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ON THE PROBLEM OF R. DE VORE

BY
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I. **Introduction.** Let $f \in L_p (1 \le p < \infty)$ be 2π-periodic,

$$\|\Delta_t^r f\|_p = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{j=0}^{r}(-1)^j\binom{r}{j}f(x+jt)\right|^p dx\right]^{1/p} \tag{1}$$

and let us consider the rth order moduli of continuity of f

$$\omega_r(L_p; f; h) = \sup_{0 \le t \le h} \|\Delta_t^r f\|_p. \tag{2}$$

R. de Vore[1] stated the following conjectures:

(a) Let $\eta_\nu \to 0$ decreasingly,

$$1 > \eta_{\nu+1}/\eta_\nu > \theta_1 > 0 \qquad (\nu = 1, 2, \ldots) \tag{3}$$

and for some $\alpha < 2$ let

$$\|\Delta_\eta^2 f\|_p = \mathcal{O}(\eta^\alpha) \qquad (\eta \in \{\eta_\nu\}) \tag{4}$$

then for every $h \ge 0$

$$\omega_2(L_p; f; h) = \mathcal{O}(h^\alpha). \tag{5}$$

(b) If for the sequence $\{\eta_\nu\}$ the condition (3) does not hold for at least one $\theta_1 > 0$ then there exists an $f \in L_p$ for which (4) is valid but (5) is violated.

R. de Vore himself settled problems (a) and (b) for the limiting case $p \to \infty$, i.e. for the space C.

In the present note we prove conjecture (a) for $p=2$ and conjecture (b) for all $1 \le p \le \infty$, in both cases also for higher order moduli of continuity.

II. **Proof of conjecture** (a) **for** $p=2$.

LEMMA 1. *Let* $\tau(x)=x$ *for* $0 \le x \le 1$ *and* $\tau(x)=1$ *for* $x>1$. *Let further* $f(x) \sim a_0/2 + \sum (a_k \cos kx + b_k \sin kx)$ *then*

$$c_1(r)[\omega_r(L_2; f; h)]^2 \le \sum_{k=1}^{\infty} [\tau(kh)]^{2r}(a_k^2+b_k^2) \overset{\text{def}}{\equiv} \phi_{2r}(f; h) \le c_2(r)[\omega_r(L_2; f; h)]^2. \tag{6}$$

Lemma 1 was proved in our paper [1].

[1] Oral communication in January 1972, at a time when both R. de Vore and the author were visiting professors at the University of Alberta, Edmonton (Canada). (See [2]).

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LEMMA 2. *We have*

$$(7)\qquad [\|\Delta_t^r f\|_2]^2 \ge 2^{2r-1}\pi^{-2r}\sum_{k\le\pi/t}(a_k^2+b_k^2)(kt)^{2r}.$$

Proof. We refer to [1] for the relation

$$\|\Delta_t^r f\|^2 = 2^{2r-1}\sum_{k=1}^{\infty}(a_k^2+b_k^2)\sin^{2r} kt/2.$$

From this we obtain (7) by taking $\sin u > 2u/\pi$ $(0\le u\le \pi/2)$ in consideration.

LEMMA 3. *We have for every* $f\in L_2$ *and all natural integers* $\rho<r$

$$(8)\qquad c_3(r,\rho)h^{2\rho}\int_h^\infty u^{-2\rho-1}[\omega_r(L_2;f;u)]^2\,du \le [\omega_\rho(L_2;f;h)]^2 \le$$

$$\le c_4(r,\rho)h^{2\rho}\int_h^\infty u^{-2\rho-1}[\omega_r(L_2;f;u)]^2\,du \qquad (h>0).$$

Lemma 3 was proved under the heading "Theorem 3" in our paper [1].

LEMMA 4. *Let* $\{t_\nu\}$ *be a sequence of positive numbers satisfying*

$$(9)\qquad 1 > t_{\nu+1}/t_\nu > \theta_1 > 0 \qquad (\nu=1,2,\ldots)$$

then for every natural r *and every* $f\in L_2$ *we have*

$$(10)\qquad [\omega_r(L_2;f;t_1)]^2 \le c_5(r,\theta_1)\sum_{\nu=1}^{\infty}\|\Delta_{t_\nu}^r f\|_2^2.$$

Proof. By (7) we have

$$(11)\qquad \sum_{\nu=1}^{\infty}\|\Delta_{t_\nu}^r f\|_2^2 \ge c_6(r)\sum_{k=1}^{\infty}(a_k^2+b_k^2)k^{2r}\sum_{kt_\nu\le\pi}t_\nu^{2r}.$$

For $kt_1\le\pi$ we have clearly

$$k^{2r}\sum_{kt_\nu\le\pi}t_\nu^{2r} > (kt_1)^{2r} \ge [\tau(kt_1)]^{2r} \qquad (kt_1\le\pi).$$

In the case $kt_1>\pi$ there exists, as a consequence of (9), an index μ for which $\pi>kt_\mu\ge\theta_1\pi$. Consequently

$$k^{2r}\sum_{kt_\nu\le\pi}t_\nu^{2r} > k^{2r}t_\mu^{2r} > (\theta_1\pi)^{2r} \ge (\theta_1\pi)^{2r}[\tau(kt_1)]^{2r}$$

(We used here that $\tau(kt_1)=1$). Combining both cases

$$(12)\qquad k^{2r}\sum_{kt_\nu\le\pi}t_\nu^{2r} > c_7(r,\theta_1)[\tau(kt_1)]^{2r}.$$

We obtain (10) from (11), (12) and the first half of (6), Q.E.D.

THEOREM 1. *Let* $\psi(\delta)$ *be a nondecreasing function. For proper choices of* $\theta_2>1$, $\theta_3>1$ *let*

$$(13)\qquad \theta_2\psi(h) \le \psi(2h) \le \theta_3\psi(h) \qquad (h>0),$$

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and for an $f \in L_2$ let

(14) $$\|\Delta_\eta^r f\|_2 \le \psi(\eta) \qquad (\eta \in \{\eta_\nu\})$$

where $\{\eta_\nu\}$ is a positive nullsequence satisfying (3) *then*

(15) $$\omega_r(L_2; f; h) < c_8(\theta_1, \theta_2, \theta_3, r)\psi(h) \qquad (0 < h \le \delta_1)$$

and for every $\rho < r$

(16) $$\omega_\rho(L_2; f; h) \le c_9(\theta_1, \theta_2, \theta_3, r, \rho) \cdot \left\{h^{2\rho} \int_h^\infty u^{-2\rho-1}\psi^2(u)\, du\right\}^{1/2}.$$

REMARK. In particular, if

$$\|\Delta_\eta^r f\|_2 = \mathcal{O}(\eta^\alpha) \qquad (\eta \in \{\eta_\nu\})$$

then we have for every $\alpha < \rho \le r$

$$\omega_\rho(L_2; f; h) = \mathcal{O}(h^\alpha).$$

This statement implies the case $p=2$ of conjecture (a).

Proof of Theorem 1. First we construct a suitable subsequence $\{n_m^*\} \subset \{\eta_\nu\}$. Let $\eta_1^* = \eta_1$. After η_m^* is constructed, let $\eta_{m+1}^* = \eta_s$ be the greatest η satisfying $\eta_s < \theta_1 \eta_m^*$. Then clearly $\eta_{s-1}/\eta_m^* > \theta_1$ so that

(17) $$1 > \theta_1 > \eta_{m+1}^*/\eta_m^* = \eta_s/\eta_{s-1} \cdot \eta_{s-1}/\eta_m^* > \theta_1^2.$$

We apply now Lemma 3 with $t_\nu = \eta_{n-1+\nu}^*$ $(\nu = 1, 2, \ldots)$ and consider that $\{\eta_m^*\} \subset \{\eta_\nu\}$ so that by (14)

$$\|\Delta_{\eta^*}^r f\| \le \psi(\eta^*) \qquad (\eta^* \in \{\eta_m^*\}).$$

We obtain

(18) $$[\omega_r(L_2; f; \eta_n^*)]^2 \le c_3(r, \theta_1^2) \sum_{m=n}^\infty \psi^2(\eta_m^*) = c_3(r, \theta_1^2)\{\psi^2(\eta_n^*) + \sum_{k=0}^\infty \sigma_k\},$$

where

(19) $$\sigma_k = \sum_{2^{-k-1}\eta_n^* \le \eta_m^* < 2^{-k}\eta_n^*} \psi^2(\eta_m^*).$$

As a consequence of (3), the number of terms to which the sum (19) is extended does not exceed $-\log_{\theta_1} 2 + 1 = c_7(\theta_1)$ so that

(20) $$\sigma_k \le c_{10}(\theta_1)\psi^2(2^{-k}\eta_n^*) \le c_{10}(\theta_1)\theta_2^{-2k}\psi^2(\eta_n^*).$$

Here we made use of the facts that $\psi(\Delta)$ is nondecreasing and satisfies (13). We obtain from (18) and (20)

(21) $$\omega_r(L_2; f; \eta_n^*) \le c_{11}(r, \theta_1, \theta_2)\psi(\eta_n^*).$$

For an arbitrary $0 < h \le \eta_1^*$ let $\eta_{j+1}^* \le h < \eta_j^*$ then by monotonicity of ω_r

(22) $$\omega_r(L_2; f; h) \le \omega_r(L_2; f; \eta_j^*) \le c_{11}(r, \theta_1, \theta_2)\psi(\eta_j^*).$$

We fix a sufficiently great integer q so that $2^{-q}<\theta_1^2$. We obtain from (17)

$$h/\eta_j^* \geq \eta_{j+1}^*/\eta_j^* \geq \theta_1^2 > 2^{-q}$$

and from (13) follows, since $\psi(\delta)$ is nondecreasing,

$$\psi(\eta_j^*) \leq \theta_3^q \psi(2^{-q}\eta_j^*) \leq \theta_3{}^q \psi(h). \tag{23}$$

From (22) and (23) we conclude that (15) is valid. We obtain (16) by combining (15) with Lemma 3. This ends our proof.

III. **Proof of conjecture** (b)

LEMMA 5. *Let* $f_k(x)=\sin kx$ *then we have*

$$\omega_r(L_p; f_k; h) = \begin{cases} 2^r A_p \left(\sin \dfrac{kh}{2}\right)^r & (0 \leq h \leq \pi/k) \\ 2^r A_p & (h \geq \pi/k) \end{cases} \tag{24}$$

where

$$A_p = \|f_1\|_p. \tag{25}$$

Proof. Lemma 5 is a trivial consequence of the relation

$$\Delta_t^r f_k\left(x-\frac{rt}{2}\right) = 2^r \sin\left(kx-r\frac{\pi}{2}\right)\sin^r \frac{kt}{2}.$$

Let us observe that A_p is increasing, so that

$$\frac{2}{\pi} = A_1 \leq A_p \leq A_\infty = 1. \tag{26}$$

Let r be an arbitrary integer and let $0<s<r$. We consider a system of non-overlapping open intervals (x_ν, X_ν) in (0, 1) and we assume that

$$(X_\nu/x_\nu)^{(r-s)^s/r} > 4^\nu \qquad (\nu = 1, 2, \ldots). \tag{27}$$

Finally, let us consider the function

$$g(x) = \sum_{\nu=1}^{\infty} 2^{-\nu-1-r} X_\nu^s \sin[X_\nu^{-\alpha} x_\nu^{-1+\alpha}]x, \tag{28}$$

where $[X_\nu^{-\alpha}x_\nu^{-1+\alpha}]\geq 1$ is the integer part of $X_\nu^{-\alpha}x_\nu^{-1+\alpha}$ and $\alpha=s/r$.

LEMMA 6. *For every* $1\leq p \leq \infty$ *and for every* $h>0$ *satisfying*

$$h \notin (x_\nu, X_\nu) \qquad (\nu = 1, 2, \ldots) \tag{29}$$

we have for the function g defined as above

$$\omega_r(L_p; g; h) \leq h^s \tag{30}$$

but we have

$$\overline{\lim_{h\to 0}}\, h^{-s}\omega_r(L_p; g; h) = \infty. \tag{31}$$

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Proof. Let

$$g_h(x) = \sum_{x_\nu > h} 2^{-\nu-1-r} X_\nu^s \sin[X_\nu^{-\alpha} x_\nu^{-1+\alpha}]x \tag{32}$$

and

$$G_h(x) = \sum_{X_\nu < h} 2^{-\nu-1-r} X_\nu^s \sin[X_\nu^{-\alpha} x_\nu^{-1+\alpha}]x. \tag{33}$$

If $h \le x_\nu$ then a fortiori $h < X_\nu^\alpha x_\nu^{1-\alpha}$ so that by Lemma 5 using the fact that $\alpha = s/r$

$$\begin{aligned}\omega_r(L_p;\, g_h;\, h) &\le \sum_{x_\nu \ge h} 2^{-\nu-1-r} X_\nu^s \cdot 2^r A_p \sin^r \frac{1}{2} \frac{h}{X_\nu^\alpha x_\nu^{1-\alpha}} \le \\ &\le A_p \sum_{x_\nu \ge h} 2^{-\nu-1} X_\nu^s \left(\frac{h}{X_\nu^\alpha x_\nu^{1-\alpha}}\right)^r = A_p \sum_{x_\nu > h} 2^{-\nu-1} (h/x_\nu)^{r-s} h^s \\ &\le \tfrac{1}{2} A_p h^s \le \tfrac{1}{2} h^s \qquad (1 \le p \le \infty).\end{aligned} \tag{34}$$

In the last link of this chain of inequalities we applied (26).

In turn, we obtain from Lemma 5 and (26)

$$\omega_r(L_p;\, G_h;\, h) \le \sum_{X_\nu \le h} 2^{-\nu-1-r} X_\nu^s \cdot 2^r A_p \le \sum 2^{-\nu-1} h^s = \tfrac{1}{2} h^s. \tag{35}$$

Now if (29) holds then

$$g(x) = g_h(x) + G_h(x) \tag{36}$$

so that (30) is a consequence of (36), (34) and (35). This proves the first half of our statement.

Let now $h_k = X_k^\alpha x_k^{1-\alpha} \in (x_k, X_k)$. Then

$$g(x) = g_{h_k}(x) + G_{h_k}(x) + 2^{-k-1-r} X_k^s f_{[h_k^{-1}]}(x). \tag{37}$$

From Lemma 5 we get using (26)

$$\omega_r(L_p;\, f_{[h_k^{-1}]};\, h_k) = 2^r A_p \sin^r([h_k^{-1}]h_k) \ge 2^r \frac{2}{\pi} \sin^r \frac{1}{2} \ge c_{12}(r). \tag{38}$$

We have in consequence of (37), (34), (35) and (38), taking (27) in consideration

$$\begin{aligned}h_k^{-s} \omega_r(L_p;\, g;\, h_k) &\ge -1 + c_{12}(r) 2^{-k-1-r} X_k^s h_k^{-s} = \\ &= -1 + c_{12}(r) 2^{-k-1-r} (X_k/x_k)^{(r-s)^s/r} > -1 + c_{13}(r) 2^k.\end{aligned} \tag{39}$$

This shows that (31) holds. Lemma 6 is proven.

THEOREM 2. *Let $\{\delta_k \to 0\}$ be a decreasing sequence and*

$$\overline{\lim_{k\to\infty}}\, \delta_{k+1}/\delta_k = 0, \tag{40}$$

then there exists for every $1 \le p \le \infty$, for every natural integer r, and for every

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$0<s<r$ *a function* $g(x)$ *for which we have*

$$\omega_r(L_p;\, g;\, \delta_k) \le \delta_k^s \qquad (k = 1, 2, \ldots) \tag{41}$$

but

$$\overline{\lim_{h\to\infty}}\, h^{-s}\omega_r(L_p;\, g;\, h) = \infty. \tag{42}$$

REMARK. Let $\rho<r$ be a natural integer. Applying the elementary relation

$$\omega_r(L_p;\, g;\, h) \le 2^{r-\rho}\omega_\rho(L_p;\, g;\, h)$$

we infer from (42) that

$$\overline{\lim_{h\to\infty}}\, h^{-s}\omega_\rho(L_p;\, g;\, h) = \infty \qquad (\rho = 1, 2, \ldots, r)$$

Taking $r=\rho=2$ this shows that de Vore's conjecture (b) is true for every $1\le p\le\infty$.

Proof. By (40) we can construct a sequence of nonoverlapping open intervals $\{(x_\nu, X_\nu)\}$ so that

(a) $X_\nu>x_\nu$ are two consecutive terms of the sequence $\{\delta_k\}$

(b) (27) is satisfied.

Let us consider the function $g(x)$, defined as in (28), which is related to this interval sequence $\{(x_\nu, X_\nu)\}$.

By our construction no δ_k is situated inside any of the open intervals (x_ν, X_ν). Applying the first half of Lemma 6 we see that (41) is valid. Moreover, (42) was proved as the second half of Lemma 6. Theorem 2 is proved.

LITERATURE

1. G. Freud, *On the L_2-continuity moduli of functions*, Periodica Mathematica (Budapest), in print.

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