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PROJECTIVE TENSOR PRODUCTS AND THE DUNFORD–PETTIS PROPERTY

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We show the existence, in $c_0 \otimes_\gamma c_0$, of a shrinking, subsymmetric basic sequence (z_i) having no c_0 subsequences. In particular, the coefficient functionals for (z_i) are weakly null and the subspace $\overline{sp}(z_i)$ fails to have the Dunford-Pettis property.

PRELIMINARIES

Throughout, we will use λ and γ for the least and greatest crossnorms repsectively. The sequence spaces c_0 and l_1 will be given their usual meanings and norms. The unit vectors for these spaces will be given by (e_n) and (e_n^*). It is known that the sequences $(e_i \otimes e_j)$ and $(e_i^* \otimes e_j^*)$ are respective *conditional* bases for $c_0 \otimes_\gamma c_0$ and $l_1 \otimes_\gamma l_1$. The assumed ordering for both of these bases is the "upper rectangular" ordering given in [**3**]. A vector $x = \sum_{ij} \beta_{ij} e_i^* \otimes e_j^*$ in $l_1 \otimes_\lambda l_1$ will be written in matrix form (β_{ij}) for short. Again, the sum is in the [**3**] ordering.

We recall that the definition of $\lambda(w)$ for $w = (\beta_{ij})$ in $l_1 \otimes_\lambda l_1$ is given by:

$$\lambda(w) = \sup \sum_i \left| \sum_j \eta_j \beta_{ij} \right| = \sup \sum_j \left| \sum_i \zeta_i \beta_{ij} \right|.$$

Here, the suprema are taken over (η_j) and (ζ_i) in *ball* l_∞.

The spaces $c_0 \otimes \gamma c_0$ and $l_1 \otimes_\lambda l_1$ are among many spaces identifiable as "matrix spaces". Such spaces are studied extensively in [**5**]. In that paper, the authors define spaces of infinite, finitely non-zero, matrices endowed with a variety of "matrix norms". The "*n th* main triangle projection" on the completion of such a space is given by:

$$T_n((a_{ij})) = \sum_{i+j \leqslant n+1} a_{ij} u_{ij};$$

where u_{ij} has 1 in position ij; and 0 elsewhere.

Of particular interest to us are the matrix spaces $M_{\lambda_{11}}$ and $M_{\lambda_{11}^*}$ which are isometric to $l_1 \otimes_\lambda l_1$ and $c_0 \otimes_\gamma c_0$ respectively. (See [**5**], p. 45, for the definition of

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conjugate norms α^*; and p. 47 for the definition of the norms λ_{pq}). In Proposition 1.2 of [**5**], the authors show that:

$$\|T_n\| \geqslant C \log(n)$$

for some $C > 0$ independent of n; where the underlying space is $M_{\lambda_{11}}$. Later, they show (Lemma 6.1, p. 64) that $\lambda_{11}^*(b_n) = \|T_n\|$ for the matrix b_n defined by:

$$b_n(i,j) = \begin{cases} 1 & \text{for } n \geqslant i \geqslant j \geqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

The result given in the abstract will be proved by a sequence of propositions, beginning with the fullowing duality result.

Proposition 1. *The dual space of $c_0 \otimes_\gamma c_0$ is identifiable with $l_1 \otimes_\lambda l_1$. In particular, the basis $(e_i \otimes e_j)$ is shrinking.*

Proof: It is well known [**1**] that $(c_0 \otimes_\gamma c_0)^*$ is canonically identifiable with $L(c_0, l_1)$, the space of continuous linear operators from c_0 into l_1. The action of the identifying mapping is accomplished by extending the transformation:

$$T(x \otimes y) = (Tx)(y) \quad ;\text{for } T \in L(c_0, l_1).$$

It is easily established directly or by using [**2**] that each element of $L(c_0, l_1)$ is a weakly compact operator; hence compact, since l_1 has the Schur property. Thus, because $c_0^* = l_1$ has Grothendieck's approximation property, the algebraic tensor product $l_1 \otimes l_1$ (under the injective norm) is canonically identifiable with a norm *dense* linear subspace of $L(c_0, l_1)$ [**1**].

That $(e_i \otimes e_j)$ is shrinking follows from the established duality and the fact that $(e_i^* \otimes e_j^*)$ are the coefficient functionals. ∎

We now define the basic sequence $(z_i) \subseteq c_0 \otimes_\gamma c_0$ as follows:

$$z_i = e_i \otimes \sum_{j=1}^{i} e_j .$$

Since (z_i) is a block sequence of a shrinking basis, it is shrinking as well. Also, it is clearly normalized. Moreover, by our duality result and the Hahn-Banach theorem, the norm in $\overline{sp}(z_i)$ is given by:

$$\gamma\left(\sum_i \alpha_i z_i\right) = \sup\left|\sum_i \alpha_i w(z_i)\right|.$$

Here, the α_i's are scalars and the sup is over all $w = (\beta_{ij}) \in$ *ball* $l_1 \otimes_\lambda l_1$.

Before our next result, recall that a basic sequence is called *sub-symmetric* if it is unconditional and equivalent to each of its subsequences.

Proposition 2. *The basic sequence (z_i) is sub-symmetric.*

Proof: We first show unconditionality. To do this, we let N be a fixed integer, $I = \{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, \dots, N\}$, and $(\alpha_i)_{i \leqslant N}$ be arbitrary scalars. By the above remarks on the γ norm, we can show that

$$\gamma\left(\sum_{i_p \in I} \alpha_{i_p} z_{i_p}\right) \leqslant \gamma\left(\sum_{i \leqslant N} \alpha_i z_i\right),$$

if given $w = (\beta_{ij}) \in$ *ball* $l_1 \otimes_\lambda l_1$, there is a $\hat{w}$ in the same ball for which:

$$\left| w\left(\sum_{i_p \in I} \alpha_{i_p} z_{i_p}\right)\right| = \left|\hat{w}\left(\sum_{i \leqslant N} \alpha_i z_i\right)\right|.$$

So, let such a w be given. Then from the definition of the λ norm, the vector $\hat{w} = \left(\hat{\beta}_{ij}\right)$ formed from w by replacing all rows in $\{1, 2, \dots, N\} - I$ with zeros is still in ball $l_1 \otimes_\lambda l_1$ and satisfies the above equation. Thus (z_i) is unconditional with unconditional constant $\leqslant 1$.

Next, we first show that for a given increasing sequence (i_k) of integers, any fixed integer K, and scalars $(\alpha_k)_{k \leqslant K}$ that:

$$\gamma\left(\sum_{k=1}^{K} \alpha_k z_k\right) \leqslant \gamma\left(\sum_{k=1}^{K} \alpha_k a_{i_k}\right) \quad (*).$$

For this it suffices to show that given $w = (\beta_{ij}) \in$ *ball* $l_1 \otimes_\lambda l_1$, there is $\hat{w} = \left(\hat{\beta}_{ij}\right)$ in the same ball satisfying the equations:

$$w(z_k) = \hat{w}\left(z_{i_k}\right); 1 \leqslant k \leqslant K.$$

This is easily done by "spreading out" the matrix for w and defining $\hat{w}$'s coordinates by

$$\hat{\beta}_{i_p i_q} = \beta_{pq}, 1 \leqslant p, q \leqslant K;$$
$$\hat{\beta}_{ij} = 0 \text{ for all other positions.}$$

Clearly, $\hat{w}$ satisfies the equations and, by the definition of the λ norm, $\lambda(\hat{w}) = \lambda(w)$. To get the other half of inequality $(*)$ it suffices to solve the equations:

$$w\left(z_{i_k}\right) = \hat{w}(z_k), \qquad 1 \leqslant k \leqslant K;$$

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where $\lambda(w) \leqslant 1$ is given and $\hat{w}$ satisfying $\lambda(\hat{w}) \leqslant 1$ is to be found. To get $\hat{w} = \left(\hat{\beta}_{pq}\right)$, we define

$$\hat{\beta}_{pq} = \sum_{i_{q-1}+1 \leqslant t \leqslant i_q} \beta_{i_p t}; \text{ where } i_0 \equiv 0.$$

It is straightforward that $\hat{w}$ satisfies the required equations. To see that $\lambda(\hat{w}) \leqslant 1$, suppose that $(\eta_k) \in$ *ball* l_∞ is arbitrary. We define $(\xi_j) \in$ *ball* l_∞ by: $\xi_j = \eta_k$ for $i_{k-1} + 1 \leqslant j \leqslant i_k$. Then

$$\sum_i \left| \sum_k \eta_k \hat{\beta}_{ik} \right| = \sum_i \left| \sum_j \xi_j \beta_{ij} \right| \leqslant 1.$$

Thus, by taking the supremum over (η_k), we get $\lambda(\hat{w}) \leqslant 1$ as desired. ∎

The following is the crucial step in our proof:

PROPOSITION 3. *The basic sequence (z_i) has no c_0 subsequences.*

PROOF: This is shown in [5] (Lemma 6.1, p. 64). Specifically, the authors show the existence of an independent constant $C > 0$ for which:

$$\gamma\left(\sum_{i=1}^{n} z_i\right) \geqslant C \log(n); \text{for all } n.$$

Thus, (z_i) is not a c_0 sequence; and, by Proposition 2, it has no c_0 subsequences. ∎

PROPOSITION 4. *If $Z = \overline{sp}(z_i)$, then the coefficient functionals $(f_i) \subseteq Z^*$ are $\sigma(Z^*, Z^{**})$ null. In particular, Z fails to have the Dunford–Pettis property.*

PROOF: Let $F \in Z^{**}$ have norm equal to one. If K is a fixed integer we choose $z \in$ *ball* Z for which $|f_i(z) - F(f_i)| \leqslant 2^{-i}$ for $i \leqslant K$. The triangle inequality yields:

$$\left\|\sum_{i=1}^{K} F(f_i) z_i\right\| \leqslant \|F\| + \|z\| \leqslant 2.$$

By the unconditionality of (z_i) and the log inequality given above, the sequence $(F(f_i))$ is an element of c_0; whence, (f_i) is weakly null. Finally, since (z_i) is shrinking, it is weakly null; so Z fails the Dunford–Pettis property by Grothendieck's criterion [4]. ∎

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