An Inversion Formula for the Generalised Laplace Transform

By R. S. Varma

(*Received 21st May*, 1947. *Read 7th June*, 1947.)

1. Recently [1] I have given a generalisation of the Laplace integral

$$\phi(s) = \int_0^\infty e^{-st} f(t) dt \tag{1}$$

in the form

$$\phi_m^k(s) = \int_0^\infty (2st)^{-\frac{1}{4}} W_{k,m}(2st) f(t)\, dt \tag{2}$$

where $W_{k,m}(x)$ stands for Whittaker Functions.

For $k = \frac{1}{4}$, $m = \pm\frac{1}{4}$, $(2st)^{-\frac{1}{4}} W_{k,m}(2st)$ degenerates into e^{-st} and we get the Laplace integral (1).

The object of this paper is to give an inversion formula [2] for the transform based on (2).

2. Multiply both sides of (2) by s^{-l} and integrate with respect to s from 0 to ∞ : we then have

$$\int_0^\infty s^{-l} \phi_m^k(s)\, ds = \int_0^\infty \int_0^\infty (2st)^{-\frac{1}{4}} W_{k,m}(2st)\, s^{-l} f(t)\, dt\, ds$$

$$= \int_0^\infty (2t)^{l-1} f(t)\, dt \int_0^\infty y^{-l-\frac{1}{4}} W_{k,m}(y)\, dy$$

$$= \frac{2^{l-1}\Gamma(-l+m+\frac{5}{4})\,\Gamma(-l-m+\frac{5}{4})}{\Gamma(-l-k+\frac{7}{4})}$$

$$\times\, {}_2F_1\left(-l+m+\tfrac{5}{4},\ -l-m+\tfrac{5}{4},\ -l-k+\tfrac{7}{4};\ \tfrac{1}{2}\right)$$

$$\times \int_0^\infty t^{l-1} f(t)\, dt, \qquad R(-l \pm m + \tfrac{5}{4}) > 0.$$

If now we apply the Mellin Inversion Formula to the integral

$$\int_0^\infty t^{l-1} f(t)\, dt = \chi(l),$$

---

[1] R. S. Varma: "A generalisation of Laplace's transform," *Current Science* (Bangalore, India), 16, (Jan., 1947), 17-18. The results of this paper were first communicated by me at the Indian Science Congress, Nagpur (1945).

[2] *Cf.* Titchmarsh, *Introduction to the theory of Fourier integrals*, (Oxford, 1937), 316, (11.6.5).

we have

$$\tfrac{1}{2}\left\{f(t+0)+f(t-0)\right\}=\frac{1}{\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}\frac{(2t)^{-l}\,\Gamma(-l-k+\frac{7}{4})\,\psi(l)}{\Gamma(-l+m+\frac{5}{4})\Gamma(-l-m+\frac{5}{4})}\,dl,$$

where

$$_2F_1(-l+m+\tfrac{5}{4},\,-l-m+\tfrac{5}{4};\,-l-k+\tfrac{7}{4};\,\tfrac{1}{2})\,\psi(l)=\int_0^\infty s^{-l}\,\phi_m^k(s)\,ds,$$

provided that

(i) $x^{c-1}f(x)$ belongs to $L(0,\infty)$,

(ii) $x^{-l}\phi_m^k(x)$ also belongs to $L(0,\infty)$,

$l=c\pm iT, \qquad -\infty<T<\infty,$

(iii) $f(x)$ is of bounded variation in the neighbourhood of the point $x=t$.

The change in the order of integration involved above is obviously justified on account of the assumptions (i) and (ii) and the inequalities

$$W_{k,m}(z)=\begin{cases}O(z^{\pm m+\frac{1}{2}}) & \text{for small } |z|\\ O(e^{-\frac{1}{2}z}z^k) & \text{for large } |z|\end{cases}$$

3. For $k=\frac{1}{4}$, $m=\pm\frac{1}{4}$, we get, when $\phi(s)$ is the Laplace transform of $f(t)$, the result [1]

$$\tfrac{1}{2}\{f(t+0)+f(t-0)\}=\frac{1}{\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}\frac{t^{-l}}{\Gamma(1-l)}\int_0^\infty s^{-l}\phi(s)\,dl\,ds.$$

[1] *Cf.* Titchmarsh, *op. cit.*, 316, (11.7.2).

THE UNIVERSITY,
LUCKNOW, INDIA.