A CONDITION OF HALO TYPE FOR THE DIFFERENTIATION OF CLASSES OF INTEGRALS

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1. Introduction. We shall consider a space S, a σ-algebra $\mathbf{M}$ of subsets of S, a measure μ defined on $\mathbf{M}$, and the μ-integrals of certain μ-integrable functions f. To each point x of a certain set E of S we associate certain ones of the sets $V \in \mathbf{M}$ and form the quotients $\int_V f(x)d\mu(x)/\mu(V)$ for each such set V. In case these quotients tend to $f(x)$ as the sets V converge to x in accordance with a definition we adopt in §2, then we say that the integral of f is *differentiable* or *derivable at* x. It is of interest to assert conditions that ensure the differentiability of a given integral or class of integrals at μ-almost all points of E.

The authors of (**1**; **2**; and **5**) considered questions of differentiability of Lebesgue integrals in Euclidean space with respect to Lebesgue measure. The sets V they associated with a point x consisted of all closed intervals containing x, with sides parallel to the coordinate axes, and convergence to x was taken with respect to their diameters tending to zero. The corresponding derivatives, when they exist, are known as *strong derivatives*. These writers expressed various kinds of *halo* conditions for the strong differentiability of certain classes of Lebesgue integrals; cf. §2. Their results depend in an essential way on the special nature of Euclidean space and Lebesgue measure (e.g., their proofs made use of similar figures and the fact that their measures are proportional). Here we propose to allow S, $\mathbf{M}$, μ, and the subfamily of $\mathbf{M}$ associated with a point of $E \subset S$ to be quite general. We shall show that a form of the halo condition mentioned above is sufficient to ensure the differentiability of a certain class of μ-integrals at μ-almost all points of a set $E \subset S$. Because of the generality of the hypotheses, our results are somewhat less sharp than those obtained in (**1**; **2**; and **5**). A comparison is made at the end of §3 showing the nature of the gap that exists between the results obtained in (**5**) and those that follow from the general considerations of the present paper when applied to the situation considered in (**5**).

2. Setting, fundamental definitions, and terminology. We employ the symbols $\cup$, $\cap$, $\bigcup$, $\bigcap$, $\subset$, and $\supset$ with their customary set-theoretical meanings. If A and B are sets, then by $A - B$ we shall mean the set of those points belonging to A but not to B; and we define $A \triangle B = (A - B) \cup (B - A)$.

We let S denote a fixed non-empty set, $\mathbf{M}$ a fixed non-empty σ-algebra of subsets of S, and μ a fixed measure defined on $\mathbf{M}$. We assume that μ is totally σ-finite; cf. (**3**, p. 31). We let μ^* denote the outer measure defined on the class $\mathbf{S}$ of all subsets of S by the relation

$$\mu^*(A) = \inf_{A \subset M \in \mathbf{M}} \mu(M)$$

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for any set $A \subset S$; as is well known, μ^* agrees with μ on $\mathbf{M}$. We let $\mathbf{N}$ denote the class of those members of $\mathbf{M}$ that are of μ-measure zero, and we let $\mathbf{N}^*$ denote the family of subsets of S that are of μ^*-measure zero. It is well known that $\mathbf{N}^*$ consists of the class of all subsets of members of $\mathbf{N}$. We agree to write $A \subset B$ (mod $\mathbf{N}$) or (mod $\mathbf{N}^*$) if and only if $\mu(A - B) = 0$ or $\mu^*(A - B) = 0$. respectively; $A = B$ (mod $\mathbf{N}$) or (mod $\mathbf{N}^*$) if and only if $A \subset B$ and $B \subset A$ (mod $\mathbf{N}$) or (mod $\mathbf{N}^*$), respectively; equivalently, $\mu(A \triangle B) = 0$ or $\mu^*(A \triangle B) = 0$, respectively.

It is well known that if A is any set, then there exists a set $M \in \mathbf{M}$ such that $A \subset M$ and no other set M' belonging to $\mathbf{M}$ and containing A has smaller μ-measure than $\mu(M)$. Such a set M is called a *μ-cover* of A; it need not be unique, but if M' is any other μ-cover of A, then $\mu(M \triangle M') = 0$. We shall let $\bar{A}$ denote any one μ-cover of A.

We let $\mathfrak{B}$ denote a *derivation basis* **(4)** defined as follows. We assume that to each point x of a fixed subset E of S there correspond sequences, in the sense of Moore–Smith, of sets of finite positive μ-measure (and so belonging to $\mathbf{M}$) that are said to *converge to* x; a typical sequence may be denoted by $\{M_\iota(x)\}$. We assume that every cofinal subsequence of an x-converging sequence also converges to x. The family of all the sequences $\{M_\iota(x)\}$, $x \in E$, is the derivation basis $\mathfrak{B}$. The set E is called the *domain of* $\mathfrak{B}$; the family $\mathbf{D}$ consisting of all sets occurring in the totality of these sequences is called the *spread of* $\mathfrak{B}$.

If λ is a numerical-valued function defined on $\mathbf{D}$ and $x \in E$, then we define

$$D_* \lambda(x) = \sup\left[\lim_\iota \sup \frac{\lambda(M_\iota(x))}{\mu(M_\iota(x))}\right],$$

where the expression in brackets denotes the limit superior for any one x-converging sequence $\{M_\iota(x)\}$, and the supremum is taken among all sequences converging to x. In exactly similar fashion we define

$$D_* \lambda(x) = \inf\left[\lim_\iota \inf \frac{\lambda(M_\iota(x))}{\mu(M_\iota(x))}\right].$$

We call $D^*\lambda(x)$ and $D_*\lambda(x)$ the *upper* and *lower* *$\mathfrak{B}$-derivatives of λ at x*, respectively. If $D^*\lambda(x) = D_*\lambda(x)$ (finite or infinite), then we say that the *$\mathfrak{B}$-derivative* $D\lambda(x) = D^*\lambda(x) = D_*\lambda(x)$ *exists at* x. In case λ is the μ-integral of a μ-measurable function f and $D\lambda(x)$ exists and coincides with $f(x)$, we shall say that $\mathfrak{B}$ *differentiates* λ *at* x or λ is *$\mathfrak{B}$-differentiable at* x. If $\mathbf{K}$ is any class of μ-integrals, we shall say that $\mathfrak{B}$ *differentiates* $\mathbf{K}$ if and only if $\mathfrak{B}$ differentiates each member of $\mathbf{K}$ at μ-almost all points of E. In particular, if $\mathfrak{B}$ differentiates the μ-integrals of the characteristic functions of all $\mathbf{M}$-sets, then we say that $\mathfrak{B}$ possesses the *μ-density property*.

$\mathfrak{B}^*$ is said to be a *subbasis* of $\mathfrak{B}$ if and only if $\mathfrak{B}^*$ is a subfamily of $\mathfrak{B}$ that contains all the cofinal subsequences of each of its sequences and that associates each of its sequences with the same points as $\mathfrak{B}$ does itself. It follows that the domain and spread of $\mathfrak{B}^*$ are subsets of the domain and spread of $\mathfrak{B}$, respectively. The spread of any subbasis $\mathfrak{B}^*$ of $\mathfrak{B}$ whose domain E^* contains X (mod $\mathbf{N}^*$) is called a *$\mathfrak{B}$-fine covering of* X. A $\mathfrak{B}$-fine covering of X may be equally well defined as any subfamily $\mathbf{V}$ of the spread of $\mathfrak{B}$ that contains, for μ-almost all $x \in X$, the sets belonging to at least one sequence $\{M_\iota(x)\}$. If $\mathbf{V}$ is such a subfamily of the spread of $\mathfrak{B}$ that for μ-almost all $x \in X$ and each sequence $\{M_\iota(x)\}$ in $\mathfrak{B}$ there exists an index ι' such that $M_\iota(x) \in \mathbf{V}$ whenever $\iota > \iota'$, then we say that $\mathbf{V}$ is a *full $\mathfrak{B}$-fine covering of* X. It is easily seen that the intersection of a $\mathfrak{B}$-fine covering of X and a full $\mathfrak{B}$-fine covering of X is again a $\mathfrak{B}$-fine covering of X.

Henceforth we let $\mathfrak{B}$ denote a derivation basis with domain $E \subset S$.

3. Halo conditions that ensure $\mathfrak{B}$-differentiability of classes of μ-integrals. If $M \in \mathbf{M}$, $0 < \alpha < 1$, and ψ is a non-negative μ-integral λ (that is, ψ is the indefinite μ-integral of some non-negative μ-measurable function), then we define $\mathbf{H}(\alpha, M, \psi)$ as the family of all sets V in the spread of B for which $\psi(V \cap M) > \alpha\mu(V)$ and $S(\alpha, M, \psi) = \cup \mathbf{H}(\alpha, M, \psi)$. We call $S(\alpha, M, \psi)$ a *ψ-halo of* M (this differs slightly from the definition given in (**4**, §2.5)). We note that $S(\alpha, M, \psi)$ may fail to be μ-measurable. In case ψ coincides with μ, we agree to abbreviate these expressions to $\mathbf{H}(\alpha, M)$ and $S(\alpha, M)$, respectively. It is readily seen that $\mathbf{H}(\alpha, M, \psi) \subset \mathbf{H}(\alpha', M', \psi)$ and $S(\alpha, M, \psi) \subset S(\alpha', M', \psi)$ whenever $0 < \alpha' \leqslant \alpha < 1$, $M \subset M' \in \mathbf{M}$, and ψ is one of the admissible integrals.

For a given family of sets $\mathbf{F}$, we define $P_{\mathbf{F}}(x)$ as the number of members of $\mathbf{F}$ to which x belongs, where x denotes an arbitrary point of S; we define $\epsilon_{\mathbf{F}}(x) = P_{\mathbf{F}}(x) - 1$ for each such x. We also define $\theta\mathbf{F}$ as the set of points in S belonging to more than one member of $\mathbf{F}$; thus

$$\theta\mathbf{F} = \{x: P_{\mathbf{F}}(x) > 1\} = \{x: \epsilon_{\mathbf{F}}(x) \geqslant 1\}.$$

If ψ is an integral of the type just described, then we say that $\mathfrak{B}$ *possesses the Vitali ψ-property* if and only if for each set $X \subset E$ of finite μ^*-measure, each $\mathfrak{B}$-fine covering $\mathbf{V}$ of X, and each $\epsilon > 0$, there exists a countable subfamily $\mathbf{F}$ of $\mathbf{V}$ such that, putting $T = \cup\mathbf{F}$,

(i) $\mu(\bar{X} - \bar{X} \cap T) < \epsilon$;
(ii) $\psi(T - T \cap \bar{X}) < \epsilon$;
(iii) $\int_T \epsilon_{\mathbf{F}}(x) d\mu(x) = \int_{\theta\mathbf{F}} \epsilon_{\mathbf{F}}(x) d\mu(x) = \sum_{V \in \mathbf{F}} \psi(V) - \psi(T) < \epsilon$.

We say that a μ-integral ψ is *μ-finite* if and only if $\psi(M)$ is finite whenever $M \in \mathbf{M}$ and $\mu(M)$ is finite.

The following is proved as (**4**, Theorem 1.43) and so is stated here without proof.

3.1. THEOREM. *If ψ is a non-negative μ-finite μ-integral and $\mathfrak{B}$ possesses the Vitali ψ-property, then $\mathfrak{B}$ differentiates ψ at μ^*-almost all points of E.*

If ψ is such a μ-integral that $0 \leqslant \psi_0(M) \leqslant \psi(M)$ holds for each set $M \in \mathbf{M}$, then we say that ψ *dominates* ψ_0. The following is an immediate consequence of

Theorem 3.1 and the obvious fact that if $\mathfrak{B}$ possesses the Vitali ψ-property, then it possesses the Vitali ψ_0-property in case ψ dominates ψ_0.

3.2. Corollary. *If ψ is a non-negative μ-integral and $\mathfrak{B}$ possesses the Vitali ψ-property, then $\mathfrak{B}$ differentiates each μ-finite μ-integral dominated by ψ at μ^*-almost all points of E.*

3.3. Corollary. *If ψ_0 is a μ-finite μ-integral such that $\psi_0(M) = \int_M f(x)d\mu(x)$ for each $M \in \mathbf{M}$, $\psi(M) = \int_M |f(x)|d\mu(x)$ for each such M, and $\mathfrak{B}$ possesses the Vitali ψ-property, then $\mathfrak{B}$ differentiates ψ_0 at μ^*-almost all points of E.*

Proof. Clearly ψ is a μ-finite μ-integral, and ψ dominates each of the integrals obtained by integrating the positive and negative functions corresponding to f. Thus $\mathfrak{B}$ differentiates each of these integrals, and so their difference ψ_0, at μ^*-almost all points of E.

If $\psi = \mu$, we have the following special result.

3.4. Corollary. *If $\mathfrak{B}$ possesses the Vitali μ-property, then $\mathfrak{B}$ differentiates the class of μ-integrals of all bounded measurable functions.*

Proof. From an inspection of (i), (ii), and (iii) above it is obvious that if $\mathfrak{B}$ possesses the Vitali μ-property and $0 < k < \infty$, then $\mathfrak{B}$ possesses the Vitali $k\mu$-property. If ψ_0 is the μ-integral of a bounded function f, then $0 \leqslant |f(x)| \leqslant k$ holds for some k, whenever $x \in S$, whence $k\mu$ dominates the μ-integral of $|f|$ and Corollary 3.3 applies.

If ψ is a non-negative μ-finite μ-integral, then we shall say that $\mathfrak{B}$ *possesses the vanishing ψ-halo property* if and only if whenever $\epsilon > 0$, $0 < \alpha < 1$, and M_0 is an $\mathbf{M}$-set of finite μ-measure, there exists $\eta > 0$ such that

$$\mu^*(S(\alpha, M, \psi)) < \epsilon$$

whenever $M_0 \supset M \in \mathbf{M}$ and $\mu(M) < \eta$ (this is similar to the *halo evanescence property of* (**4**, §2.5)).

3.5 Theorem. *If $\mathfrak{B}$ possesses the Vitali μ-property and the vanishing ψ-halo property, then $\mathfrak{B}$ possesses the Vitali ψ-property.*

Proof. We take an arbitrary set $X \subset E$ of finite μ^*-measure, any $\mathfrak{B}$-fine covering $\mathbf{V}$ of X, and any $\epsilon > 0$. By hypothesis, $\mathfrak{B}$ possesses the Vitali μ- property, so that for each positive integer n, there exists a countable family $\mathbf{F}_n \subset \mathbf{V}$ such that if $S_n = \cup \mathbf{F}_n$, then

$$(1) \qquad \begin{gathered} \mu(\bar{X} - \bar{X} \cap S_n) < 2^{-n}, \qquad \mu(S_n - S_n \cap \bar{X}) < 2^{-n}, \\ \int_{S_n} \epsilon_{\mathbf{F}_n}(x)d\mu(x) = \sum_{V \epsilon \mathbf{F}_n} \mu(V) - \mu(S_n) < 2^{-n}. \end{gathered}$$

We let

$$D_n = \theta \mathbf{F}_n, \qquad M_0 = \bigcup_{n=1}^{\infty} S_n.$$

Since $\epsilon_{\mathbf{F}_n}(x) \geqslant 1$ whenever $x \in D_n$ and $D_n \subset S_n \subset M_0$ for $n = 1, 2, \ldots$, it follows from the third inequality of (1) that

$$\mu(D_n) \leqslant \int_{S_n} \epsilon_{\mathbf{F}_n}(x) d\mu(x) < 2^{-n} \tag{2}$$

for $n = 1, 2, \ldots$. From the second inequality of (1) we see that

$$\mu(S_n) < \mu(\bar{X}) + 2^{-n}$$

for $n = 1, 2, \ldots$, whence

$$\mu(M_0) < \mu(\bar{X}) + 1 < \infty. \tag{3}$$

We choose any positive number α such that $\alpha < \epsilon/(\mu(\bar{X}) + 1)$. For $n = 1, 2, \ldots$ we let $\mathbf{K}_n$ denote the subfamily of $\mathbf{F}_n$ such that $V \in \mathbf{K}_n$ if and only if $\psi(V \cap D_n) \leqslant \alpha\mu(V)$, and we let $K_n = \cup \mathbf{K}_n$. From the properties of $\epsilon_{\mathbf{K}_n}$, $P_{\mathbf{K}_n}$, and (1) we now obtain

$$\begin{aligned} \int_{K_n} \epsilon_{\mathbf{K}_n}(x) d\psi(x) &= \int_{\theta \mathbf{K}_n} \epsilon_{\mathbf{K}_n}(x) d\psi(x) \leqslant \int_{\mathbf{K}\theta_n} P_{\mathbf{K}_n}(x) d\psi(x) \\ &\leqslant \textstyle\sum_{V \in \mathbf{K}_n} \psi(V \cap \theta\mathbf{K}_n) \leqslant \sum_{V \in \mathbf{K}_n} \psi(V \cap D_n) \\ &\leqslant \alpha \textstyle\sum_{V \in \mathbf{K}_n} \mu(V) \leqslant \alpha \sum_{V \in \mathbf{K}_n} \mu(V) \\ &\leqslant \alpha(\mu(S_n) + 2^{-n}) < \alpha(\mu(\bar{X}) + 2^{-n+1}) < \epsilon. \end{aligned} \tag{4}$$

We let $\mathbf{G}_n = \mathbf{F}_n - \mathbf{K}_n$, $\mathbf{G}_n = \cup \mathbf{G}_n$, note that $\mathbf{G}_n \subset \mathbf{H}(\alpha, D_n, \psi)$ and hence $G_n \subset S(\alpha, D_n, \psi)$, use the fact that $S_n = G_n \cup K_n$ and the relations (1) to see that $(\bar{X} - \bar{X} \cap K_n) \subset (\bar{X} - \bar{X} \cap S_n) \cup G_n$, and infer that for $n = 1, 2, \ldots$,

$$\begin{aligned} \mu(\bar{X} - \bar{X} \cap K_n) &\leqslant \mu(\bar{X} - \bar{X} \cap S_n) + \mu(G_n) < 2^{-n} + \mu^*(S(\alpha, D_n, \psi)); \\ \mu(K_n - K_n \cap \bar{X}) &\leqslant \mu(S_n - S_n \cap \bar{X}) < 2^{-n}. \end{aligned} \tag{5}$$

Recalling (3) and the definition of the vanishing ψ-halo property, we may determine $\eta > 0$ so that if $M_0 \supset M \in \mathbf{M}$ and $\mu(M) < \eta$, then $\mu^*(S(\alpha, M, \psi)) < \epsilon/2$. Since $\psi(M_0) < \infty$ because of (3) and our assumptions concerning ψ, we may suppose that η is chosen small enough so that $0 \leqslant \psi(M) < \epsilon$ for each set M such that $M_0 \supset M \in \mathbf{M}$ and $\mu(M) < \eta$.

We now select a positive integer N so large that $2^{-N} < \eta$ and also $2^{-N} < \epsilon/2$. From (2) and (5) it follows that $\mu(\bar{X} - \bar{X} \cap K_N) < \epsilon$ and

$$\psi(K_N - K_N \cap \bar{X}) < \epsilon.$$

Combining these relations with (4) it is clear that $\mathbf{K}_N$ is a subfamily of $\mathbf{V}$ satisfying the conditions required for $\mathfrak{B}$ to possess the Vitali ψ-property, as we wished to prove.

At this point we introduce fixed functions ϕ and σ taking non-negative values on the set of all non-negative real numbers. We assume that there exists a non-negative number t_0 such that ϕ and σ both increase strictly on $\{t: t_0 < t < \infty\}$, and we assume additionally the existence of a number $\lambda > 1$ and a positive integer N such that $\lambda^N > t_0$ and

$$\sum_{n=N}^{\infty} \lambda^n/\sigma(\lambda^n)$$

converges; finally, we assume for convenience that $\phi(t) = \sigma(t) = 0$ whenever $0 \leqslant t \leqslant t_0$. It will be apparent later that the behaviour of ϕ and σ on

$$\{t: 0 \leqslant t \leqslant t_0\}$$

does not affect the results we achieve; they would be equally valid provided merely that ϕ and σ enjoy appropriate Borel measurability properties on $\{t: 0 \leqslant t \leqslant t_0\}$ and are bounded on that set.

We now define $\mathbf{L}$ as the class of μ-integrals of all those μ-measurable functions f for which $\int_M \phi(\sigma(|f(x)|))d\mu(x)$ and $\int_M |f(x)|d\mu(x)$ are finite whenever $M \in \mathbf{M}$ and $\mu(M) < \infty$.

3.6. THEOREM. *Let $\mathfrak{V}$ possess the Vitali μ-property and assume additionally the following: if M_0 is any $\mathbf{M}$-set of finite μ-measure, there exists a constant $C > 1$ such that $\mu^*(S(\alpha, M)) \leqslant C\phi(1/\alpha)\mu(M)$ whenever $0 < \alpha < 1/(C + t_0)$, $M_0 \supset M \in \mathbf{M}$ and $\mu(M) < 1/C$.*

Then $\mathfrak{V}$ differentiates the class $\mathbf{L}$.

Remark. This halo condition is similar to that introduced in **(5**, p. 226**)**. There, however, the condition was shown to be necessary rather than sufficient for the differentiability of certain integrals. The theorem proved there utilized special properties of Euclidean space and Lebesgue measure.

The conditions on α and C are imposed simply to bring $1/\alpha$ within the part of the domain of ϕ where ϕ is positive-valued. The condition on the halo is the important thing; if it holds for some positive value of C, then it holds for any larger value, and we have simply taken a value sufficiently large for our purposes.

Proof. By virtue of Theorems 3.5, 3.1, and Corollary 3.3, it is sufficient to show that for each non-negative member ψ of $\mathbf{L}$, $\mathfrak{V}$ possesses the vanishing ψ-halo property. Accordingly, we take an arbitrary member ψ of $\mathbf{L}$, an arbitrary $\mathbf{M}$-set M_0 of finite μ-measure, and an arbitrary positive number ϵ. We determine a non-negative measurable function f such that $\psi(X) = \int_X f(x)d\mu(x)$ whenever $X \in \mathbf{M}$ and select a constant C in accordance with the hypotheses above. Since $\psi \in \mathbf{L}$, there exists a constant $\eta' > 0$ such that

$$\int_M \phi(\sigma(f(x)))d\mu(x) < \epsilon/2C \tag{1}$$

whenever $M_0 \supset M \in \mathbf{M}$ and $\mu(M) < \eta'$.

We consider an arbitrary number α, $0 < \alpha < 1$. We have to show that there exists $\eta > 0$ such that $\mu^*(S(\alpha, M, \psi)) < \epsilon$ whenever $M_0 \supset M \in \mathbf{M}$ and $\mu(M) < \eta$. Since, as was noted earlier, $S(\alpha', M, \psi) \subset S(\alpha, M, \psi)$ whenever $0 < \alpha \leqslant \alpha' < 1$, it is clearly sufficient for us to prove that $\mu^*(S(\alpha, M, \psi)) < \epsilon$ under the assumption that $0 < \alpha < 1/(C + t_0)$.

We now choose $\lambda > 1$ and a positive integer N in accordance with the properties of σ; we may evidently select a positive integer $K \geqslant N$ so that

$$(2) \qquad \sum_{n=K+1}^{\infty} (\lambda^n/\sigma(\lambda^n)) < \alpha/2\lambda.$$

We let η'' denote the smaller of the two numbers $1/C$ and $\epsilon/2C\phi(2\lambda^K/\alpha)$. Clearly $0 < \alpha/2\lambda^K < \alpha < 1/(C + t_0)$; consequently it follows from our hypotheses that

$$(3) \qquad \mu^*(S(\alpha/2\lambda^K, M)) \leqslant C\phi(2\lambda^K/\alpha)\mu(M) < \epsilon/2$$

whenever $M_0 \supset M \in \mathbf{M}$ and $\mu(M) < \eta''$. We now let η be the smaller of η' and η'', take an arbitrary $\mathbf{M}$-set $M \subset M_0$ with $\mu(M) < \eta$, and observe that (1) and (3) both hold.

Now we define

$$M'_0 = M \cap \{x : 0 \leqslant f(x) < \lambda^K\},$$

$$M'_n = M \cap \{x : \lambda^n \leqslant f(x) < \lambda^{n+1}\},$$

and $\qquad \alpha_n = 1/\sigma(\lambda^n)$

for $n = K + 1, K + 2, \ldots$. From the strictly increasing nature of ϕ and σ on $\{t : t_0 < t < \infty\}$ it follows that

$$(4) \qquad M'_n = M \cap \{x : \phi(\sigma(\lambda^n)) \leqslant \phi(\sigma(f(x)) < \phi(\sigma(\lambda^{n+1}))\}$$

for $n = K + 1, K + 2, \ldots$. From (2) it follows easily that for each such n, $0 < \alpha_n < \alpha < 1/(C + t_0)$; consequently from (4) and our hypotheses we obtain

$$\mu^*(S(\alpha_n, M'_n)) \leqslant C\phi(\sigma(\lambda^n))\mu(M'_n) \leqslant C\int_{M'_n}\phi(\sigma(f(x)))d\mu(x)$$

for $n = K + 1, K + 2, \ldots$. Thus, with the help of (2) we see that

$$(5) \qquad \sum_{n=K+1}^{\infty} \mu^*(S(\alpha_n, M'_n)) \leqslant C\sum_{n=K+1}^{\infty} \int_{M'_n} \phi(\sigma(f(x)))d\mu(x)$$
$$\leqslant C \int_M \phi(\sigma(f(x)))d\mu(x) < \epsilon/2.$$

We define χ so that $\chi(X) = \int_{X-M'_0} f(x)d\mu(x)$ whenever $X \in \mathbf{M}$. Thus

$$(6) \qquad \psi(X) = \psi(X - M'_0) + \psi(X \cap M'_0) \leqslant \chi(X) + \lambda^K\mu(X \cap M'_0)$$

for each such set X.

We consider next any set V belonging to the spread of $\mathfrak{B}$ such that $V \notin \mathbf{H}(\alpha_n, M'_n)$ for $n = K + 1, K + 2, \ldots$. Then

$$\mu(M'_n \cap V) \leqslant \alpha_n\,\mu(V) = (1/\sigma(\lambda^n))\mu(V)$$

and therefore

$$(7) \qquad \frac{\lambda^{n+1}\mu(M'_n \cap V)}{\mu(V)} \leqslant \frac{\lambda^{n+1}}{\sigma(\lambda^n)}$$

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for $n = K+1, K+2, \ldots$. Using (7) and (2) we obtain

$$\frac{\chi(M \cap V)}{\mu(V)} = \frac{\int_{(M-M'_0)\cap V} f(x)d\mu(x)}{\mu(V)}$$
$$= \sum_{n=K+1}^{\infty} \frac{\int_{M'_n \cap V} f(x)d\mu(x)}{\mu(V)} \leqslant \lambda \sum_{n=K+1}^{\infty} \frac{\lambda^n}{\sigma(\lambda^n)} < \frac{\alpha}{2}.$$

From this, we conclude that $V \notin \mathbf{H}(\alpha/2, M, \chi)$. Hence if $V \in \mathbf{H}(\alpha/2, M, \chi)$, then $V \in \mathbf{H}(\alpha_n, M'_n)$ for some positive integer $n \geqslant K + 1$; consequently,

$$S(\alpha/2, M, \chi) \subset \bigcup_{n=K+1}^{\infty} S(\alpha_n, M'_n)$$

and so from (5) it follows that

$$\mu^*(S(\alpha/2, M, \chi)) < \epsilon/2. \tag{8}$$

If V is any set belonging to $\mathbf{H}(\alpha, M, \psi)$, then we see from (6) that either $V \in \mathbf{H}(\alpha/2, M, \chi)$ or $V \in \mathbf{H}(\alpha/2\lambda^K, M)$; therefore,

$$S(\alpha, M, \psi) \subset S(\alpha/2, M, \chi) \cup S(\alpha/2\lambda^K, M).$$

From (3) and (8) we conclude that $\mu^*(S(\alpha, M, \psi)) < \epsilon$. This proves that $\mathfrak{B}$ possesses the vanishing ψ-halo property, as required.

It is of some interest to apply the present theory to the situation studied in **(5)**, and to compare the results obtained. The authors of **(5)** took for S a unit hypercube in Euclidean space of r dimensions, and for μ they had Lebesgue measure L in that space. Their derivation basis $\mathfrak{B}$ associated with each point $x \in S$ the family of all closed intervals containing x, with sides parallel to the coordinate axes, and convergence to x was taken with respect to the diameters tending to zero. For this basis $\mathfrak{B}$, they showed that for any α, $0 < \alpha < 1$, and any L-measurable set M,

$$L(S(\alpha, M)) = L^*(S(\alpha, M)) \leqslant C\phi(1/\alpha)L(M) \tag{I}$$

where $\phi(t) = t(\log^+ t)^{r-1}$ for each $t > 0$, and that $\mathfrak{B}$ differentiates the integrals of all L-measurable functions f such that $|f|(\log^+|f|)^{r-1}$ is L-summable over S.

If we take $\sigma(t) = t(\log^+ t)^p$ for $t > 0$, where p is any constant greater than 1, it is easily checked that

$$\sum_{n=1}^{\infty} \lambda^n/\sigma(\lambda^n)$$

converges for any $\lambda > 1$, so that σ satisfies the requirements of this section. It is well known that $\mathfrak{B}$ possesses the L-Vitali property (which is equivalent to the L-density property); because of (I) ,Theorem 3.6 ensures the $\mathfrak{B}$-differentiability of the integral of any L-measurable function f such that $\phi(\sigma(|f|))$ is L-summable over S. In this particular case, it is easily confirmed by routine computation that this includes all functions f such that $|f|(\log^+|f|)^{r-1+p}$ is L-summable over S.

Thus Theorem 3.6 falls somewhat short of the result obtained in **(5)**, which is the best possible at all; however, that result depends on the special properties of Lebesgue measure and similar figures in Euclidean space.

REFERENCES

1. J. C. Burkill, *On the differentiability of multiple integrals*, J. London Math. Soc., *26* (1951), 244–249.

2. H. Busemann and W. Feller, *Zur Differentiation der Lebesgueschen Integrale*, Fund. Math., *22* (1934), 226–256.

3. P. Halmos, *Measure theory* (New York, 1950).

4. C. Hayes and C. Y. Pauc, *Full individual and class differentiation theorems in their relations to halo and Vitali properties*, Can. J. Math., *7* (1955), 221–274.

5 B. Jessen, J. Marcinkiewicz and A. Zygmund, *Note on the differentiability of multiple integrals*, Fund. Math., *25* (1935), 217–234.

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