mathbf{Z}_2$, the integers or the 2-adic integers, we have a chain of isomorphisms of the form

$$\begin{aligned}
\mathrm{Ext}^1_{R[G]}(E_{R,-}, E_{R,+}) &\cong \bigoplus_{i=1}^{r} \mathrm{Ext}^1_{R[G]}(\mathrm{Ind}_{H_i}^{G}(R_-), E_{R,+}) \\
&\cong \bigoplus_{i=1}^{r} \mathrm{Ext}^1_{R[H_i]}(R_-, E_{R,+}) \cong \bigoplus_{i=1}^{r} \mathrm{Ext}^2_{R[H_i]}(R_+, E_{R,+}) \\
&\cong \bigoplus_{i=1}^{r} H^2(H_i\ ;\ E_{R,+}) \cong \bigoplus_{i=1}^{r}\bigoplus_{j=1}^{r} H^2\big(H_i\ ;\ \mathrm{Ind}_{H_j}^{G}(R_+)\big) \\
&\cong \bigoplus_{i=1}^{r}\bigoplus_{j=1}^{r}\bigoplus_{z\in H_i\backslash G/H_j} H^2\big(H_i\ ;\ \mathrm{Ind}^{H_i}_{H_i\cap zH_jz^{-1}}(R_+)\big) \\
&\cong \bigoplus_{H_i\sim H_j}\ \bigoplus_{z\in H_i\backslash G/H_j, H_i=zH_jz^{-1}} H^2(H_i\ ;\ R_+) \\
&\cong \bigoplus_{H_i\sim H_j}\ \bigoplus_{z\in H_i\backslash G/H_j, H_i=zH_jz^{-1}} \mathbf{Z}/2 \cong S^{+}_{\mathbf{Z},\mathcal{T}}(G)\otimes \mathbf{Z}/2 \\
&\cong S_{\mathbf{Z}/2,\mathcal{T}}(G)
\end{aligned}$$

where we have abbreviated the isomorphic rings, $S^{\pm}_{\mathbf{Z}/2,\mathcal{T}}(G)$, both to $S_{\mathbf{Z}/2,\mathcal{T}}(G)$.

Let us also abbreviate $E_{\mathbf{Z},\pm}$ to $E_{\pm}$. Then we have the following diagram of isomorphisms.

$$\begin{array}{ccc}
S^{+}_{\mathbf{Z},\mathcal{T}}(G)\otimes \mathbf{Z}/2 & \stackrel{\cong}{\longrightarrow} & S^{+}_{\mathbf{Z}_2,\mathcal{T}}(G)\otimes \mathbf{Z}/2 \\
\cong\downarrow & & \cong\downarrow \\
\mathrm{Ext}^1_{\mathbf{Z}[G]}(E_-, E_+) & \stackrel{\cong}{\longrightarrow} & \mathrm{Ext}^1_{\mathbf{Z}_2[G]}(E_{\mathbf{Z}_2,-}, E_{\mathbf{Z}_2,+})
\end{array}$$

Unraveling the chain of isomorphisms it is not hard to see that the isomorphism

$$\begin{aligned}
\mathrm{Ext}^1_{\mathbf{Z}[G]}(E_-, E_+) &\cong \bigoplus_{H_i\sim H_j}\ \bigoplus_{z\in H_i\backslash G/H_j, H_i=zH_jz^{-1}} \mathbf{Z}/2 \\
&\cong \bigoplus_{i=1}^{r} H^2(H_i\ ;\ E_+)
\end{aligned}$$

sends a 1-extension of $\mathbf{Z}[G]$-modules

$$E_+ \longrightarrow X \longrightarrow E_-$$

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to the element whose i-th coordinate is the image of the generator of $H^1(H_i\,;\mathbf{Z}_-)\cong \mathbf{Z}/2$ under the composition

$$H^1(H_i\,;\mathbf{Z}_-) \longrightarrow H^1(H_i\,;E_-) \xrightarrow{\Delta} H^2(H_i\,;E_+).$$

Here the first map is induced by the H_i-map, $\mathbf{Z}_- \longrightarrow \operatorname{Ind}_{H_i}^G(\mathbf{Z}_-)$, given by sending 1 to $1\otimes_{H_i}1$ and the second map is the coboundary associated to the long exact cohomology sequence

$$\cdots \longrightarrow H^n(H_i\,;E_+) \longrightarrow H^n(H_i\,;X) \longrightarrow H^n(H_i\,;E_-) \xrightarrow{\Delta} H^{n+1}(H_i\,;E_+) \longrightarrow \cdots$$

If $\underline{\delta} = \oplus_{H_i\sim H_j}\oplus_{z\in H_i\backslash G/H_j, H_i=zH_jz^{-1}}\delta(i,j,z)$ lies in $\operatorname{Ext}^1_{\mathbf{Z}[G]}(E_-,E_+)$ then $\underline{\lambda}\in S^+_{\mathbf{Z},T}(G)$ gives an endomorphism of $H^2(H_i\;;\;E_+)$ which we shall now evaluate. The $\mathbf{Z}[H_i]$-resolution

$$\cdots \longrightarrow \mathbf{Z}[H_i] \xrightarrow{1+\tau_i} \mathbf{Z}[H_i] \xrightarrow{1-\tau_i} \mathbf{Z}[H_i] \longrightarrow \mathbf{Z} \longrightarrow 0$$

shows that the cohomology group is computed from the complex

$$\bigoplus_{j=1}^r \operatorname{Ind}_{H_j}^G(\mathbf{Z}) \xrightarrow{1+\tau_i} \bigoplus_{j=1}^r \operatorname{Ind}_{H_j}^G(\mathbf{Z}) \xrightarrow{1-\tau_i} \bigoplus_{j=1}^r \operatorname{Ind}_{H_j}^G(\mathbf{Z})$$

and that $\underline{\delta}$ is represented by

$$\bigoplus_{H_i\sim H_j}\ \bigoplus_{z\in H_i\backslash G/H_j, H_i=zH_jz^{-1}} z\bigotimes_{H_j}\delta(i,j,z)$$

where $\delta(i,j,z)\in\mathbf{Z}$ now denotes any lifting of $\delta(i,j,z)\in\mathbf{Z}/2$. This element is mapped by $\underline{\lambda}$ to

$$\sum_{\substack{H_i\sim H_j, z\in H_i\backslash G/H_j\\ H_i=zH_jz^{-1}}}\ \sum_{\substack{H_j\sim H_k, w\in H_j\backslash G/H_k\\ H_j=wH_kw^{-1}}} zw\bigotimes_{H_k}\delta(i,j,z)\lambda(j,k,w)$$

so that

$$\underline{\lambda}_*(\underline{\delta}) \equiv \underline{\lambda}\cdot\underline{\delta}\ (\text{modulo } 2)$$

where the product is that of $S_{\mathbf{Z}/2,T}(G)$.

3.2. Identify $\oplus_{H_i\sim H_j}\oplus_{z\in H_i\backslash G/H_j,\, H_i=zH_jz^{-1}}\mathbf{Z}/2$ with $S_{\mathbf{Z}/2,T}(G)$, as in Section 1, and denote by $S_{\mathbf{Z}/2,T}(G)^*$ the multiplicative group of units. We may define a map to the class group of the integral group-ring of G

$$\psi\colon S_{\mathbf{Z}/2,T}(G)^* \longrightarrow CL(\mathbf{Z}[G])$$

by the following procedure. Let $\underline{\delta}_0\in S_{\mathbf{Z}/2,T}(G)$ denote the element for which $\delta(i,j,z)=0$ except for $1\le i=j\le r$ and $z=1\in G$ when $\delta(i,i,1)=1$. Given another element, $\underline{\mu}\in S_{\mathbf{Z}/2,T}(G)^*$, we may lift $\underline{\mu}$ to $\underline{\mu}'\in S^+_{\mathbf{Z},T}(G)\cong \operatorname{End}_{\mathbf{Z}[G]}(E_+)$ and then we may form

$$\underline{\mu}'_*(\underline{\delta}_0)\in \operatorname{Ext}^1_{\mathbf{Z}[G]}(E_-,E_+).$$

This 1-extension only depends upon $\underline{\mu} \in S_{\mathbf{Z}/2,T}(G)^*$. Also, since $\underline{\mu} \in S_{\mathbf{Z}/2,T}(G)^*$, the homomorphism $\underline{\mu}'\colon E_+ \longrightarrow E_+$ will be injective, since it is an isomorphism when reduced modulo 2. The 1-extension, $\underline{\mu}'_*(\underline{\delta}_0)$, is constructed by a push-out diagram of the form

$$\begin{array}{ccccc} E_+ & \longrightarrow & E & \longrightarrow & E_- \\ {\scriptstyle \underline{\mu}'}\big\downarrow & & {\scriptstyle \tilde{\underline{\mu}}}\big\downarrow & & {\scriptstyle 1}\big\downarrow \\ E_+ & \longrightarrow & Y(\underline{\mu}) & \longrightarrow & E_- \end{array}$$

where $E = \bigoplus_{i=1}^r \mathrm{Ind}_{H_i}^G(\mathbf{Z}[H_i]) \cong \bigoplus_{i=1}^r \mathbf{Z}[G]$. The upper 1-extension is constructed by applying $\mathrm{Ind}_{H_i}^G(-)$ to the canonical 1-extension of $\mathbf{Z}[H_i]$-modules

$$0 \longrightarrow \mathbf{Z}_+\langle 1+\tau_i\rangle \longrightarrow \mathbf{Z}[H_i] \longrightarrow \mathbf{Z}_- \longrightarrow 0$$

and summing over $i = 1, \ldots, r$. Therefore

$$Y(\underline{\mu}) \cong (E_+ \oplus E)/\Big\{\big(\underline{\mu}'(x), x\big) \mid x \in E_+\Big\}$$

is a finitely generated $\mathbf{Z}[G]$-module.

Since $\underline{\mu} \in S_{\mathbf{Z}/2,T}(G)^*$ the map $\underline{\mu}'_*$ is an isomorphism on the $\mathbf{Z}/2$-vector space, $\tilde{H}^*(J\ ;\ E_+)$ (Tate cohomology), for all $J \subseteq G$. Hence $Y(\underline{\mu})$ is a finitely generated, torsion-free, cohomologically trivial $\mathbf{Z}[G]$-module and is therefore projective, defining a class

$$[Y(\underline{\mu})] - \mathrm{rank}\big(Y(\underline{\mu})\big)\big[\mathbf{Z}[G]\big] \in CL(\mathbf{Z}[G]) = \tilde{K}_0(\mathbf{Z}[G])$$

in the class-group of the integral group-ring (*cf.* [4]II).

Set

$$\psi(\underline{\mu}) = [Y(\underline{\mu})] - \mathrm{rank}\big(Y(\underline{\mu})\big)\big[\mathbf{Z}[G]\big] \in CL(\mathbf{Z}[G]).$$

PROPOSITION 3.3. *The map, ψ, of Section 3.2 is a homomorphism*

$$\psi\colon S_{\mathbf{Z}/2,T}(G)^* \longrightarrow CL(\mathbf{Z}[G])$$

which factors through the quotient by $S_{\mathbf{Z},T}(G)^*$ to give

$$\psi\colon \frac{S_{\mathbf{Z}/2,T}(G)^*}{S^+_{\mathbf{Z},T}(G)^*} \longrightarrow CL(\mathbf{Z}[G]).$$

PROOF. The canonical 1-extension

$$E_+ \longrightarrow E \longrightarrow E_-$$

represents the class

$$\underline{\delta}_0 \in \mathrm{Ext}^1_{\mathbf{Z}[G]}(E_-, E_+) \cong \bigoplus_{H_i \sim H_j} \ \bigoplus_{z \in H_i \backslash G/H_j, H_i = zH_jz^{-1}} \mathbf{Z}/2.$$

Also $\underline{\delta}_0$ is the identity element of the ring $S_{\mathbf{Z}/2,T}(G)$.

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The H_i-cohomology coboundary associated to the 1-extension

$$E_+ \longrightarrow Y(\underline{\mu}) \longrightarrow E_-$$

may, by naturality, be computed by composing

$$\underline{\mu}'_* : \hat{H}^*(H_i\,;E_+) \longrightarrow \hat{H}^*(H_i\,;E_+)$$

for $i = 1, \ldots, r$ with the coboundary associated to the canonical 1-extension. Therefore, by the discussion of Section 3.1, the $Y(\underline{\mu})$-extension corresponds to

$$\underline{\mu}'_*(\underline{\delta}_0) = \underline{\mu} \cdot \underline{\delta}_0 = \underline{\mu} \in S_{\mathbf{Z}/2,T}(G)^* \subset S_{\mathbf{Z}/2,T}(G).$$

Now suppose that we have $\underline{\mu}, \underline{\lambda} \in S_{\mathbf{Z}/2,T}(G)^*$. We may lift these elements to $\underline{\mu}', \underline{\lambda}' \in S^+_{\mathbf{Z},T}(G)$, respectively. In $CL(\mathbf{Z}[G])$, since E is free,

$$[Y(\underline{\mu})] - \operatorname{rank}\big(Y(\underline{\mu})\big)\big[\mathbf{Z}[G]\big] = [\operatorname{Coker}(\tilde{\underline{\mu}})] = [\operatorname{Coker}(\underline{\mu}')]$$

since $\operatorname{Coker}(\tilde{\underline{\mu}}) \cong \operatorname{Coker}(\underline{\mu}')$ is a finite group which is also a cohomologically trivial $\mathbf{Z}[G]$-module. Therefore the short exact sequence

$$0 \longrightarrow \operatorname{Coker}(\underline{\lambda}') \longrightarrow \operatorname{Coker}(\underline{\mu}' \cdot \underline{\lambda}') \longrightarrow \operatorname{Coker}(\underline{\mu}') \longrightarrow 0$$

shows that, in $CL(\mathbf{Z}[G])$,

$$\begin{aligned} \psi(\underline{\mu} \cdot \underline{\lambda}) &= [\operatorname{Coker}(\underline{\mu}' \cdot \underline{\lambda}')] \\ &= [\operatorname{Coker}(\underline{\mu}')] + [\operatorname{Coker}(\underline{\lambda}')] \\ &= \psi(\underline{\mu}) + \psi(\underline{\lambda}), \end{aligned}$$

as required.

Finally, if $\underline{\mu} \in S_{\mathbf{Z}/2,T}(G)^*$ is in the image of $S^+_{\mathbf{Z},T}(G)^*$ then $\underline{\mu}'$ may be chosen to be a $\mathbf{Z}[G]$-module isomorphism in which case $Y(\underline{\mu}) \cong E$ and $\psi(\underline{\mu}) = 0$. ∎

4. **Another description of ψ.**

Let us begin with some examples.

EXAMPLE 4.1. Consider the Example 2.3, where $G = Q_8$ and T consists of the centre, $\langle x^2 \rangle$. In this case

$$\frac{S_{\mathbf{Z}/2,T}(Q_8)^*}{S^+_{\mathbf{Z},T}(Q_8)^*} \cong (\mathbf{Z}/4)^* \cong CL(\mathbf{Z}[Q_8]).$$

In fact, the homomorphism ψ is an isomorphism, as may be seen in terms of Fróhlich's Hom-description of the class-group together with Remark 4.7.

A similar discussion applies to any G with a normal subgroup, $\langle \tau \rangle \lhd G$, of order two. In this case

$$S^+_{R,T}(G) \cong R[G/\langle \tau \rangle]$$

and the homomorphism

$$\psi\colon \frac{\mathbf{Z}/2[G/\langle\tau\rangle]^*}{\mathbf{Z}[G/\langle\tau\rangle]^*} \longrightarrow CL(\mathbf{Z}[G])$$

may be identified, as we shall prove in Theorem 4.4, with the Mayer-Vietoris homomorphisms of the type studied in ([4] II p. 273). Hence the image of ψ lies in $D(\mathbf{Z}[G]) \subseteq CL(\mathbf{Z}[G])$.

When $G = Q_{2^n}$, the generalised quaternion group of order 2^n, then

$$D(\mathbf{Z}[Q_{2^n}]) \cong \mathbf{Z}/2$$

and ψ is surjective (*cf.* [4] II p. 273).

The remainder of this section will be devoted to showing that Example 4.1 is typical.

4.2. Our first observation is that T is the disjoint union of G-conjugacy classes, T_s, and that, in these circumstances, there is an isomorphism of the form

$$S^{\pm}_{R,T}(G) \cong \prod_s S^{\pm}_{R,T_s}(G).$$

In addition, the homomorphism, ψ, of Proposition 3.3 is evidently equal to the sum of the ψ's for each of the factors. Therefore we shall henceforth restrict ourselves to the case when $T = \{g_iHg_i^{-1}\,;\,1 \leq i \leq r\}$ where $r = [G : N_GH]$ is the index of the normaliser of H in G and $g_1, \ldots, g_r$ are coset representatives for G/N_GH. Write τ for the generator of H.

Next we observe that $zg_jHg_j^{-1}z^{-1} = g_iHg_i^{-1}$ if and only if $g_i^{-1}zg_j \in N_GH$. On the other hand, $g_iHg_i^{-1}zg_jHg_j^{-1} = g_iHg_i^{-1}wg_jHg_j^{-1}$ if and only if $g_j^{-1}z^{-1}wg_j \in H$. Therefore there is an obvious isomorphism of abelian groups of the form

$$S^{\pm}_{R,T}(G) \cong \bigoplus_{1\leq i,j\leq r} g_iR[N_GH]/\big(\tau - (\pm 1)\big)g_j^{-1}.$$

This map sends an element whose only component is equal to $\mu(i,j,z) \in R$ to the matrix with $\mu(i,j,z)g_i^{-1}zg_jH \in R[N_GH/H]$ as its only entry, in the (i,j)-th entry. If we endow the right hand side with the multiplication given on generators by

$$\big(g_j(wH)g_k^{-1}\big) \cdot \big(g_i(zH)g_j^{-1}\big) = \big(g_i(zwH)g_k^{-1}\big)$$

and zero in all other cases one sees easily that this becomes an isomorphism of rings. Furthermore it is clear that the right-hand side is isomorphic to the ring, $M_r\Big(R[N_GH]/\big(\tau - (\pm 1)\big)\Big)^{\mathrm{op}}$, of $r \times r$ matrices with entries in the quotient, $R[N_GH]/\big(\tau - (\pm 1)\big)$, of the group-ring of N_GH with coefficients in R and the *opposite* multiplication.

Therefore we have proved the following result.

PROPOSITION 4.3. *In the notation of Section 2.1 and Section 4.2,there is an isomorphism of rings*

$$S^{\pm}_{R,T}(G) \cong M_r\Big(R[N_GH]/\big(\tau - (\pm 1)\big)\Big)^{\mathrm{op}}$$

where Λ^{op} denotes the opposite ring of the ring Λ.

Next consider the pullback diagram

$$\begin{array}{ccc} \mathbf{Z}[H] & \longrightarrow & \mathbf{Z}_+ \\ \downarrow & & \downarrow \\ \mathbf{Z}_- & \longrightarrow & \mathbf{Z}/2 \end{array}$$

which induces up to yield the pullback square

$$\begin{array}{ccc} \mathbf{Z}[N_GH] & \longrightarrow & \mathbf{Z}[N_GH]/(\tau - 1) \\ \downarrow & & \downarrow \\ \mathbf{Z}[N_GH]/(\tau + 1) & \longrightarrow & \mathbf{Z}/2[N_GH]/(\tau - 1) \end{array}$$

from which we may obtain a K-theory Mayer-Vietoris sequence by the method described in ([5] Section 3)

$$\cdots \longrightarrow K_1\big(\mathbf{Z}[N_GH]/(\tau+1)\big) \oplus K_1\big(\mathbf{Z}[N_GH]/(\tau-1)\big) \longrightarrow$$
$$K_1\big(\mathbf{Z}/2[N_GH]/(\tau-1)\big) \xrightarrow{\delta} K_0(\mathbf{Z}[N_GH]) \longrightarrow \cdots.$$

If M_G is a maximal $\mathbf{Z}$-order, containing $\mathbf{Z}[G]$, in $\mathbf{Q}[G]$ let $D(\mathbf{Z}[G])$ denote the kernel of the canonical map of class-groups, $CL(\mathbf{Z}[G]) \rightarrow CL(M_G)$. The definition of $D(\mathbf{Z}[G])$ is independent of choice of maximal order. When we tensor the second Cartesian square with the rationals the bottom right corner vanishes. Therefore the maximal order of $\mathbf{Q}[N_GH]$ is isomorphic to a direct sum of the maximal orders in $\mathbf{Z}[N_GH]/(\tau \pm 1)$. Therefore we have an inclusion

$$\mathrm{im}\Big(K_1\big(\mathbf{Z}/2[N_GH]/(\tau-1)\big) \xrightarrow{\delta} K_0(\mathbf{Z}[N_GH])\Big) \subseteq D(\mathbf{Z}[N_GH]).$$

This is because $\delta(x)$ must vanish under the map

$$K_0(\mathbf{Z}[N_GH]) \longrightarrow K_0\big(\mathbf{Z}[N_GH]/(\tau+1)\big) \oplus K_0\big(\mathbf{Z}[N_GH]/(\tau-1)\big)$$

but the corresponding map on the K-theory of maximal orders is an isomorphism. Note also that the canonical homomorphism

$$\mathrm{Ind}^G_{N_GH}\colon K_0(\mathbf{Z}[N_GH]) \longrightarrow K_0(\mathbf{Z}[G])$$

satisfies

$$\mathrm{Ind}^G_{N_GH}(D(\mathbf{Z}[N_GH]) \subseteq D(\mathbf{Z}[G]).$$

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By Proposition 4.3, a unit

$$\underline{\mu} \in S_{\mathbf{Z}/2,T}(G)^* = \mathrm{GL}_1\, S_{\mathbf{Z}/2,T}(G)$$

defines a class

$$[\underline{\mu}] \in K_1\big(S_{\mathbf{Z}/2,T}(G)\big) \cong K_1\Big(M_r\big(\mathbf{Z}/2[N_GH]/(\tau-1)\big)^{\mathrm{op}}\Big) \cong K_1\big(\mathbf{Z}/2[N_GH]/(\tau-1)\big),$$

since $K_1(\Lambda)$ is defined to be the abelianisation of the infinite general linear group of Λ. The second isomorphism is induced by Morita equivalence (see [7]). Since we are in a very low dimension, in the proof of Theorem 4.4, we shall need to know very little about Morita equivalence. If $z \in K_1\big(\mathbf{Z}/2[N_GH]/(\tau-1)\big)$ corresponds to $[\underline{\mu}] \in K_1\Big(M_r\big(\mathbf{Z}/2[N_GH]/(\tau-1)\big)^{\mathrm{op}}\Big)$, we shall need only the fact that the coboundary, $\delta(z) \in K_0(\mathbf{Z}[N_GH])$, is represented by the Mayer-Vietoris patching construction, using a matrix representation of $\underline{\mu}$, as described in ([5] Section 3).

The remainder of this section will be devoted to proving the following result.

THEOREM 4.4. *If $\underline{\mu} \in S_{\mathbf{Z}/2,T}(G)^*$ then, when $T = \{g_iHg_i^{-1}\ ;\ 1 \le i \le r\}$ and $r = [G : N_GH]$ as in Section 4.2,*

$$\psi(\underline{\mu}) = \mathrm{Ind}_{N_GH}^{G}\Big(\delta([\underline{\mu}]) - r\big[\mathbf{Z}[N_GH]\big]\Big) \in CL(\mathbf{Z}[G]) \subseteq K_0(\mathbf{Z}[G]).$$

PROOF. Firstly, we must find an alternative description of the $\mathbf{Z}[G]$-module, $Y(\underline{\mu})$, of Section 3.2 which is more obviously related to the boundary homomorphism, δ.

Consider the following commutative diagram of $\mathbf{Z}[G]$-modules in which the right-hand square is a pull-back which defines $X(\underline{\mu})$.

$$\begin{array}{ccccccc}
E_+ \cong 2E_+ & \longrightarrow & X(\underline{\mu}) & \longrightarrow & & & E_- \\
{\scriptstyle 1}\downarrow & & \downarrow & & & & \downarrow \\
E_+ & \xrightarrow{2} & E_+ & \longrightarrow & E_+ \otimes \mathbf{Z}/2 & \xleftarrow{\underline{\mu}} & E_- \otimes \mathbf{Z}/2
\end{array}$$

In the pullback square defining $X(\underline{\mu})$ we have identified $E_\pm \otimes \mathbf{Z}/2$ by the isomorphism induced by the unique identification of $\mathbf{Z}_\pm \otimes \mathbf{Z}/2$ so that $\underline{\mu}$ may be interpreted as an isomorphism of $\mathbf{Z}[G]$-modules in the bottom right-hand corner. Therefore $X(\underline{\mu})$ consists of pairs, $(e_+, e_-) \in E_+ \times E_-$, mapping to the same element in $E_+ \otimes \mathbf{Z}/2$. Hence $\ker\big(X(\underline{\mu}) \longrightarrow E_-\big)$ consists of all pairs, $(e_+, 0)$, in which e_+ is divisible by 2. Therefore both rows of the diagram are short exact sequences. In fact, the upper row is equivalent to the 1-extension which defines $Y(\underline{\mu})$ and therefore there is an isomorphism of $\mathbf{Z}[G]$-modules of the form

$$Y(\underline{\mu}) \cong X(\underline{\mu}).$$

To see this it suffices to calculate all the compositions of Section 3.1

$$H^1(H_i\,;\mathbf{Z}_-) \longrightarrow H^1(H_i\,;E_-) \xrightarrow{\delta} H^2(H_i\,;E_+)$$

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where δ is the coboundary associated to the upper row of the diagram. We must show that this family of homomorphisms is induced by composition with $\underline{\mu}$, in the sense of Section 3.1. However there are canonical isomorphisms of the form

$$H^1(H_i\,;E_-)\cong H^1(H_i\,;E_-\otimes\mathbf{Z}/2)\cong H^2(H_i\,;E_+\otimes\mathbf{Z}/2)\cong H^2(H_i\,;E_+)$$

in terms of which the family of coboundary compositions for the exact sequence

$$0\longrightarrow E_+\stackrel{2}{\longrightarrow}E_+\longrightarrow E_+\otimes\mathbf{Z}/2\longrightarrow 0$$

is induced by composition with the identity map. Therefore the family of coboundary compositions for the exact sequence of the lower row is induced by composition with $\underline{\mu}$ and consequently the same is true for the upper row.

Next we consider the (left) $\mathbf{Z}[G]$-modules, $E_\pm=\oplus_{i=1}^r\operatorname{Ind}_{g_iHg_i^{-1}}^G(\mathbf{Z}_\pm)$. These modules are isomorphic, respectively, to $\oplus_{i=1}^r\operatorname{Ind}_H^G(\mathbf{Z}_\pm)$ by means of the homomorphism which sends $g\otimes_{g_iHg_i^{-1}}v\in\operatorname{Ind}_{g_iHg_i^{-1}}^G(\mathbf{Z}_\pm)$ to $gg_i\otimes_H v\in\operatorname{Ind}_H^G(\mathbf{Z}_\pm)$ in the i-th summand. The isomorphism of $M_r\Big(\mathbf{Z}[N_GH]/\big(\tau-(\pm1)\big)\Big)^{\mathrm{op}}$ with $S^\pm_{\mathbf{Z},T}(G)$ sends the elementary matrix, $e_{ij}^{\lambda z}$ ($\lambda\in\mathbf{Z}$, $z\in N_GH$) to $\underline{\mu}=\{\mu(s,t,w)\}$ whose only non-zero coordinate is given by $\mu(i,j,g_izg_j^{-1})=\lambda$. This elementary matrix acts by sending $g\otimes_{g_iHg_i^{-1}}v$ to $gg_izg_j^{-1}\otimes_{g_jHg_j^{-1}}v$. Translated under these isomorphisms, $e_{ij}^{\lambda z}$ acts on $gg_i\otimes_H v$ in the i-th coordinate by sending it to $gg_iz\otimes_H v$ in the j-th coordinate.

Hence there are (left) $\mathbf{Z}[G]$-module isomorphisms of the form

$$E_\pm\cong\operatorname{Ind}_{N_GH}^G(E'_\pm)$$

where the (left) $\mathbf{Z}[N_GH]$-modules, $E'_\pm$ are equal to $\oplus_{i=1}^r\operatorname{Ind}_H^{N_GH}(\mathbf{Z}_\pm)$, respectively. The elements of these modules are considered as r-tuple row vectors with entries in $\operatorname{Ind}_H^{N_GH}(\mathbf{Z}_\pm)$ upon which an element of $S^\pm_{\mathbf{Z},T}(G)$, considered as an $r\times r$ matrix via Proposition 4.3, acts via right matrix multiplication. The action of $\mathbf{Z}[N_GH]$ is by means of left multiplication on each coordinate.

Acting via right multiplication as described above, a unit

$$\underline{\mu}\in S_{\mathbf{Z}/2,T}(G)^*\cong\mathrm{GL}_r(\mathbf{Z}/2[N_GH]/H)$$

defines a (left) $\mathbf{Z}/2[N_GH]/H$-automorphism, $\underline{\mu}\colon E'_-\otimes\mathbf{Z}/2\stackrel{\cong}{\longrightarrow}E'_-\otimes\mathbf{Z}/2$ which defines a class

$$[\underline{\mu}]\in K_1(\mathbf{Z}/2[N_GH]/H)=\mathrm{GL}_\infty(\mathbf{Z}/2[N_GH]/H)_{ab}.$$

By definition of the coboundary, δ, in the K-theory Mayer-Vietoris sequence ([5] Section 3)

$$\delta([\underline{\mu}])=[X'(\underline{\mu})]\in K_0(\mathbf{Z}[N_GH])$$

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where $X'(\underline{\mu})$ is defined by a pull-back in the same manner as $X(\underline{\mu})$ but with G and $E_{\pm}$ replaced by $N_G H$ and $E'_{\pm}$. From the preceding discussion, $X(\underline{\mu}) = \mathrm{Ind}_{N_G H}^{G}\big(X'(\underline{\mu})\big)$, so that in $\mathcal{CL}(\mathbf{Z}[G]) \subset K_0(\mathbf{Z}[G])$

$$\begin{aligned}\psi(\underline{\mu}) &= Y(\underline{\mu}) - r\mathbf{Z}[G] \\ &= X(\underline{\mu}) - r\mathbf{Z}[G] \\ &= \mathrm{Ind}_{N_G H}^{G}\big(X'(\underline{\mu}) - r\mathbf{Z}[N_G H]\big) \\ &= \mathrm{Ind}_{N_G H}^{G}\big(\delta([\underline{\mu}]) - r\mathbf{Z}[N_G H]\big)\end{aligned}$$

as required. ∎

The following result is immediate from the discussion concerning the subgroup, $D(\mathbf{Z}[G])$, together with the fact that the class-group of the group-ring of the group of order two is trivial.

COROLLARY 4.5. *For any G and $\mathcal{T}$ the homomorphism of Proposition 3.3*

$$\psi \colon \frac{S_{\mathbf{Z}/2,\mathcal{T}}(G)^*}{S^+_{\mathbf{Z},\mathcal{T}}(G)^*} \longrightarrow \mathcal{CL}(\mathbf{Z}[G])$$

takes values in the subgroup $D(\mathbf{Z}[G])$.

Furthermore, if the $N_G H = H$ for each $H \in \mathcal{T}$ then ψ is trivial.

4.6. *An application.* In [1] a Chinburg invariant, $\Omega_n(L/K) \in \mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big)$ is associated to any Galois extension of number fields, L/K with group $G(L/K)$. When $n = 1$ this invariant is constructed as the Euler characteristic of a 2-extension of finitely generated $\mathbf{Z}[G(L/K)]$-modules derived from the Galois modules structure of the algebraic K-groups of rings of S-integers, $\mathcal{O}_{L,S}$, in dimensions 2 and 3. In [2] we evaluated some quaternionic examples of another construction, given in ([9] Chapter 7) for the totally real case, of an invariant $\Omega_1(L/K, 3) \in \mathcal{CL}\big(\mathbf{Z}[G(L/K)]\big)$.

In [3] we extended $\Omega_1(L/K, 3)$ to all L/K and showed that $\Omega_1(L/K) = \Omega_1(L/K, 3)$. Corollary 4.5 and our constructions with Hecke algebras yield an alternative proof of this equality. In [3] it was shown that there exists a commutative diagram of 2-extensions of the type which appears in Section 1, in which (i) $\mathcal{T}$ consists of the set of non-trivial decomposition groups at infinity for L/K (ii) the Euler characteristic of the upper 2-extension defines $\Omega_1(L/K)$ and (iii) that of the lower one defines $\Omega_1(L/K, 3)$.

From equation (1) of Section 1, we shall explain how to prove that $\Omega_1(E/\mathbf{Q}) = \Omega_1(E/\mathbf{Q}, 3)$ when E is totally complex—equality in the general case following at once, by naturality. The idea is that, in the limit, $N_{G(E/\mathbf{Q})}\langle c\rangle/\langle c\rangle$ is trivial and so Corollary 4.5 should yield the result. Unfortunately, for each $E/\mathbf{Q}$, the quotient, $N_{G(E/\mathbf{Q})}\langle c\rangle/\langle c\rangle$, may be non-trivial. Therefore we have to proceed slightly differently.

Suppose $E/\mathbf{Q}$ and $M/\mathbf{Q}$ are two totally complex Galois extensions with $E \subseteq M$ and Galois groups

$$G(M/E) \lhd G(M/\mathbf{Q}) \xrightarrow{\pi} G(E/\mathbf{Q}) \cong G(M/\mathbf{Q})/G(M/E).$$

Each of the 2-extensions of $\mathbf{Z}[G(M/\mathbf{Q})]$-modules, defining $\Omega_1(M/\mathbf{Q})$ and $\Omega_1(M/\mathbf{Q},3)$, is natural with respect to passage to quotients. This sort of naturality means, for example, that if

$$X_1 \longrightarrow A_1 \longrightarrow B_1 \longrightarrow Y_1$$

is the 2-extension which defines $\Omega_1(M/\mathbf{Q})$ then the associated 2-extension of $\mathbf{Z}[G(E/\mathbf{Q})]$-modules

$$X_1^{G(M/E)} \longrightarrow A_1^{G(M/E)} \longrightarrow B_1^{G(M/E)} \longrightarrow (Y_1)_{G(M/E)}$$

defines $\Omega_1(E/\mathbf{Q})$. In fact, the diagram of Section 1 involving these 2-extensions is natural in this sense, with respect to passage to quotient groups.

The diagram of Section 1 relating the 2-extensions for $\Omega_1(M/\mathbf{Q})$ and $\Omega_1(M/\mathbf{Q},3)$ defined an element

$$\omega_{M/\mathbf{Q}} \in \mathrm{Ext}^3_{\mathbf{Z}[G(M/\mathbf{Q})]}(E_-,E_+) \cong \mathrm{Ext}^1_{\mathbf{Z}[G(M/\mathbf{Q})]}(E_-,E_+) \cong S_{\mathbf{Z}/2,T_M}\big(G(M/\mathbf{Q})\big)$$

where T_M is the set of all conjugates of complex conjugation, c, in $G(M/\mathbf{Q})$. Naturality implies that these elements fit together to define

$$\omega = \lim_{\leftarrow} \omega_{M/\mathbf{Q}} \in \lim_{\leftarrow} S_{\mathbf{Z}/2,T_M}\big(G(M/\mathbf{Q})\big).$$

Returning to $\mathbf{Q} \subset E \subseteq M$, the canonical homomorphism

$$S_{\mathbf{Z}/2,T_M}\big(G(M/\mathbf{Q})\big) \longrightarrow S_{\mathbf{Z}/2,T_E}\big(G(E/\mathbf{Q})\big)$$

is induced by considering an element of $S_{\mathbf{Z}/2,T_M}\big(G(M/\mathbf{Q})\big)$ as lifting to a $\mathbf{Z}[G(M/\mathbf{Q})]$-endomorphism of E_+ and sending it to the reduction modulo 2 of the induced $\mathbf{Z}[G(E/\mathbf{Q})]$-module endomorphism of $E_+^{G(M/E)}$, considered as an element of $S_{\mathbf{Z}/2,T_E}\big(G(E/\mathbf{Q})\big)$.

Let $G(M/E)=\{x_1,\ldots,x_r\}$ and let $G(E/\mathbf{Q})=\{y_1'c^\epsilon,\ldots,y_s'c^\epsilon\ ;\ \epsilon=0,1\}$, where c denotes complex conjugation. Lifting each y_j' to $y_j \in G(M/\mathbf{Q})$ we may set

$$T_M=\{\langle x_iy_jcy_j^{-1}x_i^{-1}\rangle \mid 1\le i\le r, 1\le j\le s\}.$$

Hence T_M is the set of all conjugates of $\langle c\rangle$ in $G(M/\mathbf{Q})$. By Proposition 4.3, we have an isomorphism of the form

$$S^+_{\mathbf{Z},T_M}\big(G(M/\mathbf{Q})\big) \cong M_{rs}(\mathbf{Z}[N_{G(M/\mathbf{Q})}\langle c\rangle/\langle c\rangle])^{\mathrm{op}}.$$

Set $H_{i,j}=\langle x_iy_jcy_j^{-1}x_i^{-1}\rangle$ so that associated pairs of suffices, (i,j), index the rows and columns of these matrices.

With this notation, the canonical homomorphism of Hecke algebras, induced by passing to $G(M/E)$-fixed points, corresponds to the the ring homomorphism

$$M_{rs}(\mathbf{Z}[N_{G(M/\mathbf{Q})}\langle c\rangle/\langle c\rangle])^{\mathrm{op}} \longrightarrow M_s(\mathbf{Z}[N_{G(E/\mathbf{Q})}\langle c\rangle/\langle c\rangle])^{\mathrm{op}}$$

which sends the $\big((i,j),(i_1,j_1)\big)$-th entry to the (j,j_1)-th entry by the map induced by $\pi\colon G(M/\mathbf{Q}) \longrightarrow G(E/\mathbf{Q})$.

However, the inverse limit

$$N = \lim_{\longleftarrow} N_{G(M/\mathbf{Q})}\langle c\rangle/\langle c\rangle$$

is trivial. For if not then the compositum of all the $M^{\langle c\rangle}$ would be a real closed field, F, and F/F^N would be a non-trivial Galois extension. However, real closed fields do not have any non-trivial Galois automorphisms (*cf.* [6] pp. 392–398), since the roots in F of a minimum polynomial over F^N are ordered and the automorphism must preserve the ordering.

Therefore, for any given $E/\mathbf{Q}$ there exists M such that the image of the homomorphism

$$S_{\mathbf{Z}/2,T_M}\big(G(M/\mathbf{Q})\big) \longrightarrow S_{\mathbf{Z}/2,T_E}\big(G(E/\mathbf{Q})\big) \cong M_s(\mathbf{Z}/2[N_{G(E/\mathbf{Q})}\langle c\rangle/\langle c\rangle])^{\mathrm{op}}$$

lies in the subring of "constants", $M_s(\mathbf{Z}/2)^{\mathrm{op}}$. The images under ψ of Section 3.2 of such elements are trivial in the class-group of the group-ring. Therefore $\Omega_1(E/\mathbf{Q}) = \Omega_1(E/\mathbf{Q},3)$, as claimed.

However, a general Galois extension, L/K, may be embedded in one of the form $E/\mathbf{Q}$ in which E is totally complex and E/L is Galois. Each of the invariants is natural in the sense that $\Omega_1(E/\mathbf{Q})$ and $\Omega_1(E/\mathbf{Q},3)$ map to $\Omega_1(L/K)$ and $\Omega_1(L/K,3)$, respectively, under the homomorphism

$$CL\big(\mathbf{Z}[G(E/\mathbf{Q})]\big) \longrightarrow CL\big(\mathbf{Z}[G(E/K)]\big) \longrightarrow CL\big(\mathbf{Z}[G(L/K)]\big),$$

which completes the proof. ∎

REMARK 4.7. Representing ψ in the Hom-description The Hom-description represents the class-group of $\mathbf{Z}[G]$ as a quotient of the Galois equivariant, idèlic-valued functions of the (complex) representation ring, $R(G)$, of G (see [8] Section 4.2). The function which represents $\psi(\underline{\mu})$ is trivial at all places except those above the prime 2. It suffices to give the Hom-description at $p = 2$ in the case of Theorem 4.4, when all the H_i are conjugate to $H = \langle\tau\rangle$. In this case we may lift $\underline{\mu}$ to a 2-adic unit, $\underline{\mu}' \in S^{+}_{\mathbf{Z}_2,T}(G)^*$, which we may interpret as as an element of $\mathrm{GL}_r(\mathbf{Z}_2[N_GH/H])$. Therefore we have a 2-adic valued determinantal (see [8] Section 4.2) function given by, $\mathrm{Det}(\underline{\mu}')$ on $R(N_GH/H)$. Therefore there is a 2-adic valued function on $R(N_GH)$ which sends an irreducible, χ, to 1 if χ is non-trivial on τ, the generator of H, and sends it to $\mathrm{Det}(\underline{\mu}')(\chi)$ otherwise (in this case χ is inflated from N_GH/H). Composing this homomorphism with the restriction map from $R(G)$ to $R(N_GH)$ gives the 2-adic part of the Hom-description of $\psi(\underline{\mu})$.

For example, when $G = Q_8$ in Section 4.1 and $\underline{\mu}$ corresponds to $1+x+y$ the associated 2-adic function is trivial on irreducible representations except the trivial one and there it takes the value 3. This is the Hom-description of the generator (see [8] Section 5.2).

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