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CLOSED AND PRIME IDEALS IN THE ALGEBRA OF BOUNDED ANALYTIC FUNCTIONS

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Let H^∞ be the Banach algebra of all bounded analytic functions in the unit disc. We present a complete description of the closed primary (respectively prime) ideals contained in a maximal ideal of the Shilov boundary of H^∞. The paper is also concerned with chains of prime ideals in H^∞.

1. Introduction

One major problem in the analysis of the ideal structure of Banach algebras is the characterisation of the closed ideals. In the disc algebra $A(\mathbb{D})$, for example, the structure of these ideals has been determined by Beurling and Rudin (see [8, p.88]). The situation in H^∞ is much more difficult. In section 2 of this paper we determine the structure of the closed ideals in H^∞ which are contained only in maximal ideals of the set of fibres M_λ, where λ runs through a compact set of Lebesgue measure zero of the unit circle T. Using this result we then give a complete characterisation of the closed primary ideals contained in a maximal ideal of the Shilov boundary of H^∞

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Another fundamental problem in H^∞ is the characterisation of the closed prime ideals. Whereas in the disc algebra every closed prime ideal is maximal, there exist in H^∞ non-maximal closed prime ideals. In view of this situation a conjecture of Alling [1] states that a non-maximal prime ideal in H^∞ is closed if and only if it coincides with the set of all functions that vanish on a nontrivial Gleason part. In this paper we give a partial answer to this problem.

In a recent paper [5], Gorkin has shown that each maximal ideal that does not belong to the Shilov boundary of H^∞ or the unit disc contains an infinite chain of prime ideals. This leads us to the question of whether the set of all prime ideals contained in a maximal ideal of H^∞ forms a chain, as is the case in the ring $H(\mathbb{D})$ of all analytic functions in the unit disc $\mathbb{D}$. The situation in H^∞ is, however, completely different. In section 4 of the present paper we prove that the prime ideals contained in a maximal ideal m are linearly ordered by set inclusion if and only if m belongs to the Shilov boundary or the open unit disc $\mathbb{D}$.

DEFINITIONS AND NOTATION. Let H^∞ be the Banach algebra of all bounded analytic functions in the open unit disc $\mathbb{D}$ under the supremum norm. Let us denote by M_λ the fibre of the maximal ideal space M over the point $z = \lambda$, $|\lambda| = 1$, that is M_λ is the set of all maximal ideals m in M such that the function $\lambda - z$ belongs to m, and by $\hat{f}$ the Gelfand transform of f.

In the sequel let X be the Shilov boundary of H^∞, $\mathcal{G}$ the set of all those maximal ideals m whose Gleason part $R(m)$ is non-trivial and $X_\lambda = X \cap M_\lambda$, $|\lambda| = 1$.

We shall call an ideal primary if it is contained in a unique maximal ideal.

If K is a compact set of Lebesgue measure zero of the unit circle $T = \{z \in \mathbb{C}: |z| = 1\}$, then we denote by P_K the $A(\mathbb{D})$ peak-function associated with the set K; that is P_K is a function in $A(\mathbb{D})$ such that $P_K(z) = 1$ for $z \in K$ and $|P_K(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus K$. It is known that such a function always exists (see Hoffman [8], p.81).

2. Closed ideals in H^∞

It is known (see Hoffman [8 , p.88]) that in the disc algebra $A(\mathbb{D})$ every closed primary ideal I which is contained in the maximal ideal $M_1 = \{f \in A(\mathbb{D}) \ : \ f(1) = 0\}$ has the form

$$I = \exp\left(-\alpha \frac{1+z}{1-z}\right) J,$$

where $\alpha \geq 0$ and $J = M_1$ is the closure of the principal ideal $(1-z)$. In the first theorem of this section we show that, in some sense, a similar result holds for the algebra H^∞ .

THEOREM 2.1. *(Hedenmalm [6, p.13]). Let I be a closed ideal in H^∞ that is contained only in a maximal ideals of the fibre M_1 . Then there exists an $\alpha \geq 0$ such that*

$$I = exp\left(-\alpha \frac{1+z}{1-z}\right) J,$$

where J is a closed ideal that contains the outer function $1-z$.

Theorem 2.1 is due to Hedenmalm [6], who gave a non constructive proof using Banach space techniques. We present in the following a constructive proof.

Proof. Let ϕ be the greatest common divisor of the inner parts of the functions in I . ϕ has the factorisation $\phi = B\phi_1\phi_2$, where B is a Blaschke product, ϕ_1 a singular inner function with discrete measure and ϕ_2 a singular inner function with continuous measure. Because I lies only in maximal ideals of the fibre M_1 , it is easy to see that $B = \phi_2 = 1$, and ϕ_1 has only one discontinuity at the point $z = 1$ (see for example, Hoffman [8, Ex.13, p.75, and Theorem, p.161]). Therefore, ϕ has the form

$$\phi(z) = \exp\left(-\alpha\frac{1+z}{1-z}\right), \quad \text{where} \quad \alpha \geq 0$$

Hence I has the form $I = \phi J$, where $J = \{f \in H^\infty : f\phi \in I\}$ is a closed ideal in H^∞. In the sequel we shall show that the outer function $1-z$ belongs to J.

In the first two steps we construct a uniformly bounded sequence $g_n = B_n G_n$ of functions in H^∞ such that g_n converges uniformly to 1 outside every neighbourhood $U_\delta(1) = \{z \in \mathbb{D} : |z-1| < \delta\}$ of the point $z = 1$ and such that $\exp\left(-\beta\frac{1+z}{1-z}\right)g_n \in J$ for some $\beta \geq 0$.

In what follows, $H(z,t) = \dfrac{e^{it}+z}{e^{it}-z}$ denotes the Herglotz kernel.

Step 1. Let $f \not\equiv 0$ be any function of J. The H^p factorisation theorem yields the representation $f = BS_\beta G_\mu$, where B is a Blaschke product, $G_\mu(z) = \exp\frac{1}{2\pi}\int_T H(z,t)d_\mu(t)$ a function in H^∞ such that the Borel measure μ has no mass at the point $z = 1$ and S_β has at most a point mass at $z = 1$; that is, S_β has the form $S_\beta(z) = \exp\left(-\beta\frac{1+z}{1-z}\right)$ $(\beta \geq 0)$.

Because the regular measure μ has no mass at the point $z = 1$, that is $\mu(\{1\}) = 0$, we can choose open arcs $E_n \subset \partial\mathbb{D}$ containing the point $z = 1$ such that

$$|\mu|(E_n) \leq \frac{1}{n}, \quad |\mu| = \mu^+ + \mu^-,$$

$$E_n \subset E_{n-1} \quad (n \in \mathbb{N}).$$

Define the functions G_n and H_n by

$$G_n(z) = \exp\frac{1}{2\pi}\int_{E_n} H(z,t)d_\mu(t), \quad H_n(z) = \exp\frac{1}{2\pi}\int_{T\setminus E_n} H(z,t)d_\mu(t).$$

It is now easy to check that the sequence G_n converges uniformly to 1 outside every neighbourhood of the point $z = 1$. Note that $G_\mu = G_n H_n$.

Let E_n' be any open arc containing $z = 1$ such that $\overline{E_n'} \subset E_n$ for a fixed $n \in \mathbb{N}$. Now we factorise the Blaschke factor B of the function f into a product $B = B_1^{(n)} B_2^{(n)}$ of two Blaschke products such that the the zeros $a_i^{(n)}$ of $B_1^{(n)}$ are accumulating only on the set $\overline{E_n'}$ and those of $B_2^{(n)}$ only on the set $T \setminus E_n'$.

Hence $P_N^{(n)}(z) = \prod_{i=N}^{\infty} - \frac{\bar{a}_i^{(n)}}{|a_i^{(n)}|} \frac{z - a_i^{(n)}}{1 - \bar{a}_i^{(n)} z}$ converges uniformly to 1 on each compact set of $\overline{\mathbb{D}} \setminus E_n$. Now we choose $N = N(n)$ so that

$$|P_{N(n)}^{(n)}(z) - 1| \le \frac{1}{n} \quad \text{on} \quad \overline{\mathbb{D}} \setminus U_{\delta_n}(1) ,$$

where $\delta_n = \operatorname{dist}(1, \partial E_n)$. Let $B_n = P_{N(n)}^{(n)}$. Then the functions $g_n = B_n G_n$ converge uniformly to 1 outside each neighbourhood of the point $z = 1$.

We remark furthermore that $||g_n|| \le \max\{1, ||f||\}$. Thus the sequence $S_\beta g_n(1-z)$ converges uniformly on $\mathbb{D}$ to $S_\beta(1-z)$, that is

(3) $$||S_\beta g_n(1-z) - S_\beta(1-z)|| \to 0 \qquad (n \to \infty) .$$

Step 2. We are now going to show that the functions $S_\beta g_n$ belong to the ideal J .

By the first step, f has the form

$$f = \tilde{B}_n B_n B_2^{(n)} S_\beta G_n H_n = S_\beta g_n (\tilde{B}_n B_2^{(n)} H_n) ,$$

where $\tilde{B}_n$ is a finite Blaschke product and where the function $h_n = \tilde{B}_n B_2^{(n)} H_n$ are bounded away from zero in a neighbourhood of the point $z = 1$. Thus the functions h_n are not contained in any maximal ideal of the fibre M_1 . Because J is contained only in maximal ideals of the fibre M_1 , we can conclude that the ideal $(J, h_n) = H^\infty$. Hence there exist functions $x_n \in H^\infty$ and $y_n \in J$ such that

$$1 = y_n + x_n h_n .$$

Multiplying both sides by $S_\beta g_n$, we see that

$$S_\beta g_n = x_n(S_\beta g_n h_n) + y_n(S_\beta g_n) = x_n f + y_n(S_\beta g_n) \in J ,$$

which proves the assertion of the second step.

Hence the functions $S_\beta g_n(1-z)$ are in J. The ideal J being closed, we can conclude then from (3) that

(4) $$S_\beta(1-z) \in J .$$

<u>Step 3.</u> Since the greatest common divisor of the inner parts of the functions in J is invertible, we remark that for every $\varepsilon > 0$ there exists a function $f \in J$ such that $f = S_\varepsilon g$, where the Borel measure associated to the singular inner part of the function $g \in H^\infty$ has no mass at the point $z = 1$ and where $S_\varepsilon(z) = \exp\left(-\varepsilon \frac{1+z}{1-z}\right)$.

Since the function $f = S_\varepsilon g$ can now be factorised, as in the first step, into the product $f = BS_\varepsilon G_\mu$, where $\mu(\{1\}) = 0$, we can conclude from (4) that $S_\varepsilon(1-z) \in J$ for every $\varepsilon > 0$, in particular for $\varepsilon = \frac{1}{n}$.

Since the functions $S_{1/n}(z) = \sqrt[n]{\exp\left(-\frac{1+z}{1-z}\right)}$ converge uniformly to 1 outside every neighbourhood of 1 in $\mathbb{D}$, we have

$$||S_{1/n}(1-z) - (1-z)|| \underset{n\to\infty}{\to} 0 .$$

Thus, since J is closed, the outer function $1-z$ belongs to J. □

REMARK. Because the closure of the ideal $(1-z)$ coincides with the ideal $M = \{f \in H^\infty : \hat{f} \equiv 0 \text{ on } M_1\} = \{f \in H^\infty : \lim_{z\to 1} f(z) = 0\}$ (note that $||f(1-z^n)-f|| \to 0$ for $f \in M$), Theorem 2.1 implies that $M \subset J$.

Using the theorem of Beurling and Rudin about the characterisation of the closed ideals in the disc algebra $A(\mathbb{D})$ (see Hoffman [8], p.85), we can generalise Hedenmalm's result with our methods in the following sense:

THEOREM 2.2. *Let* I *be a closed ideal in* H^∞ *that is contained only in maximal ideals of the set of fibres* M_λ, *where* λ *runs through a compact set* $K \subset T$ *of Lebesgue measure zero. Then there exists an inner function* ϕ *whose boundary singularities are contained in the set* K *such that*

$$I = \phi J ,$$

where J *is a closed ideal that contains the outer function* $1-p_K$, p_K *being the* $A(\mathbb{D})$ *peak function of the set* K .

REMARK. Because the closure of the ideal $(1-p_K)$ coincides with the ideal $M = \{f \in H^\infty: \hat{f} \equiv 0 \text{ on } \bigcup_{\lambda \in K} M_\lambda\} = \{f \in H^\infty: \lim_{\substack{z\to\lambda \\ \lambda\in K}} f(z) = 0\}$, Theorem 2.2 implies that $M \subset J$. Thus the situation in H^∞ is similar to that in the disc algebra $A(\mathbb{D})$, where, under equivalent assumptions, J coincides with the ideal $M = \{f \in A(\mathbb{D}): f \equiv 0 \text{ on } K\}$.

Proof. Let ϕ be the greatest common divisor of the inner parts of the functions in I . If λ is a boundary singularity of ϕ , then there exists a sequence z_n in $\mathbb{D}$ converging to λ such that $\phi(z_n) \to 0$. Hence ϕ lies in a maximal ideal m of the fibre M_λ (see Hoffman [8, p.161]). Under the assumptions of the theorem, it is now clear that I has the form $I = \phi J$, where the boundary singularities of ϕ are contained in the set K and where J is the closed ideal $\{f \in H^\infty: f\phi \in I\}$.

In the next two steps the proof continues in the same manner as before; we have now only to factorise the functions $f \in J$ in the following form:

(1) $$f = BS_\nu G_\nu , \quad \nu = \nu(f) , \quad \mu = \mu(f) ,$$

here B is a Blaschke product, S_ν a singular inner function and $G_\mu(z) = \exp \frac{1}{2\pi} \int_T H(z,t)d_\mu(t)$ a zero free function in H^∞ such that $\mu(K) = 0$. Thus we can conclude that $S_\nu(1-p_K) \in J$.

In order to prove the third step, we remark that by the Beurling-Rudin theorem and the fact that the greatest common divisor of the inner parts of the functions of J is invertible, the closure of the ideal generated by the functions

$$\{S_{\nu(f)}(1-p_K) : f \in J\} \quad , \quad S_{\nu(f)} \text{ as in (1)} ,$$

coincides with the closure of the principal ideal generated by $1-p_K$.

Hence the function $1-p_K$ belongs to J. □

Using Theorem 2.1 we can give a complete characterisation of the closed primary ideals contained in a maximal ideal of the Shilov boundary. Thus we solve a problem raised by Hoffman (see [6, p.74]).

Before we proceed, we present some background. Let L^∞ be the Banach algebra of all essentially bounded, complex valued functions on the unit circle T under the supremum norm $||\cdot||_\infty$ and $A_1 = H^\infty|M_1$ the restriction algebra of H^∞ to the fibre M_1. It is well known (see Hoffman [8, p.187]) that A_1 is isometrically isomorphic to the quotient algebra H^∞/M, where M is the closed ideal $M = \{f \in H^\infty : \hat{f} \equiv 0 \text{ on } M_1\}$. Let $H^\infty + C = \{f + g : f \in H^\infty, g \in C\}$. A theorem Axler [2, p.567] states that for each $f \in L^\infty$ there exists a Blaschke product B such that $Bf \in H^\infty + C$. Axler's proof shows that this result can be extended to sequences of functions. Using this theorem and its proof, one can show that

$$J = \{f \in L^\infty : \hat{B}\hat{f}|X_1 \in I|X_1 \text{ for some Blaschke product } B\}$$

is a closed ideal in L^∞ whenever I is a closed ideal in H^∞ containing M. A slightly different version of the following was shown to me by Gorkin.

THEOREM 2.3. *Let* I *be a closed primary ideal contained in a maximal ideal* m *of the Shilov boundary* X *of* H^{∞} . *Then* I *is maximal.*

Proof. Since I is primary, the greatest common divisor of the inner parts of the functions in I is invertible. This follows from the fact that for every noninvertible singular inner function ϕ there exists a sequence (z_n) in $\mathbb{D}$ such that $\phi(z_n) \to 0$ (see Hoffman [8, Ex.13, p. 73]) and that the closure of $\{z_n\}$ in M contains infinitely many points $m \in M\backslash\mathbb{D}$ (see Garnett [4, p.190]). Without loss of generality we may assume that $I \subset m \in X_1$, where $X_1 = M_1 \cap X$. Theorem 2.1 implies that I contains the ideal

$$M = \{f \in H^{\infty}: \ \hat{f} \equiv 0 \ \text{on} \ M_1\} \ .$$

By the remark above, the ideal

$$J = \{f \in L^{\infty}: \ \hat{B}\hat{f}|X_1 \in I|X_1 \ \text{for some Blaschke product}\ B\}$$

is then closed in L^{∞} . Because I is assumed to be primary, it is obvious that J has this property too.

On the other hand, every closed primary ideal in $L^{\infty} \cong C(X)$ is known to be maximal. Hence J is maximal; from which we can conclude that $J \cap H^{\infty} = m$. Thus for every function $f \in m$ there exists a function $g \in I$ and a Blaschke product B such that $(\hat{B}\hat{f}-\hat{g})|X_1 \equiv 0$. Since X_1 is the Shilov boundary of A_1 , we have $\hat{B}\hat{f}-\hat{g} \equiv 0$ on M_1 ; hence $Bf-g \in M \subset I$. Since $B \notin m$ and $I \subset m$ is primary, the ideal (I,B) is the whole algebra H^{∞} . Hence there exist functions $x \in H^{\infty}$ and $y \in I$ such that $1 = y+xB$. Multiplying by f we have $f = fy+x(Bf) \in I$. Thus $I = m$. □

We remark that a characterisation of the closed primary ideals contained in a maximal ideal whose Gleason part $R(m)$ is nontrivial is known (see Hedenmalm [6, p.14]). Indeed, if $I \subset m$ is such an ideal, then I has the form

$$I = I_n = \{f \in H^{\infty}: \ \hat{f} \circ \Phi \in (z-z_0)^n H^{\infty}\} \ ,$$

where Φ is the analytic disc from $\mathbb{D}$ onto $R(m)$ such that $\Phi(z_o) = m$.

On the other hand, however, a characterisation of the closed primary ideals contained in a maximal ideal $m \in M\setminus(g \cup X)$, that is a maximal ideal whose Gleason part is trivial, but which does not belong to the Shilov boundary, is still unknown. But we conjecture that the following is true:

Conjecture 1. Let I be a closed primary ideal contained in a maximal ideal $m \in M\setminus(g \cup X)$. Then I is maximal.

In the final part of section 2, we consider ideals of the form

$$I_m = \{f \in H^\infty : \hat{f} \equiv 0 \text{ in a neighbourhood of } m \text{ in the topological space } X_1\},$$

where m is a maximal ideal of $X_1 = X \cap M_1$. Note that $X_1 \neq M_1$. In view of the results in the algebra $L^\infty \cong C(X)$, we could expect that every ideal of the form I_m would be dense in the corresponding maximal ideal. But this does not hold. Indeed, we have the following result:

PROPOSITION 2.4. *Let A be a uniform algebra, M its maximal ideal space and X its Shilov boundary. Suppose that for every $x \in X$ the ideal $I_x = \{f \in A: \hat{f} \equiv 0$ in a neighbourhood of x in the topological space $X\}$ is dense in x. Then $X = M$.*

Proof. Assume that there exists an element $m \in M\setminus X$. Since $\bar{I}_x = x$ for all $x \in X$, we can conclude that there exists for each x a function $f_x \in I_x$, $\hat{f}_x \equiv 0$ in a neighbourhood $U(x)$ of x, such that $\hat{f}_x(m) \neq 0$. Hence $X = \bigcup_{x \in X} U(x)$. Because X is compact, there exist thus finitely many functions $f_{x_1},\dots,f_{x_n} \in A$ such that the function

$$g = f_{x_1} \cdots f_{x_n}$$

vanishes identically on X, and hence on M, because X is the Shilov boundary of A. Buth this contradicts the fact that $\hat{g}(m) =$

$\hat{f}_{x_1}(m)\cdots\hat{f}_{x_n}(m) \neq 0$ by construction.

Thus $X = M$. □

3. Closed prime ideals in H^∞

In a recent paper [*10*] we have proved that each prime ideal which contains an interpolating Balschke product is primary. The next proposition will show that such an ideal cannot be closed unless it is maximal.

THEOREM 3.1. *Let* P *be a closed prime ideal containing an interpolating Blaschke product. Then* P *is maximal.*

Proof. Let m be a maximal ideal that contains P . Because P contains an interpolating Blaschke product, it is by [*10*] or [9, p.52], primary. Hence P is a closed primary ideal which lies in a maximal ideal m whose Gleason part is nontrivial. Thus, by the remark after Theorem 2.3, P has the form

$$P = \{f \in H^\infty: \ \hat{f}\circ\Phi \in (z-z_o)^n H^\infty\}$$

for an integer $n \in \mathbb{N}$. Because the order of the zero m of the interpolating Blaschke product is 1, it is easily seen that $n = 1$. Hence

$$P = \{f \in H^\infty: \ \hat{f}\circ\Phi \in (z-z_o)\, H^\infty\} .$$

But the last ideal coincides with m , thus $P = m$. □

Remark. After this work was finished, I learned that Theorem 3.1 has also been proved by Gorkin [5] (independently).

In the next proposition we consider closed prime ideals contained in a maximal ideal of the Shilov boundary.

PROPOSITION 3.2. *Let* P *be a closed prime ideal contained in a maximal ideal* m *of the Shilov boundary* X *of* H^∞ . *Then* P *contains the ideal* $I_m = \{f \in H^\infty: \ \hat{f} \equiv 0$ *in a neighbourhood of* m *in the topological space* $X_\lambda\}$ *for some* $\lambda \in T$.

Proof. Without loss of generality we may assume that $P \subset m \in X_1$, where $X_1 = M_1 \cap X$. Let f be any function of the ideal I_m and U a neighbourhood of m on which f vanishes identically. Because the restriction algebra A_1 of H^∞ to M_1 is regular on X_1 (Hoffman [8, p.189]), there exists a function $g \in H^\infty$ such that

$$\hat{g} \equiv 0 \text{ on } X_1 \setminus U \text{, but } \hat{g}(m) \neq 0$$

Thus $\hat{f}\hat{g} \equiv 0$ on X_1 and hence on M_1, because X_1 is the Shilov boundary of A_1.

Because P is prime, it is easily seen that the greatest common divisor of the inner parts of the functions in P is invertible. In order to apply Theorem 2.1, we have yet to show that the ideal P is contained only in maximal ideals of the fibre M_1.

Assume that there also exists a maximal ideal m of the fibre M_α, $\alpha \neq 1$, that contains P. Then the factorisation

$$f = \left(B_1 \exp \frac{1}{2\pi} \int_E H(z,t)d_\mu(t)\right) \left(B_2 \exp \frac{1}{2\pi} \int_{T\setminus E} H(z,t)d_\mu(t)\right),$$

where E is an open arc such that $1 \in E$, $\alpha \notin \bar{E}$ and B_1 (respectively B_2) are Blaschke products whose zeros accumulate only at $\bar{E}$ (respectively at $T \setminus E$), shows that P cannot be prime.

Hence by Theorem 2.1 we conclude that P contains the ideal $\{f \in H^\infty : \hat{f} \equiv 0 \text{ on } M_1\} = \overline{(1-z)}$; in particular $fg \in P$. Since P is prime and $g \notin m$, we get $f \in P$. This yields the assertion $I_m \subset P$. □

We are now able to characterise the closed prime ideals contained in a maximal ideal of the Shilov boundary of H^∞.

THEOREM 3.3. *Every closed prime ideal in H^∞ contained in a maximal ideal m of the Shilov boundary is maximal.*

Proof. We proceed as in the proof of Theorem 2.3. Let P be the closed prime ideal and m the maximal ideal containing it. We remark that by Proposition 3.2 P contains the ideal

$$M = \{f \in H^\infty : \hat{f} \equiv 0 \text{ on } M_1\} .$$

Thus we can conclude that the ideal

$$J = \{f \in L^\infty : \hat{B}\hat{f}|X_1 \in P|X_1 \text{ for some Blaschke product } B\}$$

is closed. By Proposition 3.2 and the fact that $A^1 = H^\infty|M_1$ is regular on X_1, we see that J is primary. Hence J is, as a closed primary ideal in L^∞, maximal. This implies that $J \cap H^\infty = m$. By exactly the same arguments as in the proof of Theorem 2.3 there exists for every $f \in m$ a function $g \in P$ and a Blaschke product B such that $Bf-g \in M \subset P$. Since P is prime and $B \notin P \subset m$, we see that $f \in P$. Thus $P = m$. □

Note that, unlike conjecture 1, there exist non-maximal closed prime ideals which are contained in a maximal ideal $m \notin X$ whose Gleason part is trivial. Indeed, by a result of Budde [3, p.11], every nontrivial Gleason part $R(m)$ contains a maximal ideal, whose Gleason part is trivial, in its closure. Hence the ideal

$$P = \{f \in H^\infty : \hat{f} \equiv 0 \text{ on } R(m)\} ,$$

which is a closed prime ideal, is such an example.

It is therefore of great interest to characterise the closed prime ideals in H^∞. In view of this, Alling conjectured:

Conjecture. (Alling [1]). Let P be a non-maximal closed prime ideal in H^∞. Then there exists a maximal ideal m whose Gleason part is nontrivial such that

$$P = \{f \in H^\infty : \hat{f} \equiv 0 \text{ on } R(m)\} .$$

Our next proposition together with Proposition 3.1 and Theorem 3.3 will yield a partial solution to the conjecture of Alling.

DEFINITION. An ideal $I \subset H^\infty$ is called free, if the functions in I have no common zeros in $\mathbb{D}$.

Let $m \in M$ and $\hat{f}(m) = 0$. Then

$$\operatorname{ord}(f,m) = \sup\{n \in \mathbb{N} : f = f_1 \cdots f_n,\ \hat{f}_i(m) = 0,\ i = 1,\dots,n\}$$

will denote the order of the zero of f at m.

PROPOSITION 3.4. *Let P be a free prime ideal in H^∞ and m be a maximal ideal that contains P. Then the following assertions are equivalent:*

(1) P does not contain any interpolating Blaschke product.

(2) For every $f \in P$ we have $\operatorname{ord}(f,m) = \infty$.

(3) $P \subset \{f \in H^\infty : \hat{f} \equiv 0$ on $R(m)\}$.

Proof. (1) ⇒ (2): Since P does not contain any interpolating Blaschke product, every function $f \in P$ can be factorised into a product $f = f_1 f_2$ of two functions in H^∞ such that $\hat{f}_1(m) = \hat{f}_2(m) = 0$ (see [10] or [9, p.53, theorem 5.5]). The fact that P is prime implies that at least one of the factors f_1 or f_2 lies in P. So continuing, we get the assertion (2).

(2) ⇒ (3): Follows directly from a theorem of Hoffman (see [3, p.403, Lemma 1.2]).

(3) ⇒ (1): If P contained an interpolating Blaschke product B, then P would be primary by [10] or [9, p.52]. Furthermore, the Gleason part $R(m)$ of m would be nontrivial. Thus (3) cannot hold. □

Remark. If we take the converse of the assertions in Proposition 3.4, we obtain a characterisation of those primary prime ideals in H^∞ which are contained in a maximal ideal $m \in G$:

PROPOSITION 3.4'. *Let P be a prime ideal in H^∞ and m a maximal ideal that contains P. Then the following assertions are equivalent:*

(1) P contains an interpolating Blaschke product.

(2) There exists an $f \in P$ such that $\operatorname{ord}(f,m) = 1$.

(3) *P is primary and* $m \in g$.

Now we can state the main result of this section.

THEOREM 3.5. *Let P be a non-maximal closed prime ideal in* H^∞ . *Then P is contained in the ideal of all those functions that vanish on a Gleason part, which is disjoint from the Shilov boundary, that is*

$$P \subset \{f \in H^\infty : \hat{f} \equiv 0 \text{ on } R(m)\} , \quad m \notin X .$$

Proof. Follows directly from Proposition 3.1, 3.4 and Theorem 3.3. □

4. Chains of prime ideals in H^∞

In a recent paper [5], Gorkin has shown that each maximal ideal $m \in M\backslash(X \cup \mathbb{D})$ contains an infinite chain of prime ideals. It is now of great interest to ask whether all the prime ideals contained in a maximal ideal form a chain; a situation which occurs, as is known (see Henriksen [7, p.716]), in the ring $H(\mathbb{D})$ of all analytic functions in the unit disc.

Our next theorem now gives a complete answer to this problem.

THEOREM 4.1. *The set of prime ideals contained in a maximal ideal m of* H^∞ *is linearly ordered (by set inclusion) if and only if m belongs to the unit disc* $\mathbb{D}$ *or to the Shilov boundary X* .

Proof. Since the case that m belongs to the unit disc is trivial, we show in the first step that the prime ideals contained in a maximal ideal m of the Shilov boundary X are linearly ordered. Note that H^∞ is a pseudoBezout ring; that is, any two functions in H^∞ have a greatest common divisor (gcd) (see v.Renteln [11, p.519]).

Let P and Q be two prime ideals contained in a maximal ideal $m \in X$. Suppose that there exists a function $f \in P \setminus Q$ and a function $g \in Q \setminus P$. Let $d = gcd(f,g)$. Hence there exist two bounded analytic functions F and G which have no proper common divisor such that

$$f = dF \quad (1) \qquad g = dG \quad (2) .$$

Because $g \notin P$, we can conclude from (2) that $d \notin P$. Thus the primeness of P and (1) yield that $F \in P$. In the same manner we conclude that $G \in Q$. Hence the functions F and G are both in m (remark that $P,Q \subset m$). But $gcd\ (F,G) = 1$; thus the ideal (F,G) contains (by v. Renteln [11, p.523]) an inner function ϕ; in particular $\phi \in m$. However $|\hat{\phi}(m)| = 1$ for every $m \in X$ and all inner function (see Hoffman [8, p.179]). Thus we have either $P \subset Q$ or $Q \subset P$.

In the final step, we show that the prime ideals contained in a maximal ideal m that does not belong to the Shilov boundary or to the unit disc, are not linearly ordered. Indeed, we shall construct two prime ideals $P \subset m$ and $Q \subset m$ such that $P \not\subset Q$ and $Q \not\subset P$.

Let $m \in M \setminus (X \cup \mathbb{D})$. We choose two functions f and g of m such that $gcd(f,g) = 1$. For example, let f be a Blaschke product in m, which exists by a theorem of D.J. Newman (see Hoffman [8, p.179]), and g the outer function $z-\alpha$, where $\alpha \in T$ is suitably chosen.

We define the following multiplicatively closed subsets of H^∞:

$$S_f = \{hf^n:\ h \in H^\infty,\ h \notin m,\ n \in \mathbb{N} \cup \{0\}\},$$
$$S_g = \{hg^n:\ h \in H^\infty,\ h \notin m,\ n \in \mathbb{N} \cup \{0\}\}.$$

Since $gcd(f,g) = 1$, it is easy to see that $(g) \cap S_f = \emptyset$ and $(f) \cap S_g = \emptyset$.

By the lemma of Krull each super ideal $P \supset (g)$, (respectively $Q \supset (f)$), which is maximal with respect to $P \cap S_f = \emptyset$, (respectively $Q \cap S_g = \emptyset$), is a prime ideal. By construction P and Q are contained in the maximal ideal m. But on the other hand, $f \in S_f$ implies $f \notin P$ and $g \in S_g$ implies $g \notin Q$. Thus $P \not\subset Q$ and $Q \not\subset P$. □

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