BULL. AUSTRAL. MATH. SOC. 20H20
VOL. 28 (1983), 151-157. (10J05, 20C32, 20E06, 26B10)

FREE k-TUPLES IN LINEAR GROUPS

ARNALDO MANDEL AND JAIRO Z. GONÇALVES

Let $\mathrm{PGL}(P)$ be the group of projective linear transformations of the n-dimensional projective space P over a field F. A topology is given on F, and it is assumed that F is locally compact; $\mathrm{PGL}(P)$ is endowed with the quotient topology from the canonical projection map $\mathrm{GL}(n, F) \to \mathrm{PGL}(P)$, where $\mathrm{GL}(n, F) \subseteq M_n(F) \simeq F^{n^2}$.

For any given k, it is shown that the set of k-tuples $(g_1, g_2, \ldots, g_k) \in \mathrm{PGL}(P)^k$ which freely generate a free subgroup of $\mathrm{PGL}(P)$ is dense in $\mathrm{PGL}(P)^k$ and has a nonvoid interior.

1. Introduction

A k-tuple of members of a group is free if it freely generates a free subgroup.

In [1], Chang, Jennings and Ree gave sufficient conditions for a pair of parabolic complex linear transformations to be free. Their results imply that the set of free pairs of members of $\mathrm{PSL}(2, \mathbb{C})$ contains an open subset of $\mathrm{PSL}(2, \mathbb{C})^2$. Tits [7] has provided a criterion that, when satisfied by a k-tuple $A = (A_1, \ldots, A_k)$ in $\mathrm{PGL}(n, F)$, F a locally compact field, implies the existence of an integer r such that

Received 30 June 1983. The authors wish to thank the referee for pointing out Epstein's paper as well as for other useful comments.

$\left\{A_1^r, \ldots, A_k^r\right\}$ is free.

We address the question of how large, topologically, is the set of free k-tuples in $\mathrm{PGL}(n, F)$, $n > 1$, when F is endowed with a topology. On the light of Tits' methods, we shall work in the following setting.

Let F be a locally compact field, endowed with a nontrivial valuation $|\ |$; V is an n-dimensional vector space over F , and P is the projective space of V . We denote by π both canonical projections $\pi : V \to P$ and $\pi : \mathrm{GL}(V) \to \mathrm{PGL}(P)$. The space V is endowed with the only topology compatible with that of F , and so is $\mathrm{GL}(V)$. The topologies on P and $\mathrm{PGL}(P)$ are those induced by the projection π . Thus, if one chooses a basis on V , and identifies $V = F^n$, $\mathrm{GL}(V) = \mathrm{GL}(n, F) \subseteq F^{n^2}$, the identifications are homeomorphisms, where F^n and F^{n^2} have the product topology.

A simple observation shows:

PROPOSITION 1. *The set of free k-tuples of* $\mathrm{PGL}(P)$ *is dense in* $\mathrm{PGL}(P)^k$. (See also Epstein [3] for a stronger result.)

We elaborate on Tits' [7] methods in order to get:

THEOREM 2. *The set of free k-tuples of* $\mathrm{PGL}(P)$ *has nonvoid interior in* $\mathrm{PGL}(P)^k$.

2. Proofs

Proof of Proposition 1. It is enough to show that the free k-tuples of $\mathrm{GL}(n, F)$ are dense in $\mathrm{GL}(n, F)^k$.

A k-tuple of matrices in $\mathrm{GL}(n, F)$ satisfies a given group identity, provided its entries simultaneously satisfy a set of n^2 polynomial equations. Thus the set of k-tuples which do not satisfy that identity is open and dense in $\mathrm{GL}(n, F)^k$.

It follows that the set of free k-tuples is a countable intersection (indexed by the members of the free group in k-generators) of dense open

https://doi.org/10.1017/S0004972700026204 Published online by Cambridge University Press

sets of $\mathrm{GL}(n, F)^k$. Thus it is dense by Baire's theorem applied to $\mathrm{GL}(n, F)^k$ and the proposition is proved.

In order to prove the next theorem it is necessary to recall some known facts and make some topological considerations.

As with Tits, our main tool for proving a k-tuple is free is the following criterion due to Macbeath [5], Lyndon and Ullman [4] and Tits [7].

LEMMA 2.1. *Let G be a group acting on a set P on the left and let $g = (g_1, \ldots, g_k)$ be a k-tuple of members of G . Suppose that there exists a k-tuple $(P_1, \ldots, P_k)$ of subsets of P and a $p \in P-P_1 \cup P_2 \cup \ldots \cup P_k$ such that for all i, j , $1 \le i$, $j \le k$, $i \neq j$ and all $m \in Z^*$, $g_i^m(P_j \cup \{p\}) \subseteq P_i$. Then $(g_1, \ldots, g_k)$ is free.*

Proof. Let F_k be the free group generated by $\bar{g}_1, \ldots, \bar{g}_k$ and let $f : F_k \to \langle g_1, \ldots, g_k\rangle$ be the epimorphism determined by $f(\bar{g}_i) = g_i$, $1 \le i \le k$. It follows by induction that if $x = x_n x_{n-1} \ldots x_1$ is a reduced word of F_k , $f(x)(p) \in P_i$, where i is such that $x_n = g_i$ or $x_n = \bar{g}_i^{-1}$. Hence, if $n \ge 1$, $f(x)(p) \neq p$, so $f(x) \neq 1$, $\operatorname{Ker} f = 1$ and the result follows.

Following Tits [7], we associate with each transformation $g \in \mathrm{PGL}(P)$ two linear subspaces of P , $A(g)$ and $A'(g)$ as follows. Choose a representative $\bar{g}$ of g and let $f(t) = \prod_{i=1}^{n} (t-\lambda_i)$ be its characteristic polynomial. Set $\Omega = \{\lambda_i : |\lambda_i| = \sup\{|\lambda_j|, 1 \le j \le n\}\}$, $f_1(t) = \prod_{\lambda_i \in \Omega} (t-\lambda_i)$ and $f_2(t) = \prod_{\lambda \notin \Omega} (t-\lambda_i)$. We define $A(g)$ and $A'(g)$ as the subspaces of P which correspond to the kernels of $f_1(\bar{g})$ and $f_2(\bar{g})$, respectively.

DEFINITION. A k-tuple $(g_1, \ldots, g_k)$ of $\mathrm{PGL}(P)^k$ satisfies *Tits' criterion* if

(a) $A(g_i)$ and $A\left(g_i^{-1}\right)$ are points,

(b) $i \neq j$ implies that

$$\left(A(g_i) \cup A\left(g_i^{-1}\right)\right) \cap \left(A'(g_j) \cup A'\left(g_j^{-1}\right)\right) = \emptyset .$$

In [7], Lemma 3.12, it is proved

LEMMA 2.2. *If $(g_1, \ldots, g_k)$ satisfies Tits' criterion there exist in P compact neighborhoods U_i of $A(g_i)$ and U_i' of $A\left(g_i^{-1}\right)$, $1 \le i \le k$, $p \in P - \bigcup_i (U_i \cup U_i')$ and $r > 0$ such that for all integers $m \ge r$,*

$$g_i^m(U_i) \subseteq \operatorname{int} U_i , \quad g_i^{-m}(U_i') \subseteq \operatorname{int} U_i' \tag{1}$$

and, for all $j \neq i$,

$$\begin{aligned} g_i^m(U_j \cup U_j' \cup \{p\}) &\subseteq \operatorname{int} U_i , \\ g_i^{-m}(U_j \cup U_j' \cup \{p\}) &\subseteq \operatorname{int} U_i' , \end{aligned} \tag{2}$$

where int *denotes topological interior (condition (1) is not explicitly in* Tits' *proof, but can be achieved in the same way as condition (2)).*

Now we need a topological lemma.

LEMMA 2.3. *Let K, O be subsets of P, K compact, O open. Then $A(K, O) = \{T \in \mathrm{PGL}(P) : T(K) \subseteq O\}$ is open in $\mathrm{PGL}(P)$.*

Proof. Let us consider the diagram

$$\begin{array}{ccc} \mathrm{GL}(V) \times V^* & \xrightarrow{\phi} & V^* \\ \Big\downarrow{\scriptstyle \pi\times\pi} & & \Big\downarrow{\scriptstyle \pi} \\ \mathrm{PGL}(P) \times P & \xrightarrow{\bar\phi} & P \end{array}$$

where $\phi(t, v) = Tv$ and $\bar\phi$ is defined such that the above diagram

https://doi.org/10.1017/S0004972700026204 Published online by Cambridge University Press

commutes. As can be seen easily $\bar{\phi}$ is well defined.

We claim that $\bar{\phi}$ *is continuous.*

Let A be an open set contained in P. Then, since ϕ and π are continuous and $\pi \times \pi$ is an open map, it follows that $(\pi \times \pi)\left(\phi^{-1}\left(\pi^{-1}(A)\right)\right)$ is open in $\mathrm{PGL}(P) \times P$ and the claim is proved.

Now let $C(P)$ be the set of all continuous maps of P into itself, endowed with the compact open topology. Define $\hat{\phi} : \mathrm{PGL}(P) \to C(P)$ by $\hat{\phi}(T)(x) = \bar{\phi}(T, x)$. Since $\bar{\phi}$ is continuous, it follows that $\hat{\phi}$ is continuous too (see [2], Theorem XII 3.1). The set $A(K, O)$ is the inverse image by $\hat{\phi}$ of a basic open set of $C(P)$, thus it is open.

Proof of Theorem 2. Let $\left(f_1, \ldots, f_k\right)$ be a k-tuple satisfying Tits' conditions and let $r, U_1, \ldots, U_k, U_1', \ldots, U_k', p$ be the objects associated with this k-tuple in Lemma 2.2; let $P_i = U_i \cup U_i'$, $1 \le i \le k$.

Following the notation of Lemma 2.3, let, for $1 \le i$, $j \le k$, $i \ne j$,

$$K_{ij} = A\left(P_j \cup \{p\}, \text{ int } U_i\right),$$

$$K_{ij}' = A\left(P_j \cup \{p\}, \text{ int } U_i'\right),$$

$$L_i = A\left(U_i, \text{ int } U_i\right),$$

$$L_i' = A\left(U_i', \text{ int } U_i'\right),$$

$$M_i = \left(\bigcap_{j \ne i} K_{ij}\right) \cap L_i, \quad M_i' = \left(\bigcap_{j \ne i} K_{ij}'\right) \cap L_i',$$

$$W_i = M_i \cap \left\{g \in \mathrm{PGL}(P) : g^{-1} \in M_i'\right\}.$$

It follows from Lemma 2.3 that M_i and M_i' are open neighborhoods of f_i^r and f_i^{-r}, respectively. Since $x \mapsto x^{-1}$ is a homeomorphism of $\mathrm{PGL}(P)$, W_i is an open neighborhood of f_i^r.

The open set $W_1 \times \ldots \times W_k$ is comprised of free k-tuples only.

https://doi.org/10.1017/S0004972700026204 Published online by Cambridge University Press

This follows from Lemma 2.1 as the construction of the W_i ensures that if $(g_1, \ldots, g_k) \in W_1 \times \ldots \times W_k$, then

$$g_i^m(P_j \cup \{p\}) \subseteq \begin{cases} U_i , & m > 0 , \\ U_i' , & m < 0 . \end{cases}$$

One verifies this as follows: $g_i(P_j \cup \{p\}) \subseteq U_i$, as $g_i \in K_{ij}$, and since $g_i \in L_i$, $g_i^m(P_j \cup \{p\}) = g_i^{m-1}g_i(P_j \cup \{p\}) \subseteq U_i$, for all positive m. Negative powers m can be similarly analysed.

3. Concluding remarks

Actually, we had intended to show that the set of free k-tuples is open. Theorem 2 is an approximation to that, as we have not succeeded in our original purpose (nor have we been able to disprove the conjectured result).

Indeed, we conjecture:

(a) the set of free k-tuples is open in $\mathrm{PGL}(P)^k$; or a less optimistic form of (a);

(b) the set of free k-tuples contains a dense open set of $\mathrm{PGL}(P)^k$.

In a companion paper [6], we shall give a modified proof of Theorem 2 which is quite computational, and allows one to explicitly describe some open sets as free k-tuples. There we also show that one should not get to carried away with optimism about (a), namely, there is no Zariski-open set of free k-tuples, at least when F has infinitely many roots of unity.

References

[1] Bomshik Chang, S.A. Jennings and Rimhak Ree, "On certain pairs of matrices which generate free groups", *Canad. J. Math.* 10 (1958), 279-284.

[2] James Dugundji, *Topology* (Allyn and Bacon, Boston, 1966).

[3] D.B.A. Epstein, "Almost all subgroups of a Lie group are free", *J. Algebra* **19** (1971), 261-262.

[4] R.C. Lyndon and J.L. Ullman, "Pairs of real 2-by-2 matrices that generate free products", *Michigan Math. J.* **15** (1968), 161-166.

[5] A.M. Macbeath, "Packings, free products and residually finite groups", *Proc. Cambridge Philos. Soc.* **59** (1963), 555-558.

[6] Arnaldo Mandel and Jairo Z. Gonçalves, "Construction of open sets of free k-tuples of matrices", submitted.

[7] J. Tits, "Free subgroups in linear groups", *J. Algebra* **20** (1972), 250-270.

IMEUSP
Cx. Postal 20.570,
01000 - São Paulo - SP,
Brazil.