

A HYBRID MEAN VALUE OF THE INVERSION OF L -FUNCTIONS AND GENERAL QUADRATIC GAUSS SUMS*

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Abstract. The main purpose of this paper is, using the estimates for character sums and the analytic method, to study the $2k$ -th power mean of the inversion of Dirichlet L -functions with the weight of general quadratic Gauss sums, and give two interesting asymptotic formulas.

§1. Introduction

Let $q \geq 2$ be an integer, χ denotes a Dirichlet character modulo q . For any integer n , we define the general quadratic Gauss sums $G(n, \chi; q)$ as follows:

$$(1) \quad G(n, \chi; q) = \sum_{a=1}^q \chi(a) e\left(\frac{na^2}{q}\right),$$

where $e(y) = e^{2\pi iy}$. This sum is important, because it is a generalization of the classical Gauss sums and quadratic Gauss sums. But about the properties of $G(n, \chi; q)$, we know very little at present, we do not know even how large $|G(n, \chi; q)|$ is. The main purpose of this paper is, using the estimates for character sums and the analytic method, to study the asymptotic distribution of the $2k$ -th power mean

$$(2) \quad \sum_{p \leq Q} \frac{1}{p \cdot (p-1)^2} \sum_{\chi \bmod p} \frac{|G(n, \chi; p)|^4}{|L(1, \chi)|^{2k}},$$

and give a sharp asymptotic formula, where Q is any positive number with $Q \geq 2$, p denotes an odd prime, $L(s, \chi)$ denotes the Dirichlet L -function corresponding to character $\chi \bmod p$. In fact, we shall prove the following:

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THEOREM 1. *Let p denote an odd prime with $p \leq Q$. Then for any fixed positive integer k and integer n with $|n| \leq Q$, we have the asymptotic formula*

$$\sum_{p \leq Q} \frac{1}{p(p-1)^2} \sum_{\chi \bmod p} \frac{|G(n, \chi; p)|^4}{|L(1, \chi)|^{2k}} = 3 \cdot C(k) \cdot \pi(Q) + O\left(Q^{\frac{1}{2}+\epsilon}\right),$$

where $C(k) = \prod_p \left[1 + \frac{(C_k^1)^2}{p^2} + \frac{(C_k^2)^2}{p^4} + \dots + \frac{(C_k^k)^2}{p^{2k}}\right]$, $\sum_{\chi \bmod p}$ denotes the summation over all characters modulo p , \prod_p denotes the product over all primes, $\pi(Q)$ denotes the number of all primes p satisfying $2 \leq p \leq Q$, ϵ denotes any fixed positive number, and $C_k^i = \frac{k!}{(k-i)!i!}$.

THEOREM 2. *Let $Q \geq 3$ be a real number. Then for any fixed positive integer k and any integer n with $|n| \leq Q$, we have the asymptotic formula*

$$\sum'_{p \leq Q} \frac{1}{p^2(p-1)^2} \sum_{\chi \bmod p} \frac{|G(n, \chi; p)|^6}{|L(1, \chi)|^{2k}} = 10 \cdot C(k) \cdot \pi_1(Q) + O\left(Q^{\frac{1}{2}+\epsilon}\right),$$

where \sum' denotes the summation over all primes p with $p \leq Q$ and $p \equiv 3 \pmod 4$, $\pi_1(Q)$ denotes the number of all primes p satisfying $2 \leq p \leq Q$ and $p \equiv 3 \pmod 4$.

§2. Some lemmas

In order to complete the proof of the theorems, we need following several lemmas.

LEMMA 1. *Let $y \geq 2$ be a real number, k be any positive integer. Then for*

$$r(n) = \sum_{n_1 n_2 \dots n_k = n} \mu(n_1) \dots \mu(n_k),$$

we have an estimate

$$\sum_{q \leq Q} \sup_{\substack{a, x \\ x \leq y}} \left| \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} r(n) \right| \ll_{A, k} y (\ln y)^{-A} + y^{1-\frac{1}{2k}} Q (\ln(yQ))^4,$$

where $\mu(n)$ is the Möbius function, A is any positive number, $\ll_{A, k}$ denote the constants implied by the symbols \ll depend only on parameter A and k .

Proof. (See reference [5]).

LEMMA 2. *Let p denote an odd prime with $p \leq Q$, χ denote a Dirichlet character modulo p . Then for any positive integer k , we have an estimate*

$$\sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} \left| \sum_{\chi \bmod p} \frac{\chi(a)}{|L(1, \chi)|^{2k}} \right| = O\left(Q^{\frac{1}{2} + \epsilon}\right),$$

where ϵ denotes any fixed positive number.

Proof. For convenience, firstly we write

$$A(\chi, y) = \sum_{\frac{p}{a} < n \leq y} \chi(n)r(n), \quad B(\chi, y) = \sum_{p < n \leq y} \chi(n)r(n),$$

where a is any positive integer with $1 \leq a < p$. If $\text{Re}(s) > 1$ and $\chi \neq \chi_0$ (principal character mod p), then we have

$$(3) \quad \frac{1}{L^k(s, \chi)} = \sum_{n=1}^{\infty} \frac{\chi(n)r(n)}{n^s}.$$

From (3) and Abel’s identity we have

$$(4) \quad \frac{1}{L^k(s, \chi)} = \sum_{n \leq N} \frac{\chi(n)r(n)}{n^s} + s \int_N^{+\infty} \frac{A(\chi, y)}{y^{s+1}} dy.$$

By the property of Dirichlet L -function and $L(1, \chi) \neq 0$ we know that (4) is also correct if $s = 1$. So we have

$$(5) \quad \begin{aligned} \frac{1}{L^k(1, \chi)} &= \sum_{1 \leq n < \frac{p}{a}} \frac{\chi(n)r(n)}{n} + \int_{\frac{p}{a}}^{+\infty} \frac{A(\chi, y)}{y^2} dy \\ &= \sum_{1 \leq n < p} \frac{\chi(n)r(n)}{n} + \int_p^{+\infty} \frac{B(\chi, y)}{y^2} dy. \end{aligned}$$

Hence from (3), (5) and note that $\frac{1}{L(1, \chi_0)} = 0$ we get

$$\sum_{\chi \bmod p} \frac{\chi(a)}{|L(1, \chi)|^{2k}} = \sum'_{\chi \bmod p} \chi(a) \left| \sum_{n=1}^{\infty} \frac{\chi(n)r(n)}{n} \right|^2$$

$$\begin{aligned}
 &= \sum'_{\chi \bmod p} \chi(a) \left(\sum_{1 \leq n < \frac{p}{a}} \frac{\chi(n)r(n)}{n} + \int_{\frac{p}{a}}^{+\infty} \frac{A(\chi, y)}{y^2} dy \right) \\
 &\quad \times \left(\sum_{1 \leq m < p} \frac{\bar{\chi}(m)r(m)}{m} + \int_p^{+\infty} \frac{B(\bar{\chi}, y)}{y^2} dy \right) \\
 &= \sum'_{\chi \bmod p} \chi(a) \left(\sum_{1 \leq n < \frac{p}{a}} \frac{\chi(n)r(n)}{n} \right) \left(\sum_{1 \leq m < p} \frac{\bar{\chi}(m)r(m)}{m} \right) \\
 &\quad + \sum'_{\chi \bmod p} \chi(a) \left(\sum_{1 \leq n < \frac{p}{a}} \frac{\chi(n)r(n)}{n} \right) \left(\int_p^{+\infty} \frac{B(\bar{\chi}, y)}{y^2} dy \right) \\
 &\quad + \sum'_{\chi \bmod p} \chi(a) \left(\sum_{1 \leq m < p} \frac{\bar{\chi}(m)r(m)}{m} \right) \left(\int_{\frac{p}{a}}^{+\infty} \frac{A(\chi, y)}{y^2} dy \right) \\
 &\quad + \sum'_{\chi \bmod p} \chi(a) \left(\int_{\frac{p}{a}}^{+\infty} \frac{A(\chi, y)}{y^2} dy \right) \left(\int_p^{+\infty} \frac{B(\bar{\chi}, y)}{y^2} dy \right) \\
 &\equiv M_1 + M_2 + M_3 + M_4,
 \end{aligned}$$

where $\sum'_{\chi \bmod p}$ denotes the summation over all nonprincipal characters mod p . So from the above we have

$$\begin{aligned}
 (6) \quad &\sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} \left| \sum_{\chi \bmod p} \frac{\chi(a)}{|L(1, \chi)|^{2k}} \right| \\
 &\leq \sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} (|M_1| + |M_2| + |M_3| + |M_4|).
 \end{aligned}$$

Now we shall estimate each term in expression (6). From the orthogonality relation for characters modulo p we know that for $(p, mn) = 1$,

$$\sum_{\chi \bmod p} \chi(n)\bar{\chi}(m) = \begin{cases} p - 1, & \text{if } n \equiv m \pmod{p}; \\ 0, & \text{otherwise.} \end{cases}$$

From this identity we can easily get

$$(7) \quad M_1 = \sum_{\chi \bmod p} \chi(a) \left(\sum_{1 \leq n < \frac{p}{a}} \frac{\chi(n)r(n)}{n} \right) \left(\sum_{1 \leq m < p} \frac{\bar{\chi}(m)r(m)}{m} \right) + O(p^\epsilon)$$

$$\begin{aligned}
 &= (p-1) \sum_{1 \leq n < \frac{p}{a}} \sum_{\substack{1 \leq m < p \\ an \equiv m(p)}} \frac{r(n)r(m)}{nm} + O(p^\epsilon) \\
 &= (p-1) \sum_{1 \leq n < \frac{p}{a}} \frac{r(n)r(an)}{an^2} + O(p^\epsilon) \\
 &\leq \frac{pr(a)}{a} \sum_{1 \leq n < \frac{p}{a}} \frac{|r(n)|^2}{n^2} + O(p^\epsilon) \\
 &\leq \frac{pr(a)}{a} \sum_{\substack{n=1 \\ (n,p)=1}}^\infty \frac{|r(n)|^2}{n^2} + O(p^\epsilon).
 \end{aligned}$$

Note that

$$\sum_{\substack{n=1 \\ (n,p)=1}}^\infty \frac{r^2(n)}{n^2} = \prod_{\substack{p_1 \\ p_1 \neq p}} \left[1 + \frac{(C_k^1)^2}{p_1^2} + \frac{(C_k^2)^2}{p_1^4} + \dots + \frac{(C_k^k)^2}{p_1^{2k}} \right] \ll C,$$

an absolute constant, and the estimate $r(n) \leq d_k(n) \ll p^\epsilon$, from (7) we immediately get

$$(8) \quad \sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} |M_1| \ll \sum_{p \leq Q} \frac{1}{p^{\frac{1}{2}}} \sum_{a=1}^{p-1} \frac{r(a)}{a} + \sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} p^\epsilon \ll Q^{\frac{1}{2}+\epsilon}.$$

$$\begin{aligned}
 (9) \quad &\sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} |M_2| \\
 &= \sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} \left| \sum'_{\chi \bmod p} \chi(a) \left(\sum_{n < \frac{p}{a}} \frac{\chi(n)r(n)}{n} \right) \left(\int_p^{+\infty} \frac{B(\bar{\chi}, y)}{y^2} dy \right) \right| \\
 &\leq \sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} \left| \sum'_{\chi \bmod p} \chi(a) \left(\sum_{n < \frac{p}{a}} \frac{\chi(n)r(n)}{n} \right) \left(\int_p^{Q^{2k}} \frac{B(\bar{\chi}, y)}{y^2} dy \right) \right| \\
 &\quad + \sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} \left| \sum'_{\chi \bmod p} \chi(a) \left(\sum_{n < \frac{p}{a}} \frac{\chi(n)r(n)}{n} \right) \left(\int_{Q^{2k}}^{+\infty} \frac{B(\bar{\chi}, y)}{y^2} dy \right) \right| \\
 &\leq \sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} \left| \sum_{\chi \bmod p} \chi(a) \left(\sum_{n < \frac{p}{a}} \frac{\chi(n)r(n)}{n} \right) \left(\int_p^{Q^{2k}} \frac{\sum_{p < m \leq y} \bar{\chi}(m)r(m)}{y^2} dy \right) \right|
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{p \leq Q} \frac{1}{p^{\frac{1}{2}}} \sum_{a=1}^{p-1} \sum_{n < \frac{p}{a}} \frac{|r(n)|}{n} \left| \int_{Q^{2^k}}^{+\infty} \frac{1}{y^2} \left(\sum_{\substack{p < m \leq y \\ m \equiv na(p)}} r(m) \right) dy \right| + O\left(Q^{\frac{1}{2}+\epsilon}\right) \\
 \leq & \sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} \left| \sum_{\chi \bmod p} \chi(a) \left(\sum_{n < \frac{p}{a}} \frac{\chi(n)r(n)}{n} \right) \left(\int_p^{Q^{2^k}} \frac{\sum_{p < m \leq y} \bar{\chi}(m)r(m)}{y^2} dy \right) \right| \\
 & + O\left(Q^{\frac{1}{2}+\epsilon} \int_{Q^{2^k}}^{+\infty} \frac{1}{y^2} \sum_{p \leq Q} \sup_{\substack{a, z \\ z \leq y}} \left| \sum_{\substack{m \leq z \\ m \equiv a(p)}} r(m) \right| dy \right) + O\left(Q^{\frac{1}{2}+\epsilon}\right).
 \end{aligned}$$

So from Lemma 1 (taking $A = 2$) and (9) we have

$$\begin{aligned}
 (10) \quad & \sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} |M_2| \\
 \leq & \sum_{p \leq Q} \frac{1}{p^{\frac{1}{2}}} \int_p^{Q^{2^k}} \frac{1}{y^2} \left[\sum_{a=1}^{p-1} \sum_{n < \frac{p}{a}} \sum_{\substack{p < m \leq y \\ an \equiv m(p)}} \frac{r(n)r(m)}{n} \right] dy + O\left(Q^{\frac{1}{2}+\epsilon}\right) \\
 & + O\left(Q^{\frac{1}{2}+\epsilon} \int_{Q^{2^k}}^{+\infty} \frac{1}{y^2} \left[y(\ln y)^{-A} + y^{1-\frac{1}{2^k}} Q(\ln(yQ))^4 \right] dy \right) \\
 = & O\left(\sum_{p \leq Q} \frac{1}{p^{\frac{1}{2}}} \sum_{a=1}^{p-1} \sum_{n < \frac{p}{a}} \frac{1}{n} \int_p^{Q^{2^k}} \frac{y \cdot \frac{1}{p} \cdot p^\epsilon}{y^2} dy \right) \\
 & + O\left(Q^{\frac{1}{2}+\epsilon} \int_{Q^{2^k}}^{+\infty} \left[y^{-1}(\ln y)^{-A} + y^{-1-\frac{1}{2^k}} Q(\ln(yQ))^4 \right] dy \right) \\
 = & O\left(\sum_{p \leq Q} p^{-\frac{1}{2}+\epsilon} \right) + O\left(Q^{\frac{1}{2}+\epsilon}\right) \\
 = & O\left(Q^{\frac{1}{2}+\epsilon}\right).
 \end{aligned}$$

Applying the same method of proving (10), we also have the estimate

$$\begin{aligned}
 (11) \quad & \sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} |M_3| \\
 = & \sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} \left| \sum'_{\chi \bmod p} \chi(a) \left(\sum_{m \leq p} \frac{\bar{\chi}(m)r(m)}{m} \right) \left(\int_{\frac{p}{a}}^{+\infty} \frac{A(\chi, y)}{y^2} dy \right) \right|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} \left| \sum'_{\chi \bmod p} \chi(a) \left(\sum_{m \leq p} \frac{\bar{\chi}(m)r(m)}{m} \right) \left(\int_{\frac{p}{a}}^{Q^{2^k}} \frac{A(\chi, y)}{y^2} dy \right) \right| \\
 &\quad + \sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} \left| \sum'_{\chi \bmod p} \chi(a) \left(\sum_{m \leq p} \frac{\bar{\chi}(m)r(m)}{m} \right) \left(\int_{Q^{2^k}}^{+\infty} \frac{A(\chi, y)}{y^2} dy \right) \right| \\
 &\leq \sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} \left| \sum_{\chi \bmod p} \chi(a) \left(\sum_{m < p} \frac{\bar{\chi}(m)r(m)}{m} \right) \left(\int_{\frac{p}{a}}^{Q^{2^k}} \frac{\sum_{\frac{p}{a} < n \leq y} \chi(n)r(n)}{y^2} dy \right) \right| \\
 &\quad + \sum_{p \leq Q} \frac{1}{p^{\frac{1}{2}}} \sum_{a=1}^{p-1} \int_{Q^{2^k}}^{+\infty} \frac{1}{y^2} \left| \sum_{m < p} \frac{r(m)}{m} \sum_{\substack{\frac{p}{a} < n \leq y \\ an \equiv m(p)}} r(n) \right| dy + O\left(Q^{\frac{1}{2}+\epsilon}\right) \\
 &\leq \sum_{p \leq Q} \frac{1}{p^{\frac{1}{2}}} \sum_{a=1}^{p-1} \int_{\frac{p}{a}}^{Q^{2^k}} \frac{1}{y^2} \left| \sum_{m < p} \sum_{\substack{\frac{p}{a} < n \leq y \\ an \equiv m(p)}} \frac{r(m)r(n)}{m} \right| dy + O\left(Q^{\frac{1}{2}+\epsilon}\right) \\
 &\quad + O\left(Q^{\frac{1}{2}+\epsilon} \int_{Q^{2^k}}^{+\infty} \frac{1}{y^2} \sum_{p \leq Q} \sup_{\substack{a, z \\ z \leq y}} \left| \sum_{\substack{n \leq z \\ n \equiv a(p)}} r(n) \right| dy \right) \\
 &= O\left(\sum_{p \leq Q} \frac{p}{p^{\frac{1}{2}}} \sum_{m < p} \frac{1}{m} \int_{\frac{p}{a}}^{Q^{2^k}} \frac{y \cdot \frac{1}{p} \cdot p^\epsilon}{y^2} dy\right) + O\left(Q^{\frac{1}{2}+\epsilon}\right) \\
 &= O\left(Q^{\frac{1}{2}+\epsilon}\right).
 \end{aligned}$$

where we have used the estimate $r(n) \ll n^\epsilon$ and for any fixed positive integers h and m , the number of the solutions of equation $an = hp + m$ (for a and n) is $\ll d(hp + m) \ll p^\epsilon$. Similarly, we can also get the estimate

$$(12) \quad \sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} \left| \sum'_{\chi \bmod p} \chi(a) \int_{\frac{p}{a}}^{+\infty} \frac{A(\chi, y)}{y^2} dy \int_p^{+\infty} \frac{B(\bar{\chi}, z)}{z^2} dz \right| = O\left(Q^{\frac{1}{2}+\epsilon}\right).$$

Combining (6), (8), (10), (11) and (12) we obtain the estimate

$$\sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} \left| \sum_{\chi \bmod p} \frac{\chi(a)}{|L(1, \chi)|^{2k}} \right| = O\left(Q^{\frac{1}{2}+\epsilon}\right).$$

This proves Lemma 2.

LEMMA 3. Let $Q \geq 2$, then for any fixed integer k , we have the asymptotic formula

$$\sum_{p \leq Q} \frac{1}{\phi(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \frac{1}{|L(1, \chi)|^{2k}} = \frac{1}{2} \cdot C(k) \cdot \pi(Q) + O\left(Q^{\frac{1}{2}+\epsilon}\right).$$

Proof. For prime power p^α , note that $\mu(1) = 1$, $\mu(p) = -1$ and $\mu(p^i) = 0$ if $i > 1$. So that we have the identity

$$r(p^\alpha) = \sum_{n_1 n_2 \cdots n_k = p^\alpha} \mu(n_1) \mu(n_2) \cdots \mu(n_k) = \begin{cases} (-1)^\alpha C_k^\alpha, & \text{if } \alpha \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

and the asymptotic formula

$$\begin{aligned} \sum_{\substack{n=1 \\ (n,p)=1}}^\infty \frac{r^2(n)}{n^2} &= \prod_{\substack{p_1 \\ p_1 \neq p}} \left[1 + \frac{(C_k^1)^2}{p_1^2} + \frac{(C_k^2)^2}{p_1^4} + \cdots + \frac{(C_k^k)^2}{p_1^{2k}} \right] \\ &= C(k) + O\left(\frac{1}{p^2}\right), \end{aligned}$$

from the orthogonality relation for characters and the method of proving Lemma 2 we can easily get

$$\begin{aligned} &\sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left| \sum_{n \leq Q^{2k}} \frac{r(n)\chi(n)}{n} \right|^2 \\ &= \frac{\phi(p)}{2} \sum_{\substack{n \leq Q^{2k} \\ (n,p)=1}} \frac{r^2(n)}{n^2} + \frac{\phi(p)}{2} \sum_{n \leq Q^{2k}} \sum_{\substack{m \leq Q^{2k} \\ m \equiv \pm n(p) \\ m \neq n}} \frac{r(n)r(m)}{nm} + O(Q^\epsilon) \\ &= \frac{\phi(p)}{2} \cdot C(k) + O(Q^\epsilon), \end{aligned}$$

$$\sum_{p \leq Q} \frac{1}{\phi(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left(\sum_{n \leq Q^{2k}} \frac{r(n)\bar{\chi}(n)}{n} \right) \int_{Q^{2k}}^\infty \frac{A(y, \chi)}{y^2} dy \ll Q^{\frac{1}{2}+\epsilon}$$

and

$$\sum_{p \leq Q} \frac{1}{\phi(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left| \int_{Q^{2k}}^\infty \frac{A(y, \chi)}{y^2} dy \right|^2 \ll Q^{\frac{1}{2}+\epsilon},$$

where $A(y, \chi) = \sum_{Q^{2^k} < n \leq y} r(n)\chi(n)$. Note that the identity

$$\frac{1}{L^k(1, \chi)} = \sum_{n=1}^{+\infty} \frac{r(n)\chi(n)}{n} = \sum_{n \leq Q^{2^k}} \frac{r(n)\chi(n)}{n} + \int_{Q^{2^k}}^{\infty} \frac{A(y, \chi)}{y^2} dy.$$

Combining the above asymptotic formula and the estimates we have

$$\begin{aligned} & \sum_{p \leq Q} \frac{1}{\phi(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \frac{1}{|L(1, \chi)|^{2k}} \\ &= \sum_{p \leq Q} \frac{1}{\phi(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left| \sum_{n \leq Q^{2^k}} \frac{r(n)\chi(n)}{n} \right|^2 \\ & \quad + \sum_{p \leq Q} \frac{1}{\phi(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left(\sum_{n \leq Q^{2^k}} \frac{r(n)\chi(n)}{n} \right) \int_{Q^{2^k}}^{\infty} \frac{A(y, \bar{\chi})}{y^2} dy \\ & \quad + \sum_{p \leq Q} \frac{1}{\phi(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left(\sum_{n \leq Q^{2^k}} \frac{r(n)\bar{\chi}(n)}{n} \right) \int_{Q^{2^k}}^{\infty} \frac{A(y, \chi)}{y^2} dy \\ & \quad + \sum_{p \leq Q} \frac{1}{\phi(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left| \int_{Q^{2^k}}^{\infty} \frac{A(y, \chi)}{y^2} dy \right|^2 \\ &= \frac{1}{2} \cdot C(k) \cdot \pi(Q) + O\left(Q^{\frac{1}{2}+\epsilon}\right). \end{aligned}$$

This proves Lemma 3.

LEMMA 4. *For any integer $q \geq 1$, we have the calculation formula*

$$G(1; q) = \frac{1}{2} \sqrt{q}(1+i) \left(1 + e^{\frac{-\pi i q}{2}}\right) = \begin{cases} \sqrt{q} & \text{if } q \equiv 1 \pmod{4}; \\ 0 & \text{if } q \equiv 2 \pmod{4}; \\ i\sqrt{q} & \text{if } q \equiv 3 \pmod{4}; \\ (1+i)\sqrt{q} & \text{if } q \equiv 0 \pmod{4}, \end{cases}$$

where $G(1; q) = \sum_{a=1}^q e\left(\frac{a^2}{q}\right)$.

Proof. This is a remarkable formula of Gauss. See the formula (30) of [1].

LEMMA 5. *Let p be an odd prime, χ be any nonprincipal even character (i.e. $\chi(-1) = 1$ and $\chi \neq \chi_0$) mod p . Then for any integer n with $(n, p) = 1$, we have the identity*

$$|G(n, \chi; p)|^2 = 2p + \binom{n}{p} G(1; p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2 - 1}{p} \right),$$

where $\binom{n}{p}$ is the Legendre symbol.

Proof. First we let $G(n; p) = \sum_{a=1}^p e\left(\frac{na^2}{p}\right)$. Then for $p \nmid n$, from the formula (29) of [1] we know that

$$(13) \quad G(n; p) = \binom{n}{p} G(1; p).$$

Applying (13) we can get that if χ is a nonprincipal even character mod p , then

$$\begin{aligned} |G(n, \chi; p)|^2 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a) \bar{\chi}(b) e\left(\frac{na^2 - nb^2}{p}\right) \\ &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(ab) e\left(\frac{n(a^2 - b^2)}{p}\right) \\ &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a) e\left(\frac{nb^2(a^2 - 1)}{p}\right) \\ &= \sum_{a=1}^{p-1} \chi(a) \left[\sum_{b=1}^p e\left(\frac{nb^2(a^2 - 1)}{p}\right) - 1 \right] \\ &= \sum_{a=1}^{p-1} \chi(a) G(n(a^2 - 1); p) - \sum_{a=1}^{p-1} \chi(a) \\ &= 2G(0; p) + \sum_{a=2}^{p-2} \chi(a) G(n(a^2 - 1); p) \\ &= 2p + \binom{n}{p} G(1; p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2 - 1}{p} \right). \end{aligned}$$

This proves Lemma 5.

§3. Proof of the theorems

In this section, we shall complete the proof of the theorems. First note that if χ is an odd character modulo p (or $\chi \neq \chi_0$ and $p|n$), then

$$G(n, \chi; p) = \sum_{a=1}^p \chi(a) e\left(\frac{a^2 n}{p}\right) = 0.$$

So for any integer n , from Lemma 5 we have

$$\begin{aligned} (14) \quad & \sum_{\substack{p \leq Q \\ (p,n)=1}} \frac{1}{p\phi^2(p)} \sum_{\chi \bmod p} \frac{|G(n, \chi; p)|^4}{|L(1, \chi)|^{2k}} \\ &= \sum_{\substack{p \leq Q \\ (p,n)=1}} \frac{1}{p\phi^2(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \frac{|G(n, \chi; p)|^4}{|L(1, \chi)|^{2k}} \\ &= \sum_{\substack{p \leq Q \\ (p,n)=1}} \frac{1}{p\phi^2(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left[2p + \binom{n}{p} G(1; p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2 - 1}{p}\right) \right]^2 \\ & \qquad \qquad \qquad \cdot \frac{1}{|L(1, \chi)|^{2k}} \\ &= \sum_{\substack{p \leq Q \\ (p,n)=1}} \frac{1}{p\phi^2(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left[4p^2 + 4p \binom{n}{p} G(1; p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2 - 1}{p}\right) \right] \\ & \qquad \qquad \qquad \cdot \frac{1}{|L(1, \chi)|^{2k}} \\ & \quad + \sum_{\substack{p \leq Q \\ (p,n)=1}} \frac{G^2(1; p)}{p\phi^2(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(ab) \left(\frac{a^2 - 1}{p}\right) \left(\frac{b^2 - 1}{p}\right) \\ & \qquad \qquad \qquad \cdot \frac{1}{|L(1, \chi)|^{2k}}. \end{aligned}$$

On the other hand, we have the identities

$$\begin{aligned} (15) \quad & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(ab) \left(\frac{a^2 - 1}{p}\right) \left(\frac{b^2 - 1}{p}\right) \\ &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a) \left(\frac{a^2 \bar{b}^2 - 1}{p}\right) \left(\frac{b^2 - 1}{p}\right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a) \left(\frac{a^2 - b^2}{p}\right) \left(\frac{b^2 - 1}{p}\right) \\
 &= 2 \left(\frac{-1}{p}\right) (p - 3) + \sum_{a=2}^{p-2} \sum_{b=1}^{p-1} \chi(a) \left(\frac{a^2 - b^2}{p}\right) \left(\frac{b^2 - 1}{p}\right),
 \end{aligned}$$

$$\begin{aligned}
 \sum_{a=2}^{p-2} \left(\frac{a^2 - b^2}{p}\right) \sum_{\chi(-1)=-1} \frac{\chi(a)}{|L(1, \chi)|^{2k}} \\
 = \sum_{a=1}^{p-1} \left(\frac{a^2 - 1}{p}\right) \sum_{\chi(-1)=-1} \frac{\chi(a)}{|L(1, \chi)|^{2k}} = 0.
 \end{aligned}$$

From the Lemma 1 of [8] we can also deduce the estimate

$$(16) \quad \sum_{b=1}^{p-1} \left(\frac{b^2 - a^2}{p}\right) \left(\frac{b^2 - 1}{p}\right) \ll \sqrt{p}, \quad a^2 \not\equiv 1 \pmod{p}.$$

Now combining (14), (15), (16), Lemma 2, Lemma 3 and Lemma 4 we have

$$\begin{aligned}
 &\sum_{p \leq Q} \frac{1}{p\phi^2(p)} \sum_{\chi \pmod{p}} \frac{|G(n, \chi; p)|^4}{|L(1, \chi)|^{2k}} \\
 &= \sum_{\substack{p \leq Q \\ (p, n)=1}} \frac{1}{p\phi^2(p)} \left(4p^2 + G^2(1; p) \cdot 2 \left(\frac{-1}{p}\right) (p - 3)\right) \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \frac{1}{|L(1, \chi)|^{2k}} \\
 &\quad + \sum_{\substack{p \leq Q \\ (p, n)=1}} \frac{1}{p\phi^2(p)} 4p \left(\frac{n}{p}\right) G(1; p) \sum_{a=1}^{p-1} \left(\frac{a^2 - 1}{p}\right) \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \frac{\chi(a)}{|L(1, \chi)|^{2k}} \\
 &\quad + \sum_{\substack{p \leq Q \\ (p, n)=1}} \frac{G^2(1; p)}{p\phi^2(p)} \sum_{a=2}^{p-2} \sum_{b=1}^{p-1} \left(\frac{a^2 - b^2}{p}\right) \left(\frac{b^2 - 1}{p}\right) \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \frac{\chi(a)}{|L(1, \chi)|^{2k}} \\
 &= \sum_{p \leq Q} \frac{6p^2 - 6p}{p(p - 1)^2} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \frac{1}{|L(1, \chi)|^{2k}} + O(Q^\epsilon) \\
 &\quad + O\left(\sum_{p \leq Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=2}^{p-2} \left|\sum_{\chi \neq \chi_0} \frac{\chi(a)}{|L(1, \chi)|^{2k}}\right|\right) \\
 &= 3 \cdot C(k) \cdot \pi(Q) + O(Q^{\frac{1}{2} + \epsilon}).
 \end{aligned}$$

This proves Theorem 1.

Now we prove Theorem 2. From Lemma 5 we have

$$\begin{aligned}
 (17) \quad & \sum'_{p \leq Q} \frac{1}{p^2 \phi^2(p)} \sum_{\chi \bmod p} \frac{|G(n, \chi; p)|^6}{|L(1, \chi)|^{2k}} \\
 &= \sum'_{\substack{p \leq Q \\ (p, n)=1}} \frac{1}{p^2 \phi^2(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \frac{|G(n, \chi; p)|^6}{|L(1, \chi)|^{2k}} \\
 &= \sum'_{\substack{p \leq Q \\ (p, n)=1}} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left[2p + \binom{n}{p} G(1; p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2 - 1}{p} \right) \right]^3 \frac{1}{|L(1, \chi)|^{2k}}
 \end{aligned}$$

For $(p, n) = 1$ and $p \equiv 3 \pmod{4}$,

$$\begin{aligned}
 (18) \quad & \sum_{\chi \neq \chi_0} \frac{|G(n, \chi; p)|^6}{|L(1, \chi)|^{2k}} = \sum_{\chi \neq \chi_0} \frac{\overline{|G(n, \chi; p)|^6}}{|L(1, \chi)|^{2k}} \\
 &= \sum_{\chi \neq \chi_0} \left| \sum_{a=1}^{p-1} \bar{\chi}(a) e\left(\frac{-na^2}{p}\right) \right|^6 \cdot \frac{1}{|L(1, \bar{\chi})|^{2k}} \\
 &= \sum_{\chi \neq \chi_0} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{-na^2}{p}\right) \right|^6 \cdot \frac{1}{|L(1, \chi)|^{2k}} \\
 &= \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left[2p + \binom{-n}{p} G(1; p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2 - 1}{p} \right) \right]^3 \cdot \frac{1}{|L(1, \chi)|^{2k}}.
 \end{aligned}$$

Note that $\left(\frac{-1}{p}\right) = -1$, from (15), (16), (17), (18), Lemma 2, Lemma 3 and Lemma 4 we obtain

$$\begin{aligned}
 & \sum'_{p \leq Q} \frac{1}{p^2 \phi^2(p)} \sum_{\chi \bmod p} \frac{|G(n, \chi; p)|^6}{|L(1, \chi)|^{2k}} \\
 &= \frac{1}{2} \left[\sum'_{p \leq Q} \frac{1}{p^2 \phi^2(p)} \sum_{\chi \neq \chi_0} \frac{|G(n, \chi; p)|^6}{|L(1, \chi)|^{2k}} + \sum'_{p \leq Q} \frac{1}{p^2 \phi^2(p)} \sum_{\chi \neq \chi_0} \frac{\overline{|G(n, \chi; p)|^6}}{|L(1, \chi)|^{2k}} \right]
 \end{aligned}$$

$$\begin{aligned}
&= \sum'_{\substack{p \leq Q \\ (p,n)=1}} \frac{1}{p^2 \phi^2(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left[(2p)^3 + 6pG^2(1;p) \left(\sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p} \right) \right)^2 \right] \\
&\qquad \qquad \qquad \cdot \frac{1}{|L(1, \chi)|^{2k}} \\
&= 10 \cdot C(k) \cdot \pi_1(Q) + O\left(Q^{\frac{1}{2}+\epsilon}\right).
\end{aligned}$$

This completes the proof of Theorem 2.

Note. For general integer $m \geq 3$, whether there exists an asymptotic formula for

$$\sum_{p \leq Q} \frac{1}{p^{m-1}(p-1)^2} \sum_{\chi \bmod p} \frac{|G(n, \chi; p)|^{2m}}{|L(1, \chi)|^{2k}}$$

is an unsolved problem.

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