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A BERNSTEIN-SCHOENBERG TYPE OPERATOR: SHAPE PRESERVING AND LIMITING BEHAVIOUR

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ABSTRACT. Using a new B-spline basis due to Dahmen, Micchelli and Seidel, we construct a univariate spline approximation operator of Bernstein-Schoenberg type. We show that it shares all the shape preserving properties of the usual Bernstein-Schoenberg operator and we derive a Voronovskaya type asymptotic error estimate.

1. **Introduction.** In [10] Schoenberg introduced a spline approximation operator which generalised the Bernstein polynomial and which we shall refer to as the Bernstein-Schoenberg operator. Like the Bernstein polynomial, this operator is 'variation diminishing' and therefore has certain 'shape preserving' properties, (see also [4]). The asymptotic error estimate for Bernstein polynomials due to Voronovskaya, (see [6]), has also been extended to the Bernstein-Schoenberg operator by Lee and the authors [3], extending work of Marsden and Riemenschneider, ([7], [8], [9]).

In extending the Bernstein-Schoenberg operator to higher dimensions, using simplex splines, Goodman and Lee [2] also generalised the univariate operator. More recently a different construction has been proposed [1] for spaces spanned by simplex splines which seems more natural and also leads to an approximation operator of Bernstein-Schoenberg type. In this paper, we study the univariate case of this operator, which also generalises Bernstein polynomials but is distinct from the Bernstein-Schoenberg operator. After defining this operator in this section, we shall show in Section 2 that it is variation diminishing and shares all the shape preserving properties of the Bernstein-Schoenberg operator, while in Section 3 we derive an asymptotic estimate of Voronovskaya type.

We first specialise to one dimension some of the definitions and results in [1]. Let $T_0 = \{t_0^0, \dots, t_n^0\}$ and $T_1 = \{t_0^1, \dots, t_n^1\}$ be sequences of numbers which we shall call *clouds* of *knots*. We assume $T_0 < T_1$, *i.e.*, $t_i^0 < t_j^1$ for all $i,j = 0, \dots, n$. We then define polynomials P_i^k of degree k, $k = 0, \dots, n$, $i = 0, \dots, k$, recursively as follows:

$$P_0^0(x) = 1 \quad \text{and for } k = 1, \dots, n, \ i = 0, \dots, k,$$

$$P_i^k(x) = \frac{(x - t_{k-i}^0)}{(t_{i-1}^1 - t_{k-i}^0)} P_{i-1}^{k-1}(x) + \frac{(t_i^1 - x)}{(t_i^1 - t_{k-1-i}^0)} P_i^{k-1}(x), \tag{1.1}$$

where P_{-1}^{k-1} and P_k^{k-1} are taken to be zero.

We can extend this recurrence relation to the polar forms p_i^k of P_i^k as follows:

$$p_0^0 = 1 \tag{1.2}$$

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and for $k = 1, \ldots, n$, $i = 0, \ldots, k$,

$$(1.3)\quad p_i^k(x_1, \ldots, x_k) = \frac{(x_k - t_{k-i}^0)}{(t_{i-1}^1 - t_{k-i}^0)} p_{i-1}^{k-1}(x_1, \ldots, x_{k-1}) + \frac{(t_i^1 - x_k)}{(t_i^1 - t_{k-1-i}^0)} p_i^{k-1}(x_1, \ldots, x_{k-1}).$$

The polynomials P_i^k, $i = 0, \ldots, k$ are linearly independent. Given any polynomial P of degree $k \geq 1$ with polar form p, then

$$(1.4)\qquad P = \sum_{i=0}^{k} a_i P_i^k,$$

where

$$(1.5)\qquad a_i = p(t_0^0, \ldots, t_{k-1-i}^0, t_0^1, \ldots, t_{i-1}^1).$$

For any i, the B-spline B_i^k with knots $t_0^0, \ldots, t_{k-i}^0, t_0^1, \ldots, t_i^1$ can be so normalised that it coincides with P_i^k on $[\max T_0, \min T_1]$. We call B_i^k, $i = 0, \ldots, k$, the *B-splines* of degree k corresponding to the clouds T_0 and T_1.

Now take a sequence of clouds $T_i = \{t_0^i, \ldots, t_n^i\}$, $i = 0, \ldots, m$, with $T_{i-1} < T_i$, $i = 1, \ldots, m$, and $t_0^0 = \cdots = t_n^0 = a$, $t_0^m = \cdots = t_n^m = b$. For $i = 1, \ldots, m$ we denote by $B_{i,j}^k$, $j = 0, \ldots, n$, the B-splines of degree k corresponding to the clouds T_{i-1} and T_i. Thus $B_{i,j}^k$ has knots at $t_0^{i-1}, \ldots, t_{k-j}^{i-1}, t_0^i, \ldots, t_j^i$. The B-splines $B_{i,j}^k$, $j = 0, \ldots, k$, are linearly independent on $(\max T_{i-1}, \min T_i)$, whereas $B_{\ell,j}^k$ vanishes on this interval for $j = 0, \ldots, k$ and $\ell \neq i$. Thus the B-splines $B_{i,j}^k$, $i = 1, \ldots, m$, $j = 0, \ldots, k$ are linearly independent and so form a basis for the space of splines of degree k with knots at t_j^i, $i = 0, \ldots, m$, $j = 0, \ldots, k$. (When we refer to the space of splines with given knots, we always mean those splines which vanish outside the convex hull of the set of knots.)

Now take any polynomial P of degree $k \geq 1$ with polar form p. Writing

$$(1.6)\qquad a_{i,j} := p(t_0^{i-1}, \ldots, t_{k-1-j}^{i-1}, t_0^i, \ldots, t_{j-1}^i),$$

consider the function

$$s := \sum_{i=1}^{m} \sum_{j=0}^{k} a_{i,j} B_{i,j}^k.$$

Then s is a spline function with knots $\{t_j^i : i = 0, \ldots, m, j = 0, \ldots, k\}$ which coincides with P on the intervals $[\max T_{i-1}, \min T_i]$, $i = 1, \ldots, m$. Now $P = \sum_{i=1}^m \sum_{j=0}^k b_{i,j} B_{i,j}^k$ for some coefficients $b_{i,j}$. For each i, $1 \leq i \leq m$, the B-splines $B_{i,j}^k$ are linearly independent on $(\max T_{i-1}, \min T_i)$ and so $b_{i,j} = a_{i,j}$, $j = 0, \ldots, k$. Thus we have

$$(1.7)\qquad P(x) = \sum_{i=1}^{m} \sum_{j=0}^{k} a_{i,j} B_{i,j}^k(x), \quad a \leq x \leq b.$$

In particular (1.6) and (1.7) give, for $k = n$,

$$(1.8)\qquad 1 = \sum_{i=1}^{m} \sum_{j=0}^{n} B_{i,j}^n(x), \quad a \leq x \leq b,$$

$$(1.9)\qquad x = \sum_{i=1}^{m} \sum_{j=0}^{n} \xi_{i,j} B_{i,j}^n(x), \quad a \leq x \leq b,$$

where

$$\xi_{i,j} := \frac{1}{n}(t_0^{i-1} + \cdots + t_{n-1-j}^{i-1} + t_0^i + \cdots + t_{j-1}^i). \tag{1.10}$$

We now define an operator S by

$$Sf(x) := \sum_{i=1}^{m} \sum_{j=0}^{n} f(\xi_{i,j}) B_{i,j}^n(x), \quad a \le x \le b, \tag{1.11}$$

for any function f on $[a, b]$.

By (1.8) and (1.9) we have $Sf = f$ for all linear functions f. Clearly S is a positive operator, *i.e.*, $f \ge 0 \implies Sf \ge 0$. Since $\xi_{1,0} = a$, $\xi_{m,n} = b$, $B_{i,j}^n(a) = 0$ except for $i = j = 0$ and $B_{i,j}^n(b) = 0$ except for $i = m, j = n$, we have

$$Sf(a) = f(a), \quad Sf(b) = f(b). \tag{1.12}$$

The operator S is similar to the Bernstein-Schoenberg operator but distinct from it, coinciding only in the very special case $t_0^i = \cdots = t_n^i$ for $i = 1, \ldots, m-1$, when S reduces to disjoint Bernstein operators.

2. **Shape properties of** S.

THEOREM 1. *If f is convex on $[a, b]$, then $Sf \ge f$.*

PROOF. Take $a \le x \le b$. By (1.9)

$$f(x) = f\Big(\sum_{i=1}^{m} \sum_{j=0}^{n} B_{i,j}^n(x)\xi_{i,j}\Big) \le \sum_{i=1}^{m} \sum_{j=0}^{n} B_{i,j}^n(x) f(\xi_{i,j}),$$

using the convexity of f, (1.8) and the fact that $B_{i,j}^n(x) \ge 0$, $i = 1, \ldots, m, j = 0, \ldots, n$. It follows from (1.11) that $f(x) \le Sf(x)$. ∎

We now consider what happens when we insert an extra cloud of knots. Let $T_i = \{t_0^i, \ldots, t_n^i\}$, $i = 0, 1$, be clouds of knots and $B_j^k, j = 0, \ldots, k$, the corresponding B-splines of degree k, $k = 0, \ldots, n$. We insert a new cloud $T = \{t_0, \ldots, t_n\}$ with $T_0 < T < T_1$. For $k = 0, \ldots, n$, let $B_{0,j}^k, j = 0, \ldots, k$, be the B-splines of degree k corresponding to T_0 and T, and $B_{1,j}^k, j = 0, \ldots, k$, the B-splines corresponding to T and T_1.

LEMMA 1. *For $i = 0, \ldots, n$,*

$$B_i^n = \sum_{j=0}^{n} \alpha_{i,j} B_{0,j}^n + \sum_{j=0}^{n} \beta_{i,j} B_{1,j}^n, \tag{2.1}$$

where $\alpha_{i,j} \ge 0$, $\beta_{i,j} \ge 0$, $i,j = 0, \ldots, n$, and

$$\sum_{i=0}^{n} \alpha_{i,j} = \sum_{i=0}^{n} \beta_{i,j} = 1, \quad j = 0, \ldots, n. \tag{2.2}$$

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PROOF. We have seen that the B-splines $B^n_{i,j}$, $i = 0, 1, j = 0, \ldots, n$, form a basis for all splines of degree n with knots in T_0, T and T_1. Thus (2.1) holds for some unique $\alpha_{i,j}$, $\beta_{i,j}$, $i,j = 0, \ldots, n$.

For $i = 0, \ldots, n$, we denote by P^n_i, as before, the polynomial which coincides with B^n_i on $[\max T_0, \min T_1]$. The polar form p^n_i of P^n_i satisfies the recurrence relation (1.2), (1.3). Now by (1.4), (1.5) and (2.1) we see that for $i,j = 0, \ldots, n$,

$$\alpha_{i,j} = p^n_i(t^0_0, \ldots, t^0_{n-j-1}, t_0, \ldots, t_{j-1}), \tag{2.3}$$

$$\beta_{i,j} = p^n_i(t_0, \ldots, t_{n-j-1}, t^1_0, \ldots, t^1_{j-1}). \tag{2.4}$$

We shall prove by induction on k that for $k = 0, \ldots, n$,

$$p^k_i(t^0_0, \ldots, t^0_{k-j-1}, t_0, \ldots, t_{j-1}) \geq 0, \quad i,j = 0, \ldots, k. \tag{2.5}$$

If $k = 0$, this follows trivially from (1.2). Assume that it is true for $k-1$. By (1.3) we have for $i = 0, \ldots, k, j = 1, \ldots, k$,

$$\begin{aligned} p^k_i(t^0_0, \ldots, t^0_{k-j+1}, t_0, \ldots, t_{j-1}) = {} & \frac{(t_{j-1} - t^0_{k-i})}{(t^1_{i-1} - t^0_{k-i})} p^{k-1}_{i-1}(t^0_0, \ldots, t^0_{k-j-1}, t_0, \ldots, t_{j-2}) \\ & + \frac{(t^1_i - t_{j-1})}{(t^1_i - t^0_{k-1-i})} p^{k-1}_i(t^0_0, \ldots, t^0_{k-j-1}, t_0, \ldots, t_{j-2}). \end{aligned}$$

Since $T_0 < t_{j-1} < T_1$ for $j = 1, \ldots, k$, the induction hypothesis (2.5) for $i = 0, \ldots, k$, $j = 1, \ldots, k$ is verified. Now from (1.4) and (1.5) for $P = P^k_i$ we see that

$$p^k_i(t^0_0, \ldots, t^0_{k-1}) = \delta_{i,0}$$

and so (2.5) also holds for $j = 0$, $i = 0, \ldots, k$.

Thus we have established (2.5) for $k = 0, \ldots, n$. Putting $k = n$ and recalling (2.3) shows that $\alpha_{i,j} \geq 0$ for $i,j = 0, \ldots, n$. Similarly we can show that $\beta_{i,j} \geq 0$ for $i,j = 0, \ldots, n$.

Finally we note from (1.8) and (2.1) that on $[\max T_0, \min T_1]$

$$1 = \sum_{i=0}^n B^n_i = \sum_{j=0}^n B^n_{0,j} \sum_{i=0}^n \alpha_{i,j} + \sum_{j=0}^n B^n_{1,j} \sum_{i=0}^n \beta_{i,j}.$$

But on $[\max T_0, \min T]$, $\sum_{i=0}^n B^n_{0,j} = 1$ and $B^n_{1,j} = 0, j = 0, \ldots, n$. Thus $\sum_{i=0}^n \alpha_{i,j} = 1$. Similarly $\sum_{i=0}^n \beta_{i,j} = 1$. ∎

We return now to the sequence of clouds $T_i = \{t^i_0, \ldots, t^i_n\}$, $i = 0, \ldots, m$, and the operator S defined by (1.11). For some r, $1 \leq r \leq m$, we shall insert a new cloud $T = \{t_0, \ldots, t_n\}$ with $T_{r-1} < T < T_r$. This gives a new sequence of clouds

$$\tilde{T}_i = \begin{cases} T_i, & 0 \leq i \leq r-1, \\ T, & i = r, \\ T_{i-1}, & r+1 \leq i \leq m+1. \end{cases}$$

We shall denote by $\tilde{S}$ the operator corresponding to S in (1.11) for the sequence $\tilde{T}_0, \ldots, \tilde{T}_{m+1}$.

THEOREM 2. *If f is convex on $[a,b]$, then $\tilde{S}f \leq Sf$.*

PROOF. For $i = 1,\ldots,m+1$, let $\tilde{B}^n_{i,j}, j = 0,\ldots,n$, be the B-splines corresponding to $\tilde{T}_{i-1}$ and $\tilde{T}_i$. Then by Lemma 1,

$$B^n_{r,j} = \sum_{k=0}^{n} \alpha_{j,k}\tilde{B}^n_{r,k} + \sum_{k=0}^{n} \beta_{j,k}\tilde{B}^n_{r+1,k}, \quad j = 0,\ldots,n,$$

where $\alpha_{j,k} \geq 0$, $\beta_{j,k} \geq 0$, $j,k = 0,\ldots,n$, and $\sum_{j=0}^{n} \alpha_{j,k} = \sum_{j=0}^{n} \beta_{j,k} = 1$. So for any function g on $[a,b]$ we have

$$\sum_{j=0}^{n} g(\xi_{r,j})B^n_{r,j} = \sum_{k=0}^{n} \tilde{B}^n_{r,k} \sum_{j=0}^{n} \alpha_{j,k} g(\xi_{r,j}) + \sum_{k=0}^{n} \tilde{B}^n_{r+1,k} \sum_{j=0}^{n} \beta_{j,k} g(\xi_{r,j}),$$

and so

$$Sg = \sum_{\substack{i=1\\ i\neq r}}^{m} \sum_{j=0}^{n} g(\xi_{i,j})B^n_{i,j} + \sum_{j=0}^{n} \tilde{B}^n_{r,j} \sum_{k=0}^{n} \alpha_{k,j} g(\xi_{r,k}) + \sum_{j=0}^{n} \tilde{B}^n_{r+1,j} \sum_{k=0}^{n} \beta_{k,j} g(\xi_{r,k}). \tag{2.6}$$

Putting $g(x) = x$ and recalling that in this case $Sg = g$, we have

$$x = \sum_{\substack{i=1\\ i\neq r}}^{m} \sum_{j=0}^{n} \xi_{i,j}B^n_{i,j}(x) + \sum_{j=0}^{n} \tilde{B}^n_{r,j}(x) \sum_{k=0}^{n} \alpha_{k,j}\xi_{r,k} + \sum_{j=0}^{n} \tilde{B}_{r+1,j}(x) \sum_{k=0}^{n} \beta_{k,j}\xi_{r,k}, \quad a \leq x \leq b. \tag{2.7}$$

Now the definition of $\tilde{S}g$ is of the form

$$\tilde{S}g = \sum_{\substack{i=1\\ i\neq r}}^{m} \sum_{j=0}^{n} g(\xi_{i,j})B^n_{i,j} + \sum_{j=0}^{n} g(\tilde{\xi}_{r,j})\tilde{B}^n_{r,j} + \sum_{j=0}^{n} g(\tilde{\xi}_{r+1,j})\tilde{B}^n_{r+1,j}. \tag{2.8}$$

Putting $g(x) = x$ gives

$$x = \sum_{\substack{i=1\\ i\neq r}}^{m} \sum_{j=0}^{n} \xi_{i,j}B^n_{i,j}(x) + \sum_{j=0}^{n} \tilde{\xi}_{r,j}\tilde{B}^n_{r,j}(x) + \sum_{j=0}^{n} \tilde{\xi}_{r+1,j}\tilde{B}^n_{r+1,j}(x), \quad a \leq x \leq b. \tag{2.9}$$

Comparing (2.7) and (2.9) and recalling the linear independence of the B-splines gives

$$\tilde{\xi}_{r,j} = \sum_{k=0}^{n} \alpha_{k,j}\xi_{r,k}, \quad \tilde{\xi}_{r+1,j} = \sum_{k=0}^{n} \beta_{k,j}\xi_{r,k}.$$

Thus if f is a convex function,

$$f(\tilde{\xi}_{r,j}) \leq \sum_{k=0}^{n} \alpha_{k,j} f(\xi_{r,k}),$$

$$f(\tilde{\xi}_{r+1,j}) \leq \sum_{k=0}^{n} \beta_{k,j} f(\xi_{r,k}).$$

By (2.6) and (2.8) with $g = f$ we then have $\tilde{S}f \leq Sf$. ∎

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We now return to the situation of Lemma 1 in which we insert a new cloud $T = \{t_0, \ldots, t_n\}$ between T_0 and T_1. From (2.1) we see that for constants $a_0, \ldots, a_n$,

$$\sum_{i=0}^{n} a_i B_i^n = \sum_{i=0}^{n} a_i \sum_{j=0}^{n} \{\alpha_{i,j} B_{0,j}^n + \beta_{i,j} B_{1,j}^n\}$$
$$= \sum_{j=0}^{n} B_{0,j}^n \sum_{i=0}^{n} \alpha_{i,j} a_i + \sum_{j=0}^{n} B_{1,j}^n \sum_{i=0}^{n} \beta_{i,j} a_i.$$

Thus

$$\sum_{i=0}^{n} a_i B_i^n = \sum_{i=0}^{n} a_{0,i} B_{0,i}^n + \sum_{i=0}^{n} a_{1,i} B_{1,i}^n,$$

where

$$[a_{0,0}, \ldots, a_{0,n}, a_{1,0}, \ldots, a_{1,n}]^T = A[a_0, \ldots, a_n]^T,$$

where the matrix $A = (A_{i,j})_{i=0}^{2n+1}\,{}_{j=0}^{n}$ is defined by

$$A_{i,j} = \begin{cases} \alpha_{j,i}, & i = 0, \ldots, n \\ \beta_{j,i-n-1}, & i = n+1, \ldots, 2n+1. \end{cases} \tag{2.10}$$

LEMMA 2. *If $t_0 = \cdots = t_n$, then the matrix A defined by (2.10) is totally positive.*

PROOF. Suppose that $t_0 = \cdots = t_n = t$. As in the proof of Lemma 1 we recall that the polar forms p_i^n, $i = 0, \ldots, n$, are given by the recurrence relation (1.2), (1.3). Extending (2.3) and (2.4) we define for $k = 0, \ldots, n$, $i, j = 0, \ldots, k$,

$$\alpha_{i,j}^k = p_i^k(t_0^0, \ldots, t_{k-j-1}^0, t_0, \ldots, t_{j-1}), \tag{2.11}$$
$$\beta_{i,j}^k = p_i^k(t_0, \ldots, t_{k-j-1}, t_0^1, \ldots, t_{j-1}^1). \tag{2.12}$$

From (1.3), (2.11) and (2.12) we see that for $k = 1, \ldots, n$, $i = 0, \ldots, k$,

$$\alpha_{i,j}^k = \gamma_i^k \alpha_{i-1,j-1}^{k-1} + \delta_i^k \alpha_{i,j-1}^{k-1}, \quad j = 1, \ldots, k, \tag{2.13}$$
$$\beta_{i,j}^k = \gamma_i^k \beta_{i-1,j}^{k-1} + \delta_i^k \beta_{i,j}^{k-1}, \quad j = 0, \ldots, k-1, \tag{2.14}$$

where

$$\gamma_i^k = \frac{t - t_{k-i}^0}{t_{i-1}^1 - t_{k-i}^0}, \quad i = 1, \ldots, k,$$
$$\delta_i^k = \frac{t_i^1 - t}{t_i^1 - t_{k-1-i}^0}, \quad i = 0, \ldots, k-1.$$

Now for $k = 0, \ldots, n$, we define a matrix $A^k = (A_{i,j}^k)_{i=0}^{2k+1}\,{}_{j=0}^{k}$ by

$$A_{i,j}^k = \begin{cases} \alpha_{j,i}^k, & i = 0, \ldots, k, \\ \beta_{j,i-k-1}^k, & i = k+1, \ldots, 2k+1. \end{cases}$$

We shall prove by induction on k that A^k is totally positive. For $k = 0$, the 2×1 matrix A^0 has entries equal to 1 and so A^0 is totally positive. Take $k \geq 1$ and assume that A^{k-1} is totally positive. From (2.13) and (2.14) we deduce that

$$A_{i,j}^k = A_{i-1,j-1}^{k-1} \gamma_j^k + A_{i-1,j}^{k-1} \delta_j^k, \quad i = 1, \ldots, 2k, j = 0, \ldots, k, \tag{2.15}$$

while from (1.4) and (1.5) we deduce that

$$A^k_{0,j} = p^k_j(t^0_0,\ldots,t^0_{k-1}) = \delta_{j,0}, \tag{2.16}$$

$$A^k_{2k+1,j} = p^k_j(t^1_0,\ldots,t^1_{k-1}) = \delta_{j,k+1}. \tag{2.17}$$

Let $B = (b_{i,j})_{i=0}^{k-1}{}_{j=0}^{k}$ be defined by

$$b_{i,j} = \begin{cases} \delta^k_j, & j = i, \\ \gamma^k_j, & j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\gamma^k_j \geq 0, \delta^k_j \geq 0$, B is totally positive. Now from (2.15), the matrix $(A^k_{i+1,j})_{i=0}^{2k-1}{}_{j=0}^{k}$ is equal to $A^{k-1}B$ and, since A^{k-1} is totally positive, $A^{k-1}B$ is totally positive. From (2.16) and (2.17) it follows that A^k is totally positive, which completes the proof by induction.

From (2.3), (2.4), (2.11), (2.12), we see that $A^n = A$ and thus A is totally positive. ∎

LEMMA 3. *Let $T_i = \{t^i_0,\ldots,t^i_n\}$, $i = 0,1$, be clouds with $T_0 < T_1$ and let B^n_i, $i = 0,\ldots,n$, be the corresponding B-splines. For points $\tau_1,\ldots,\tau_r$ with $\max T_0 < \tau_1 < \cdots < \tau_r < \min T_1$, the matrix $\big(B^n_j(\tau_i)\big)_{i=1}^{r}{}_{j=0}^{n}$ is totally positive.*

PROOF. For $i = 1,\ldots,r$, define a cloud $U_i = \{u^i_0,\ldots,u^i_n\}$ by $u^i_j = \tau_i, j = 0,\ldots,n$. Writing $U_0 = T_0$, $U_{r+1} = T_1$, we let $B^n_{i,j}, j = 0,\ldots,n$ be the B-splines associated with U_{i-1} and U_i, for $i = 1,\ldots,r+1$. For any constants $a_0,\ldots,a_n$ we can write

$$\sum_{j=0}^{n} a_j B^n_j = \sum_{i=1}^{r+1}\sum_{j=0}^{n} a_{i,j} B^n_{i,j},$$

where by repeated application of Lemma 2,

$$[a_{1,0,\ldots}, a_{1,n}, a_{2,0},\ldots,a_{2,n},\ldots,a_{r+1,0},\ldots,a_{r+1,n}]^T = C[a_0,\ldots,a_n]^T, \tag{2.18}$$

where the matrix $C = (C_{i,j})_{i=1}^{(r+1)(n+1)}{}_{j=0}^{n}$ is totally positive. Now for $i = 1,\ldots,r$, $B^n_{i,n}$ has a knot of multiplicity $n+1$ at τ_i and thus

$$a_{i,n} = \sum_{j=0}^{n} a_j B^n_j(\tau_i).$$

Comparing with (2.18) shows that

$$B^n_j(\tau_i) = C_{i(n+1),j}, \quad i = 1,\ldots,r, j = 0,\ldots,n,$$

so the matrix $\big(B^n_j(\tau_i)\big)_{i=1}^{r}{}_{j=0}^{n}$ is a sub-matrix of C and so is totally positive. ∎

THEOREM 3. *Take a sequence of clouds $T_i = \{t^i_0,\ldots,t^i_n\}$, $i = 0,\ldots,m$, with $T_i < T_j$ for $0 \leq i < j \leq m$. For $i = 1,\ldots,m$, $j = 0,\ldots,n$, let $B^n_{i,j}$ be the corresponding B-splines and write $B^n_{i,j} =: B^n_{(i-1)(n+1)+j}$. Then for any numbers $\tau_0 < \cdots < \tau_{p-1}$ for any $p \geq 1$, the matrix $\big(B^n_i(\tau_j)\big)_{i=0}^{m(n+1)-1}{}_{j=0}^{p-1}$ is totally positive.*

PROOF. It is sufficient to prove that for $0 \leq r < r+p-1 \leq m(n+1)-1$ and $\tau_0 < \cdots < \tau_{p-1}$,

$$\det\big(B^n_{r+i}(\tau_j)\big)_{i=0}^{p-1}{}_{j=0}^{p-1} \geq 0. \tag{2.19}$$

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Suppose that $r = (i_1 - 1)(n+1)+j_1$ and $r+p-1 = (i_2-1)(n+1)+j_2$. Thus $B^n_r = B^n_{i_1,j_1}$ and $B^n_{r+p-1} = B^n_{i_2,j_2}$. Then $\{B^n_{r+i} : i = 0,\dots,p-1\}$ span the space of splines with knots at $t_0^{i_1-1},\dots,t_{n-j_1}^{i_1-1}, t_0^{i_2},\dots,t_{j_2}^{i_2}$ and t_j^i for $i_1 \le i < i_2$ and $j = 0,\dots,n$. Suppose that this set of knots is $\{s_0,\dots,s_{p+n}\}$, where $s_0 \le s_1 \le \cdots \le s_{p+n}$. By the Schoenberg-Whitney theorem we know that the determinant in (2.19) is non-zero if and only if

$$s_i < \tau_i < s_{i+n+1}, \quad i = 0,\dots,p-1. \tag{2.20}$$

Thus it is sufficient to prove (2.19) under the assumption (2.20). Now we can vary the numbers $\tau_0,\dots,\tau_{p-1}$ continuously so that (2.20) always holds until we have $T_j < \tau_i < T_{j+1}$ whenever B^n_{r+i} has knots from T_j and T_{j+1}. Since the determinant in (2.19) varies continuously and is never zero, it is sufficient to show that (2.19) holds for this new choice of $\tau_0,\dots,\tau_{p-1}$. But in this case the matrix in (2.19) comprises diagonal blocks:

$$\begin{gathered}\left(B^n_{i_1,j_1+i}(\tau_j)\right)_{i,j=0}^{n-j_1},\\ \left(B^n_{i_1+\ell,i}(\tau_{(n+1)\ell-j_1+j})\right)_{i,j=0}^{n}, \quad \ell = 1,\dots,i_2-i_1-1,\\ \left(B^n_{i_2,i}(\tau_{p-1-j_2+j})\right)_{i,j=0}^{j_2}.\end{gathered}$$

By Lemma 3, each of these blocks has positive determinants and so (2.19) holds. ∎

In the standard manner we can deduce from Theorem 3 variation diminishing properties of the operator S. Given a sequence $a_0,\dots,a_r$ of real numbers, we denote by $S^-(a_0,\dots,a_r)$ the number of strict sign changes in the sequence. For a function f on an interval $[a,b]$ we write $V(f) = \sup S^-\big(f(x_0),\dots,f(x_r)\big)$, where the supremum is taken over all $a \le x_0 < \cdots < x_r \le b$, for all $r \ge 1$.

THEOREM 4. $V(Sf) \le V(f)$.

PROOF. It is well-known that for any totally positive matrix A, we have $S^-(Aa) \le S^-(a)$ for any vector a. It follows from (1.11) and Theorem 3 that whenever $a \le x_0 < \cdots < x_r \le b$,

$$\begin{aligned}&S^-\big(Sf(x_0),\dots,Sf(x_r)\big)\\ &\qquad \le S^-\big(f(\xi_{1,0}),\dots,f(\xi_{1,n}),f(\xi_{2,0}),\dots,f(\xi_{2,n}),\dots,f(\xi_{m,0}),\dots,f(\xi_{m,n})\big).\end{aligned} \tag{2.21}$$

Since $T_0 < T_1 < \cdots < T_m$, we have $\xi_{1,0} < \cdots < \xi_{1,n} < \xi_{2,0} < \cdots < \xi_{2,n} < \cdots < \xi_{m,0} < \cdots < \xi_{m,n}$, and so

$$S^-\big(f(\xi_{1,0}),\dots,f(\xi_{1,n}),\dots,f(\xi_{m,0}),\dots,f(\xi_{m,n})\big) \le V(f). \tag{2.22}$$

Applying (2.21) and (2.22) and taking the supremum over all $(x_0,\dots,x_r)$ with $a \le x_0 < \cdots < x_r \le b$ gives the result. ∎

COROLLARY 1. *For any linear function ℓ, $V(Sf-\ell) \le V(f-\ell)$.*

PROOF. This follows on applying Theorem 4 to the function $f-\ell$ and recalling that $S\ell = \ell$. ∎

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COROLLARY 2. *If f is an increasing (or decreasing) function, then so is Sf.*

PROOF. If f is increasing, then for any constant function c, Corollary 1 gives $V(Sf-c) \leq V(f-c) \leq 1$. So Sf is monotonic. Also by (1.12), $Sf(a) = f(a) \leq f(b) = Sf(b)$ and so Sf is increasing. The result follows similarly if f is decreasing. ∎

COROLLARY 3. *If f is convex, then so is Sf.*

PROOF. Suppose that f is convex. By Corollary 1, $V(Sf-\ell) \leq V(f-\ell) \leq 2$ for linear function ℓ. So Sf is convex or concave. Choosing ℓ with $\ell(a) = f(a)$, $\ell(b) = f(b)$, we have $f - \ell \leq 0$ on $[a,b]$. Thus $Sf(a) = f(a) = \ell(a)$, $Sf(b) = f(b) = \ell(b)$, and $Sf - \ell = S(f-\ell) \leq 0$. So Sf is convex. ∎

3. **Convergence properties of S.** For $n = 1, 2, \ldots$, we take clouds $T_0^n < \cdots < T_m^n$, where $T_i^n = \{t_{0,n}^i, \ldots, t_{n,n}^i\}$, $i = 0, \ldots, m$ and $t_{0,n}^0 = \cdots = t_{n,n}^0 = a$, $t_{0,n}^m = \cdots = t_{n,n}^m = b$. For $i = 1, \ldots, m$, $B_{i,j}^n$, $j = 0, \ldots, n$, denote the B-splines of degree n corresponding to the clouds T_{i-1}^n and T_i^n. As in (1.11) we define the operator S_n by

$$S_n f(x) := \sum_{i=1}^{m} \sum_{j=0}^{n} f(\xi_{i,j}^n) B_{i,j}^n(x), \quad a \leq x \leq b, \tag{3.1}$$

for any function f on $[a,b]$, where

$$\xi_{i,j}^n := \frac{1}{n} \sum \{t : t \in T_{i,j}^n\}, \tag{3.2}$$

where

$$T_{i,j}^n := \{t_{0,n}^{i-1}, \ldots, t_{n-1-j,n}^{i-1}, t_{0,n}^i, \ldots, t_{j-1,n}^i\}. \tag{3.3}$$

We are interested in the convergence properties of S_n as $n \longrightarrow \infty$.

Now (1.6) and (1.7) give

$$x^2 = \sum_{i=1}^{m} \sum_{j=0}^{n} \tau_{ij}^n B_{i,j}^n(x), \quad a \leq x \leq b, \tag{3.4}$$

where

$$\tau_{i,j}^n := \frac{2}{n(n-1)} \sum \{tt' : t, t' \in T_{i,j}^n, t \neq t'\}. \tag{3.5}$$

So from (3.1) we have

$$S_n(t^2)(x) - x^2 = \sum_{i=1}^{m} \sum_{j=0}^{n} a_{i,j}^n B_{i,j}^n(x), \quad a \leq x \leq b, \tag{3.6}$$

where

$$a_{i,j}^n = (\xi_{i,j}^n)^2 - \tau_{i,j}^n,$$

which after some simplification gives

(3.7) $$a^n_{i,j} = \frac{1}{n(n-1)} \sum \{t^2 : t \in T^n_{i,j}\} - \frac{1}{n-1} (\xi^n_{i,j})^2.$$

THEOREM 5. *If f is continuous on $[a,b]$, then $S_n f$ converges uniformly to f on $[a,b]$ as $n \longrightarrow \infty$.*

PROOF. By the Bohman-Korovkin theorem (see [5]), it is sufficient to verify the result for the functions $f(x) = 1$, $f(x) = x$ and $f(x) = x^2$. Since $S_n f = f$ for all linear functions f, it is sufficient to consider $f(x) = x^2$. All sets $T^n_{i,j}$ have entries lying in $[a,b]$ and so, by (3.7) there is a constant C, independent of i,j and n with $|a^n_{i,j}| \leq C/n$. Thus by (3.6), for $x \in [a,b]$,

$$|S_n f(x) - f(x)| \leq \frac{C}{n} \sum_{i=1}^{m} \sum_{j=0}^{n} B^n_{i,j}(x) = \frac{C}{n},$$

by (1.8), which completes the proof. ■

LEMMA 4. *There is a constant C such that for any $n > 4$, and $x \in [a,b]$, for the function $f_x(t) = (t-x)^4$, we have*

$$S_n f_x(x) \leq \frac{C}{n^2}.$$

PROOF. Let B_n denote the Bernstein operator of degree n on $[a,b]$, *i.e.*, $B_n = S_n$, for $m = 1$. Since f_x is convex, repeated application of Theorem 2 gives $S_n f_x \leq B_n f_x$. But direct calculation shows that $B_n f_x(x) = O(\frac{1}{n^2})$, which gives the result. ■

Now for x in $[a,b]$, write

(3.8) $$E_n(x) := S_n(t^2)(x) - x^2.$$

THEOREM 6. *Suppose that f is a bounded, measurable function on $[a,b]$ and has a second derivative at x in $[a,b]$. If $\lim_{n\to\infty} nE_n(x) =: e(x)$, then*

$$\lim_{n \to \infty} n\big(S_n f(x) - f(x)\big) = \frac{1}{2} f''(x) e(x).$$

PROOF. This follows by expanding f by Taylor's formula about x and applying Lemma 4, exactly as in the proof of Theorem 1 of [3]. ■

We would expect the limit $\lim_{n\to\infty} nE_n(x)$ to exist only if the clouds T^n_i, $i = 0, \ldots, m$, have some form of limiting distribution as $n \longrightarrow \infty$. For $i = 0, \ldots, m$, we describe the distribution of T^n_i by the function $g_{i,n}$ on $[0,1)$ defined by

(3.9) $$g_{i,n}(x) = t^i_{j,n}, \quad \frac{j}{n+1} \leq x < \frac{j+1}{n+1}.$$

We shall assume that as $n \longrightarrow \infty$, $g_{i,n}$ converges almost everywhere on $[0,1]$ to a function g_i. Clearly $g_0 = a$, $g_m = b$ and for $i = 1, \ldots, m$ $\operatorname{ess\,sup} g_{i-1} \leq \operatorname{ess\,inf} g_i$. We shall need the additional assumption that

$$g_i(x) > g_{i-1}(1-x) \quad \text{a.e. in } [0,1],\ i = 1, \ldots, m. \tag{3.10}$$

THEOREM 7. *Suppose that f is a bounded, measurable function on $[a,b]$ and has a second derivative at x in (a,b). Let $a(x)$ in $[0,1)$ and $i(x) \in \{1, \ldots, m\}$ be the unique values satisfying*

$$\int_0^{1-a(x)} g_{i(x)-1}(t)\,dt + \int_0^{a(x)} g_{i(x)}(t)\,dt = x. \tag{3.11}$$

Then

$$\lim_{n\longrightarrow\infty} n\big(S_n f(x) - f(x)\big) = \frac{1}{2} f''(x)e(x), \tag{3.12}$$

where

$$e(x) = \int_0^{1-a(x)} g_{i(x)-1}^2(t)\,dt + \int_0^{a(x)} g_{i(x)}^2(t)\,dt - x^2. \tag{3.13}$$

PROOF. By Theorem 6, it is sufficient to prove that

$$\lim_{n\longrightarrow\infty} nE_n(x) = e(x),$$

where $e(x)$ is given by (3.13). Now by (3.8) and (3.6),

$$E_n(x) = \sum_{i=1}^{m}\sum_{j=0}^{n} a_{i,j}^n B_{i,j}^n(x),$$

where the coefficients $a_{i,j}^n$ are given by (3.7). So by (3.7),

$$nE_n(x) = \sum_{i=1}^{m}\sum_{j=0}^{n} b_{i,j}^n B_{i,j}^n(x) - \frac{n}{n-1} S_n(t^2)(x),$$

where

$$\begin{aligned} b_{i,j}^n &= \frac{1}{n-1}\sum\{t^2 : t \in T_{i,j}^n\} \\ &= \frac{1}{n-1}\Big\{\sum_{\nu=0}^{n-1-j}(t_{\nu,n}^{i-1})^2 + \sum_{\nu=0}^{j-1}(t_{\nu,n}^{i})^2\Big\} \\ &= \frac{n+1}{n-1}\Big\{\int_0^{\frac{n-j}{n+1}} g_{i-1,n}^2(t)\,dt + \int_0^{\frac{j}{n+1}} g_{i,n}^2(t)\,dt\Big\}, \end{aligned} \tag{3.14}$$

by (3.9). Since, by Theorem 5, $\lim_{n\to\infty} \frac{n}{n-1} S_n(t^2)(x) = x^2$, it is sufficient to show that

$$\lim_{n\longrightarrow\infty} \sum_{i=1}^{m}\sum_{j=0}^{n} b_{i,j}^n B_{i,j}^n(x) = \int_0^{1-a(x)} g_{i(x)-1}^2(t)\,dt + \int_0^{a(x)} g_{i(x)}^2(t)\,dt. \tag{3.15}$$

We denote the right hand side of (3.15) by $F(x)$ and consider

$$\sum_{i=1}^{m}\sum_{j=0}^{n} b_{i,j}^{n}B_{i,j}^{n}(x) - S_nF(x) = \sum_{i=1}^{m}\sum_{j=0}^{n}\big(b_{i,j}^{n} - F(\xi_{i,j}^{n})\big)B_{i,j}^{n}(x). \tag{3.16}$$

First note that the functions $g_{i,n}$ are uniformly bounded, since $|g_{i,n}| \leq K := \max\{|a|,|b|\}$, $i = 1,\ldots,m$, $n = 1,2,\ldots$. By the bounded convergence theorem,

$$\lim_{n\to\infty}\int_0^1 |g_{i,n} - g_i| = 0, \quad i = 1,\ldots,m, \tag{3.17}$$

and for $i = 1,\ldots,m$,

$$\int_0^1 |g_{i,n}^2 - g_i^2| \leq 2K\int_0^1 |g_{i,n} - g_i| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Take any ε, $0 < \varepsilon < 1$. Then for all large enough n, (3.14) gives

$$\Big|b_{i,j}^{n} - \int_0^{\frac{n-j}{n}} g_{i-1}^2(t)\,dt - \int_0^{\frac{j}{n}} g_i^2(t)\,dt\Big| < \varepsilon,$$

and so, by definition of F,

$$|b_{i,j}^{n} - F(\xi_{i,j}^{n}| < \varepsilon + H, \tag{3.18}$$

where

$$H := \Big|\int_0^{1-\frac{j}{n}} g_{i-1}^2(t)\,dt + \int_0^{\frac{j}{n}} g_i^2(t)\,dt - \int_0^{1-a(\xi_{i,j}^{n})} g_{i(\xi_{i,j}^{n})-1}^2(t)\,dt - \int_0^{a(\xi_{i,j}^{n})} g_{i(\xi_{i,j}^{n})}^2(t)\,dt\Big|.$$

We shall show that

$$\Big|i(\xi_{i,j}^{n}) + a(\xi_{i,j}^{n}) - i - \frac{j}{n}\Big| < \varepsilon \tag{3.19}$$

implies

$$|b_{i,j}^{n} - F(\xi_{i,j}^{n})| < \varepsilon(1 + 4K^2). \tag{3.20}$$

Suppose that (3.19) holds. Since $\varepsilon < 1$ and $|a(\xi_{i,j}^{n}) - \frac{j}{n}| < 1$, we have $|i(\xi_{i,j}^{n}) - i| \leq 1$. If $i(\xi_{i,j}^{n}) = i$, then

$$\begin{aligned} H &\leq \Big|\int_{1-a(\xi_{i,j}^{n})}^{1-\frac{j}{n}} g_{i-1}^2(t)\,dt\Big| + \Big|\int_{a(\xi_{i,j}^{n})}^{\frac{j}{n}} g_i^2(t)\,dt\Big| \\ &\leq 2K^2\Big|\frac{j}{n} - a(\xi_{i,j}^{n})\Big| \\ &< 2K^2\varepsilon, \end{aligned}$$

and (3.20) follows from (3.18). Next suppose that $i(\xi_{i,j}^{n}) = i + 1$. Then $a(\xi_{i,j}^{n}) < \varepsilon$ and $1 - \frac{j}{n} < \varepsilon$ and so

$$\begin{aligned} H &\leq \Big|\int_0^{1-\frac{j}{n}} g_{i-1}^2(t)\,dt\Big| + \Big|\int_{1-a(\xi_{i,j}^{n})}^{\frac{j}{n}} g_i^2(t)\,dt\Big| + \Big|\int_0^{a(\xi_{i,j}^{n})} g_{i(\xi_{i,j}^{n})}^2(t)\,dt\Big| \\ &< K^2\Big(1 - \frac{j}{n}\Big) + K^2\Big|\frac{j}{n} - 1 + a(\xi_{i,j}^{n})\Big| + K^2|a(\xi_{i,j}^{n})| \\ &< 4K^2\varepsilon, \end{aligned}$$

and again (3.20) follows. We can similarly deduce (3.20) when $i(\xi_{i,j}) = i - 1$.

Now define $G\colon [0,m] \longrightarrow [a,b]$ by $G(m) = b$ and

$$G(x) = \int_0^{1-x+\nu} g_\nu(t)\,dt + \int_0^{x-\nu} g_{\nu+1}(t)\,dt, \quad \nu \le x < \nu+1,\ \nu = 0,\dots,m-1.$$

Then G is continuous and since

$$G(x) = \int_0^1 g_\nu(t)\,dt + \int_0^{x-\nu} \big(g_{\nu+1}(t) - g_\nu(1-t)\big)\,dt, \quad \nu \le x < \nu+1,\ \nu = 0,\dots,m-1,$$

we see from (3.10) that G is strictly increasing. Now for x in $[a,b)$,

$$G\big(i(x) - 1 - a(x)\big) = \int_0^{1-a(x)} g_{i(x)-1}(t)\,dt + \int_0^{a(x)} g_{i(x)}(t)\,dt = x,$$

by (3.11). Thus

$$G^{-1}(x) = i(x) - 1 + a(x), \quad a \le x < b. \tag{3.21}$$

Since G^{-1} is continuous on $[a,b]$ we can choose δ so that $|G^{-1}(x) - G^{-1}(y)| < \varepsilon$ whenever x,y in $[a,b]$ satisfy $|x-y| < \delta$. Now by (3.2) and (3.9),

$$\xi^n_{i,j} - G\Big(i - 1 + \frac{j}{n}\Big) = \frac{n+1}{n}\Big\{\int_0^{\frac{n-j}{n+1}} g_{i-1,n}(t)\,dt + \int_0^{\frac{j}{n+1}} g_{i,n}(t)\,dt\Big\} - \int_0^{1-\frac{j}{n}} g_{i-1}(t)\,dt - \int_0^{\frac{j}{n}} g_i(t)\,dt,$$

and so for all large enough n, (3.17) gives

$$\Big|\xi^n_{i,j} - G\Big(i-1+\frac{j}{n}\Big)\Big| < \delta.$$

Thus for $a \le x < b$, (3.21) gives

$$\Big|i(\xi^n_{i,j}) + a(\xi^n_{i,j}) - i - \frac{j}{n}\Big| = \Big|G^{-1}(\xi^n_{i,j}) - \Big(i - 1 + \frac{j}{n}\Big)\Big| < \varepsilon.$$

Recalling (1.8), we see from (3.20) and (3.16) that

$$\Big|\sum_{i=1}^{m}\sum_{j=0}^{n} b^n_{i,j} B^n_{i,j}(x) - S_n F(x)\Big| < \varepsilon(1 + 4K^2). \tag{3.22}$$

But F is continuous and so by Theorem 5, $\lim_{n\to\infty} S_n F(x) = F(x)$. Since ε can be arbitrarily small, (3.22) implies (3.15) and the proof is complete. ∎

EXAMPLE. As an example we take $m \ge 2$ and suppose that in the limit the elements of the clouds are in increasing order and uniformly distributed, *i.e.*, we may take for $0 \le x < 1$,

$$g_0(x) = 0,\ g_i(x) = i - 1 + x, \quad i = 1,\dots,m-1,\ g_m(x) = m - 1.$$

We first calculate from (3.11) the functions $i(x)$ and $a(x)$. If $i(x) = 1$, then

$$x = \int_0^{a(x)} g_1(t)\,dt = \int_0^{a(x)} t\,dt = \frac{1}{2}\big(a(x)\big)^2$$

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and so $a(x) = \sqrt{2x}$, $0 \leq x < \frac{1}{2}$.

Now suppose that $2 \leq i(x) \leq m-1$. Then

$$x = \int_0^{1-a(x)} \big(i(x) - 2 + t\big)\,dt + \int_0^{a(x)} \big(i(x) - 1 + t\big)\,dt$$

and a straightforward calculation gives $x + \frac{3}{2} = i(x) + \big(a(x)\big)^2$, *i.e.*, $i(x) = [x + \frac{3}{2}]$, $a(x)^2 = \{x + \frac{3}{2}\}$, $\frac{1}{2} \leq x < m - \frac{3}{2}$.

If $i(x) = m$, then

$$x = \int_0^{1-a(x)} (m - 2 + t)\,dt + \int_0^{a(x)} (m-1)\,dt$$

which gives

$$a(x) = \sqrt{2x - 2m + 3}, \quad m - \frac{3}{2} \leq x < m - 1.$$

We now apply (3.13) to find $e(x)$. For $0 \leq x < \frac{1}{2}$,

$$\begin{aligned} e(x) &= \int_0^{a(x)} t^2\,dt - x^2 = \frac{1}{3}\big(a(x)\big)^3 - x^2 \\ &= \frac{1}{3}(2x)^{3/2} - x^2. \end{aligned}$$

Next suppose $\frac{1}{2} \leq x < m - \frac{3}{2}$. Then

$$e(x) = \int_0^{1-a(x)} \big(i(x) - 2 + t\big)^2\,dt + \int_0^{a(x)} \big(i(x) - 1 + t\big)^2\,dt - x^2$$

which after some calculation gives

$$e(x) = i(x)^2 - 3i(x) + \frac{7}{3} + 2\big(a(x)\big)^2\big(i(x) - 1\big) - x^2.$$

If we suppose that $j - \frac{1}{2} \leq x < j + \frac{1}{2}$ for an integer j with $1 \leq j \leq m-2$, then $i(x) = j+1$, $a(x)^2 = x + \frac{1}{2} - j$ and we get

$$e(x) = \frac{1}{3} - (x - j)^2.$$

Finally we suppose that $m - \frac{3}{2} \leq x < m - 1$. Then

$$e(x) = \int_0^{1-a(x)} (m - 2 + t)^2\,dt + \int_0^{a(x)} (m-1)^2\,dt - x^2$$

which becomes, on recalling $a(x) = \sqrt{2x - 2m + 3}$,

$$e(x) = \frac{1}{3}\big(1 - (2x - 2m + 3)^{3/2}\big) - (m - 1 - x)^2.$$

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