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INF-SUP CONDITIONS FOR FINITE-DIFFERENCE APPROXIMATIONS OF THE STOKES EQUATIONS

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Abstract

Inf-sup conditions are proven for three finite-difference approximations of the Stokes equations. The finite-difference approximations use a staggered-mesh scheme and the schemes resulting from the backward and the forward differencings.

1. Introduction

Inf-sup conditions, which have been introduced independently by Babuška [3] and Brezzi [5], are important to study the linear boundary-value problems with a constraint such as the following.

Find $(u, p) \in X \times M$ satisfying

$$\begin{aligned} Au + B'f &= f \quad \text{in } X', \\ Bu &= g \quad \text{in } M', \end{aligned} \tag{1.1}$$

where X and M are two Hilbert spaces, X' and M' are their corresponding dual spaces, and $A \in L(X; X')$ and $B \in L(X; M')$ are two linear operators with $B' \in L(M; X')$ as the dual operator of B.

The linear operators A and B are associated with the bilinear forms

$$a(.,.) : X \times X \to \mathbb{R}, \qquad b(.,.) : X \times M \to \mathbb{R}.$$

Let $\langle .,. \rangle$ denote the duality pairing between the spaces X and X' or M and M'. Then (1.1) is equivalent to the following variational problem.

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Given $f \in X'$ and $g \in M'$, find a pair $(u, p) \in X \times M$ such that

$$\begin{aligned} a(u, v) + b(v, p) &= \langle f, v\rangle \qquad \forall v \in X, \\ b(u, q) &= \langle g, q\rangle \qquad \forall q \in M. \end{aligned} \tag{1.2}$$

The inf-sup condition related to (1.2) is

$$\exists C > 0 \quad \text{such that} \inf_{p \in M\setminus\{0\}} \sup_{u \in X} \frac{b(u, p)}{\|u\|_X \|p\|_M} \geq C. \tag{1.3}$$

The bilinear form $a(.,.)$ in (1.2) is related to the norm $\|.\|_X$ in (1.3) for most problems.

The inf-sup conditions in the continuous problems and the finite-element problems are studied extensively in many places, for example Aziz and Babuška [2] and Babuška [3], Brezzi [5] and Girault and Raviart [6]. On the other hand, conditions for three finite-difference approximations of the Stokes problem are proven for the first time by Shin in his Ph.D thesis [10]. This result is simplified and shown in this paper.

The finite-difference schemes that we are interested in this paper are a staggered-mesh scheme and the schemes that come from the backward and the forward differencings. These schemes are rather simple and hence serve well to the theoretical point of view. The proofs are done by setting a relation between a continuous space and its finite-difference approximation space and uses the inf-sup condition of the continuous space.

2. Definitions

Let Ω be in a domain in $\mathbb{R}^d$ and let Γ be its boundary. For simplicity, we focus on the case when $d = 2$, but the results in this paper will hold for any $d \geq 2$. We denote by $L^2(\Omega)$ the space of real functions defined on Ω which are integrable in the L^2 sense with the usual inner product and norm

$$(u, v)_\Omega := \iint_\Omega uv\, dA, \quad \|u\|_\Omega^2 := (u, u)_\Omega.$$

Let

$$H_0^1(\Omega) := \{u \in L^2(\Omega) \mid u_x, u_y \in L^2(\Omega) \text{ and } u|_\Gamma = 0\}$$

have respectively the inner product and norm

$$(u, v)_{1,\Omega} := \iint_\Omega \nabla u \cdot \nabla v\, dA, \quad \|u\|_{1,\Omega}^2 := (u, u)_{1,\Omega}$$

and

$$L_0^2(\Omega) := \{p \in L^2(\Omega) \mid (p, 1)_\Omega = 0\}.$$

We use the notation $\mathbf{u} = (u_i)$ for a vector. We shall often be concerned with two-dimensional vector functions with components in $L^2(\Omega)$ or $H_0^1(\Omega)$. The notation $L^2(\Omega)^2$, $H_0^1(\Omega)^2$ will be used for the product spaces. Define, for $\mathbf{u}$ and $\mathbf{v} \in L^2(\Omega)^2$,

$$(\mathbf{u}, \mathbf{v})_\Omega := \sum_{i=1}^{2} (u_i, v_i)_\Omega, \qquad \|\mathbf{u}\|_\Omega^2 := (\mathbf{u}, \mathbf{u})_\Omega$$

and for $\mathbf{u}$ and $\mathbf{v} \in H_0^1(\Omega)^2$,

$$(\mathbf{u}, \mathbf{v})_{1,\Omega} := \sum_{i=1}^{2} (u_i, v_i)_{1,\Omega}, \qquad \|\mathbf{u}\|_{1,\Omega}^2 := (\mathbf{u}, \mathbf{u})_{1,\Omega}.$$

We also make some definitions analogous to the above on discrete subsets of the unit square S in $\mathbb{R}^2$. Let

$$S := \{(x, y) \in \mathbb{R}^2 \mid 0 < x, y < 1\}$$

and T be a boundary. Let

$$h := \frac{1}{N}, \qquad \text{for some } N \in \mathbb{N},$$
$$\mathbb{R}_h^2 := \{(lh, mh) \in \mathbb{R}^2 \mid l, m \in \mathbb{N}\},$$
$$S_h := \bar{S} \cap \mathbb{R}_h^2,$$

where $\bar{S}$ is the closure of S. Define

$$S_{l,m} := \{(x, y) \in S \mid (l-1)h < x < lh, (m-1)h < y < mh\}$$

for $l, m = 1, \ldots, N$. Figure 1 shows $S_{l,m}$ when $N = 3$.

For an arbitrary discrete set Ω_h of the form

$$\Omega_h := \{(lh, mh) \in S_h \mid l_0 \le l \le l_1 \text{ and } m_0 \le m \le m_1\},$$

we define

$$\Omega_h^0 := \{(lh, mh) \in S_h \mid l_0 + 1 \le l \le l_1 - 1, m_0 + 1 \le m \le m_1 - 1\},$$
$$e(\Omega_h) := \{(lh, mh) \in S_h \mid l_0 + 1 \le l \le l_1, m_0 \le m \le m_1\},$$
$$w(\Omega_h) := \{(lh, mh) \in S_h \mid l_0 \le l \le l_1 - 1, m_0 \le m \le m_1\},$$
$$s(\Omega_h) := \{(lh, mh) \in S_h \mid l_0 \le l \le l_1, m_0 \le m \le m_1 - 1\},$$
$$n(\Omega_h) := \{(lh, mh) \in S_h \mid l_0 \le l \le l_1, m_0 + 1 \le m \le m_1\}$$

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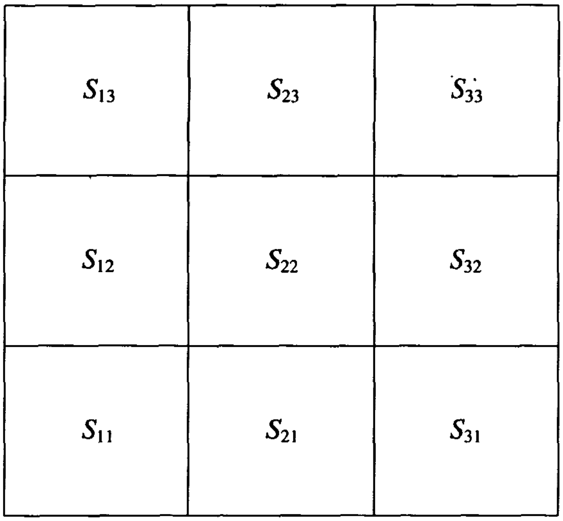

FIGURE 1. $S_{l,m}$ when $N = 3$

as the interior, east, west, south and the north sides of Ω_h and define

$$\begin{aligned} se(\Omega_h) &:= s(\Omega_h) \cap e(\Omega_h), & sw(\Omega_h) &:= s(\Omega_h) \cap w(\Omega_h) \\ ne(\Omega_h) &:= n(\Omega_h) \cap e(\Omega_h), & nw(\Omega_h) &:= n(\Omega_h) \cap w(\Omega_h). \end{aligned}$$

For the boundary Γ_h of Ω_h, we define

$$e(\Gamma_h), \qquad w(\Gamma_h), \qquad s(\Gamma_h), \qquad n(\Gamma_h)$$

as the east, west, south and north parts of Γ_h including the end points.

In this paper, we want to study both standard and staggered grids. The staggered-mesh schemes use different grids that are staggered for the pressure and the velocity. A staggered grid is shown in Figure 2. The points marked by P, I, and II are where the pressure and the first and the second components of the velocity are defined, respectively.

Let

$$\begin{aligned} S_p &:= \left\{ \left(\left(l - \frac{1}{2} \right) h, \left(m - \frac{1}{2} \right) h \right) \in S \mid l, m = 1, \dots, N \right\}, \\ S_I &:= \left\{ \left(lh, \left(m - \frac{1}{2} \right) h \right) \in S \mid l = 1, \dots, N, m = 0, \dots, N+1 \right\}, \\ S_{II} &:= \left\{ \left(\left(l - \frac{1}{2} \right) h, mh \right) \in S \mid l = 0, \dots, N+1, m = 1, \dots, N \right\}. \end{aligned}$$

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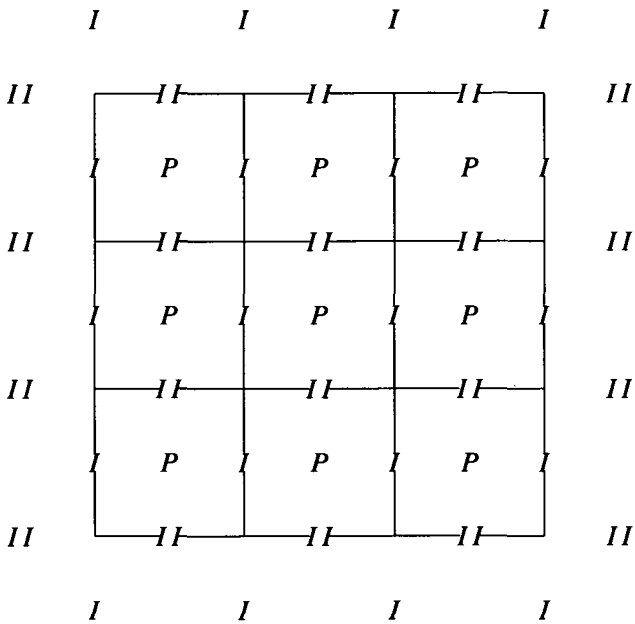

FIGURE 2. Staggered grid

Then these are the sets for P, I, and II. Figure 2 shows S_p, S_I and S_{II} when $N = 3$. Staggered-mesh schemes have been used by Amsden and Harlow [1], Brandt and Dinar [4], Harlow and Welch [7], Patankar and Spalding [8], Raithby and Schneider [9] and others.

Let $L^2(\Omega_h)$ be the space of all discrete functions defined on Ω_h with inner product and norm

$$(U, V)_{\Omega_h} := h^2 \sum_{(x,y)\in\Omega_h} U(x, y)V(x, y), \qquad \|U\|^2_{\Omega_h} := (U, U)_{\Omega_h}$$

respectively and let

$$L_0^2(\Omega_h) := \{P \in L^2(\Omega_h) \mid (P, 1)_{\Omega_h} = 0\}.$$

Then $L^2(\Omega_h)$ and $L_0^2(\Omega_h)$ and the discrete analogies of $L^2(\Omega)$ and $L_0^2(\Omega)$.

For notational convenience, we introduce

$$U_{l,m} := U(lh, mh),$$

and define the forward, backward and central differencings on the x axis and y axis, respectively, as

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$$(\delta_{x+}U)_{l,m} := \frac{U_{l+1,m} - U_{l,m}}{h}, \qquad (\delta_{y+}U)_{l,m} := \frac{U_{l,m+1} - U_{l,m}}{h},$$
$$(\delta_{x-}U)_{l,m} := \frac{U_{l,m} - U_{l-1,m}}{h}, \qquad (\delta_{y-}U)_{l,m} := \frac{U_{l,m} - U_{l,m-1}}{h},$$
$$(\delta_{x0}U)_{l,m} := \frac{U_{l+\frac{1}{2},m} - U_{l-\frac{1}{2},m}}{h}, \qquad (\delta_{y0}U)_{l,m} := \frac{U_{l,m+\frac{1}{2}} - U_{l,m-\frac{1}{2}}}{h}.$$

Define the discrete gradients as

$$\nabla_+ := (\delta_{x+}, \delta_{y+}), \qquad \nabla_- := (\delta_{x-}, \delta_{y-}), \qquad \nabla_0 := (\delta_{x0}, \delta_{y0}),$$

and let ∇_h^2 be the five-point discrete Laplacian, then $\nabla_h^2 = \nabla_- \cdot \nabla_+ = \nabla_+ \cdot \nabla_-$.

The inner product and the norm of

$$H_0^1(\Omega_h) := \{U \in L^2(\Omega_h) \mid U|_{\Gamma_h} = 0\}$$

are defined as

$$(U, V)_{1,\Omega_h} := (\nabla_+ U, \nabla_+ V)_{sw(\Omega_h)} = (\nabla_- U, \nabla_- V)_{ne(\Omega_h)}, \qquad \|U\|_{1,\Omega_h}^2 := (U, U)_{1,\Omega_h}$$

which are the sums over all points in Ω_h where difference quotients are defined. The inner product and the norm of the product spaces $L^2(\Omega_h)^2$ and $H_0^1(\Omega_h)^2$ are defined naturally from $L^2(\Omega_h)$ and $H_0^1(\Omega_h)$.

3. Inf-Sup conditions for finite-difference spaces

To show the inf-sup conditions for finite-difference spaces which come from approximations of the Stokes problem, we begin with the related theory for partial differential equations. The steady-state Stokes equations in $\mathbb{R}^d$ are

$$\begin{aligned} \nabla_h^2 \mathbf{u} + \nabla p &= f, \\ \nabla \cdot \mathbf{u} &= g \qquad \text{in } \Omega \subset \mathbb{R}^d, \end{aligned}$$

where the velocity $\mathbf{u}$ is a vector of dimension d and the pressure p is a scalar. Refer to Aziz and Babuška [2] for the proof of the next theorem.

THEOREM 1. *Let Ω be a bounded domain with a Lipschitz-continuous boundary. Then there exists a positive constant $C_p = C_p(\Omega)$ such that any $p \in L_0^2(\Omega)$ has a vector $\mathbf{u} \in H_0^1(\Omega)^2$ which satisfies*

$$\nabla \cdot \mathbf{u} = p \quad \textit{in} \quad \Omega \quad \textit{and} \quad \|\mathbf{u}\|_{1,\Omega}^2 \le C_p \|p\|_{\Omega}^2.$$

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The above theorem implies the so-called inf-sup condition for the Stokes problem.

THEOREM 2.

$$\inf_{p\in L_0^1(\Omega)\setminus\{0\}} \sup_{\mathbf{u}\in H_0^1(\Omega)^2} \frac{(\nabla\cdot\mathbf{u}, p)_\Omega^2}{\|\mathbf{u}\|_{1,\Omega}^2\|p\|_\Omega^2} \geq C_p^{-1}.$$

By the next theorem, we will get the inf-sup conditions for finite-difference spaces.

THEOREM 3. *There exist positive constant C_f, which is independent of h, such that*

(1) *any $P \in L_0^2(S_P)$ has a vector $\mathbf{U} \in H_0^1(S_I) \times H_0^1(S_{II})$ which satisfies*

$$(\nabla_0\cdot\mathbf{U}, P)_{S_P} = \|P\|_{S_P}^2, \qquad \|U_1\|_{1,S_I}^2 + \|U_2\|_{1,S_{II}}^2 \leq C_f\|P\|_{S_P}^2;$$

(2) *any $P \in L_0^2(S_h^0)$ has a vector $\mathbf{U} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))$ which satisfies*

$$(\nabla_-\cdot\mathbf{U}, P)_{S_h^0} = C_1\|P\|_{S_h^0}^2, \qquad \|U_1\|_{1,w(S_h)}^2 + \|U_2\|_{1,s(S_h)}^2 \leq C_f\|P\|_{S_h^0}^2;$$

(3) *any $P \in L_0^2(S_h^0)$ has a vector $\mathbf{U} \in H_0^1(e(S_h)) \times H_0^1(n(S_h))$ which satisfies*

$$(\nabla_+\cdot\mathbf{U}, P)_{S_h^0} = C_1\|P\|_{S_h^0}^2, \qquad \|U_1\|_{1,e(S_h)}^2 + \|U_2\|_{1,n(S_h)}^2 \leq C_f\|P\|_{S_h^0}^2.$$

Setting $C := 1/C_f$, we get the following inf-sup conditions for some finite-difference spaces.

THEOREM 4. *There exists a positive constant C, which is independent of h, such that*

(1) $$\sup_{\mathbf{U}\in H_0^1(S_I)\times H_0^1(S_{II})} \frac{(\nabla_0\cdot U, P)_{S_P}^2}{\|U_1\|_{1,S_I}^2 + \|U_2\|_{1,S_{II}}^2} \geq C\|P\|_{S_P}^2, \qquad \forall P \in L_0^2(S_p),$$

(2) $$\sup_{\mathbf{U}\in H_0^1(w(S_h))\times H_0^1(s(S_h))} \frac{(\nabla_-\cdot U, P)_{S_h}^0}{\|U_1\|_{1,w(S_h)}^2 + \|U_2\|_{1,s(S_h)}^2} \geq C\|P\|_{S_h^0}^2, \qquad \forall P \in L_0^2(S_h^0),$$

(3) $$\sup_{\mathbf{U}\in H_0^1(e(S_h))\times H_0^1(n(S_h))} \frac{(\nabla_+\cdot U, P)_{S_h}^0}{\|U_1\|_{1,e(S_h)}^2 + \|U_2\|_{1,n(S_h)}^2} \geq C\|P\|_{S_h^0}^2, \qquad \forall P \in L_0^2(S_h^0),$$

PROOF OF THEOREM 3. We first prove (1). Let $P \in L_0^2(S_P)$. Then we define the piecewise-constant function $p \in L^2(S)$ by

$$p|_{S_{l,m}} := P_{l-\frac{1}{2},m-\frac{1}{2}}, \tag{3.1}$$

for $l, m = 1, \ldots, N$. Note that

$$(p, 1)_S = (P, 1)_{S_P} = 0 \quad \text{and} \quad \|p\|_S = \|P\|_{S_P}. \tag{3.2}$$

Since $p \in L_0^2(S)$, by Theorem 1, there exists a vector $\mathbf{u} = (u_1, u_2) \in H_0^1(S)^2$ such that

$$\nabla \cdot \mathbf{u} = p \quad \text{in} \quad S \quad \text{and} \quad \|\mathbf{u}\|_{1,S}^2 \leq C_p \|p\|_S^2. \tag{3.3}$$

For $t \in [0, 1]$, define the line segments

$$x_l(t) := \begin{cases} (l-t)h, & \text{if } 1 \leq l \leq N; \\ 0 & \text{if } l = 0; \\ N & \text{if } l = N+1, \end{cases}$$

$$y_m(t) := \begin{cases} (m-t)h, & \text{if } 1 \leq m \leq N; \\ 0 & \text{if } m = 0; \\ N & \text{if } m = N+1. \end{cases}$$

If we define

$$(U_1)_{l,m-\frac{1}{2}} := \int_0^1 u_1(lh, y_m(t))\, dt,$$

for $l = 0, \dots, N$ and $m = 0, \dots, N+1$, and

$$(U_2)_{l-\frac{1}{2},m} := \int_0^1 u_2(x_l(t), mh)\, dt,$$

for $l = 0, \dots, N+1$ and $m = 0, \dots, N$, then

$$\mathbf{U} := (U_1, U_2) \in H_0^1(S_I) \times H_0^1(S_{II})$$

since $\mathbf{u} \in H_0^1(S)^2$.

Let's first show that

$$(\nabla_0 \cdot \mathbf{U}, P)_{S_P} = \|P\|_{S_P}^2.$$

By (3.2) and (3.3),

$$\|P\|_{S_P}^2 = \|p\|_S^2 = (\nabla \cdot \mathbf{u}, p)_S.$$

Hence it is sufficient to show that

$$(\nabla_0 \cdot \mathbf{U}, P)_{S_P} = (\nabla \cdot \mathbf{u}, p)_S.$$

Using change of variable and the definitions of U_1 and U_2, one gets

$$(\delta_{x0} U_1)_{l-\frac{1}{2},m-\frac{1}{2}} = \frac{1}{h^2} \iint_{S_{l,m}} \frac{\partial u_1}{\partial x}\, dA, \tag{3.4}$$

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and

$$(\delta_{y0}U_2)_{l-\frac{1}{2},m-\frac{1}{2}} = \frac{1}{h^2}\iint_{S_{l,m}} \frac{\partial u_2}{\partial y}\,dA, \tag{3.5}$$

for $l, m = 1, \ldots, N$. Hence, using (3.1), (3.4) and (3.5),

$$\begin{aligned}(\nabla_0 \cdot \mathbf{U}, P)_{S_P} &= h^2 \sum_{l,m=1}^{N} (\nabla_0 \cdot \mathbf{U})_{l-\frac{1}{2},m-\frac{1}{2}} P_{l-\frac{1}{2},m-\frac{1}{2}} \\ &= \sum_{l,m=1}^{N} \iint_{S_{l,m}} (\nabla \cdot \mathbf{u}) p\,dA = (\nabla \cdot \mathbf{u}, p)_S.\end{aligned}$$

Next we show that, for some constant C_f,

$$\|U_1\|^2_{1,S_I} + \|U_2\|^2_{1,S_{II}} \le C_f \|P\|^2_{S_P}. \tag{3.6}$$

By (3.4) and the Schwarz inequality, we have

$$\begin{aligned}\|\delta_{x-}U_1\|^2_{e(S_I)} = \|\delta_{x0}U_1\|^2_{S_P} &= h^2 \sum_{l,m=1}^{N} \left(\frac{1}{h^2}\iint_{S_{l,m}} \frac{\partial u_1}{\partial x}\,dA\right)^2 \\ &\le \frac{1}{h^2}\sum_{l,m=1}^{N} \operatorname{area}(S_{l,m}) \iint_{S_{l,m}} \left(\frac{\partial u_1}{\partial x}\right)^2 dA = \iint_S \left(\frac{\partial u_1}{\partial x}\right)^2 dA.\end{aligned} \tag{3.7}$$

To evaluate $\|\delta_{y-}U_1\|_{n(S_I)}$, we need some definitions. Let

$$D_{l,m} := \{(x_l(t), y_m(t)) \mid t \in [0, 1]\},$$

for $l, m = 1, \ldots, N$, then $D_{l,m}$ is a diagonal line segment in $S_{l,m}$ and generates an upper and a lower triangles in $S_{l,m}$, which we denote by $U_{l,m}$ and $L_{l,m}$, respectively. Figure 3 shows $D_{l,m}$, $U_{l,m}$ and $L_{l,m}$ when $N = 3$. Define

$$(U_1^D)_{l,m} := \int_0^1 u_1(x_1(t), y_m(t))\,dt,$$

for $l, m = 1, \ldots, N$.

Note that

$$\begin{aligned}&(\delta_{y-}U_1)_{l,m-\frac{1}{2}} \\ &\quad= \frac{1}{h}\left((U_1)_{l,m-\frac{1}{2}} - (U_1)_{l,m-\frac{3}{2}}\right) \\ &\quad= \frac{1}{h}\left((U_1)_{l,m-\frac{1}{2}} - (U_1^D)_{l,m} + (U_1^D)_{l,m} - (U_1^D)_{l,m-1} + (U_1^D)_{l,m-1} - (U_1^D)_{l,m-\frac{3}{2}}\right),\end{aligned}$$

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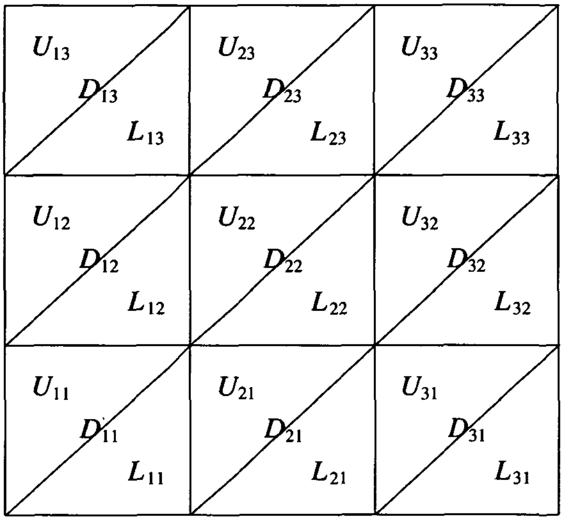

FIGURE 3. $D_{l,m}$, $U_{l,m}$ and $L_{l,m}$ when $N = 3$.

for $l = 1, \ldots, N$ and $m = 2, \ldots, N$. Similarly to (3.4), one gets

$$(\delta_{y-} U_1)_{l,m-\frac{1}{2}} = \frac{1}{h^2}\left(\iint_{L_{l,m}} \frac{\partial u_1}{\partial x}\, dA + \iint_{L_{l,m} \cup U_{l,m-1}} \frac{\partial u_1}{\partial y}\, dA - \iint_{L_{l,m-1}} \frac{\partial u_1}{\partial y}\, dA\right). \tag{3.8}$$

By the way that u_1 and U_1 are defined, for $m = 1$,

$$(U_1)_{l,m-\frac{3}{2}} = 0 = \int_0^1 u_1(x_l(t), 0)\, dt,$$

and, for $m = N + 1$,

$$(U_1)_{l,m-\frac{1}{2}} = 0 = \int_0^1 u_1(x_l(t), 1)\, dt.$$

Hence, for $m = 1$,

$$\begin{aligned}
&(\delta_{y-} U_1)_{l,m-\frac{3}{2}} \\
&\quad = \frac{1}{h}\left(\int_0^1 u_1(lh, y_m(t))\, dt - \int_0^1 u_1(x_l(t), 0)\, dt\right) \\
&\quad = \frac{1}{h}\int (u_1(lh, y_m(t)) - u_1(x_l(t), y_m(t)) + u_1(x_l(t), y_m(t)) - u_1(x_l(t), 0))\, dt \\
&\quad = \frac{1}{h^2}\left(\iint_{L_{l,m}} \frac{\partial u_1}{\partial x}\, dA + \iint_{L_{l,m}} \frac{\partial u_1}{\partial y}\, dA\right).
\end{aligned} \tag{3.9}$$

Similarly, for $m = N + 1$,

$$
\begin{aligned}
(\delta_{y-}U_1)_{l,m-\frac{1}{2}}
&= \frac{1}{h}\left(\int_0^1 u_1(x_l(t), 1)\,dt - \int_0^1 u_1(lh, y_m(t))\,dt\right) \\
&= \frac{1}{h}\int (u_1(x_l(t), 1) - u_1(x_l(t), y_m(t)) + u_1(x_l(t), y_m(t)) - u_1(lh, y_m(t)))\,dt \\
&= \frac{1}{h^2}\left(\iint_{U_{l,m}} \frac{\partial u_1}{\partial y}\,dA + \iint_{L_{l,m}} \frac{\partial u_1}{\partial x}\,dA\right).
\end{aligned} \tag{3.10}
$$

Note that, for any real numbers, x, y and z,

$$
(x+y+z)^2 \le x^2 + y^2 + z^2 + 2(xy + yz + xz) \le 3(x^2 + y^2 + z^2). \tag{3.11}
$$

Applying the Schwarz inequality to (3.8) and using (3.11), one gets

$$
\begin{aligned}
(\delta_{y-}U_1)^2_{l,m-\frac{1}{2}} \le & \frac{3}{h^4}\Bigg(\text{area}(L_{l,m}) \iint_{L_{l,m}} \left(\frac{\partial u_1}{\partial x}\right)^2 dA \\
& + \text{area}(L_{l,m} \cup U_{l,m}) \iint_{L_{l,m} \cup U_{l,m-1}} \left(\frac{\partial u_1}{\partial y}\right)^2 dA \\
& + \text{area}(L_{l,m-1}) \iint_{L_{l,m-1}} \left(\frac{\partial u_1}{\partial y}\right)^2 dA\Bigg) \\
= & \frac{3}{h^2}\Bigg(\iint_{L_{l,m}} \left(\frac{\partial u_1}{\partial x}\right)^2 dA + 2\iint_{L_{l,m} \cup U_{l,m-1}} \left(\frac{\partial u_1}{\partial y}\right)^2 dA \\
& + \iint_{L_{l,m-1}} \left(\frac{\partial u_1}{\partial y}\right)^2 dA\Bigg)
\end{aligned} \tag{3.12}
$$

for $l = 1, \ldots, N-1$ and $m = 2, \ldots, N$. Similarly (3.9) implies, for $m = 1$,

$$
(\delta_{y-}U_1)^2_{l,m-\frac{1}{2}} \le \frac{2}{h^2}\left(\iint_{L_{l,m}} \left(\frac{\partial u_1}{\partial x}\right)^2 dA + \iint_{L_{l,m}} \left(\frac{\partial u_1}{\partial y}\right)^2 dA\right) \tag{3.13}
$$

and (3.10) implies, for $m = N + 1$,

$$
(\delta_{y-}U_1)^2_{l,m-\frac{1}{2}} \le \frac{2}{h^2}\left(\iint_{U_{l,m}} \left(\frac{\partial u_1}{\partial y}\right)^2 dA + \iint_{L_{l,m}} \left(\frac{\partial u_1}{\partial x}\right)^2 dA\right). \tag{3.14}
$$

Combining equations (3.12) to (3.14) gives

$$
\|\delta_{y-}U_1\|^2_{n(S_l)} \le 6\|u_1\|^2_{1,S}. \tag{3.15}
$$

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By (3.7) and (3.15), one gets

$$\|U_1\|_{1,S_l}^2 = \|\delta_{x-}U_1\|_{e(S_l)}^2 + \|\delta_{y-}U_1\|_{n(S_l)}^2 \le 7\|u_1\|_{1,S}^2.$$

It is similar to show that

$$\|U_2\|_{1,S_{ll}}^2 \le 7\|u_2\|_{1,S}^2.$$

Thus,

$$\|U_1\|_{1,S_l}^2 + \|U_2\|_{1,S_{ll}}^2 \le 7\|\mathbf{u}\|_{1,S}^2.$$

By (3.1) and (3.3),

$$\|\mathbf{u}\|_{1,S}^2 \le C_p\|p\|_S^2 = C_p\|P\|_S^2.$$

Taking $C_f := 7C_p$, we get the inequality in (3.6), which completes the proof of statement (1) in Theorem 3.

Now let's prove the statement (2) in Theorem 3. Let $P \in L_0^2(S_h^0)$ and define

$$p|_{S_{l,m}} := P_{l,m}$$

for $l, m = 1, \ldots, N-1$, then $p \in L_0^2(S_{sw})$. Hence there exists a vector $\mathbf{u} \in H_0^1(S_{sw})^2$ such that

$$\nabla \cdot \mathbf{u} = p \text{ in } S_{sw} \qquad \text{and} \qquad \|\mathbf{u}\|_{1,S_{sw}}^2 \le C_p\|p\|_{S_{sw}}^2.$$

Define

$$x_l(t) := \begin{cases} (l-t)h, & \text{if } 1 \le l \le N-1; \\ 1, & \text{if } l = 0; \\ 1-h & \text{if } l = N, \end{cases}$$

$$y_m(t) := \begin{cases} (m-t)h, & \text{if } 1 \le m \le N-1; \\ 1, & \text{if } m = 0; \\ 1-h, & \text{if } m = N. \end{cases}$$

We define

$$(U_1)_{l,m} := \int_0^1 u_2(lh, y_m(t))\,dt$$

for $l = 0, \ldots, N-1$ and $m = 0, \ldots, N$,

$$(U_2)_{l,m} := \int_0^1 u_2(x_l(t), mh)\,dt$$

for $l = 0, \ldots, N$ and $m = 0, \ldots, N-1$, and then

$$\mathbf{U} = (U_1, U_2) \in H_0^1(w(S_h)) \times H_0^1(s(S_h)),$$

since $\mathbf{u} \in H_0^1(S_{sw})^2$. The proof that $\mathbf{U}$ satisfies the required properties is similar to the proof for statement (1). The proof for (3) is similar to the proof of (2).

4. Conclusion

The inf-sup conditions are proved for three finite-difference approximations of the Stokes problem. The finite-difference approximations use a staggered-mesh scheme and the schemes resulting from the backward and the forward differencings.

If Q_h is the Schur complement of the linear system generated by one of the finite-difference approximations that we discussed in this paper, the inf-sup conditions that we proved in this paper can be used to prove that the condition number $\kappa(Q_h)$ is independent of mesh size h and to prove the convergence estimation of the solution generated by Q_h, which we report in [12]. These results for Q_h support the use of the pressure equation method, a new fast iterative method introduced by Shin and Strikwerda [11], and other iterative methods to solve the finite-difference approximations of the Stokes and the incompressible Navier-Stokes equations, since the Schur complement Q_h plays an important role in studying those equations.

Future research on the inf-sup conditions for other finite-difference approximations and for other linear boundary-value problems with a constraint needs to be done.

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