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$E(K/k)$ AND OTHER ARITHMETICAL INVARIANTS FOR FINITE GALOIS EXTENSIONS

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§1. Introduction

Let k be an algebraic number field and K be a finite extension of k. Recently, T. Ono defined positive rational numbers $E(K/k)$ and $E'(K/k)$ for K/k. In [7], he investigated some relations between $E(K/k)$ and other cohomological invariants for K/k. He obtained a formula when K is a normal extension of k. In our paper [3], we obtained a similar formula for $E'(K/k)$ in the case of normal extensions K/k. Both proofs essentially use Ono's results on the Tamagawa number of algebraic tori, on which the formulae themselves do not depend. Hence, in [8], T. Ono posed a problem to give direct proofs of these formulae.

In this paper, we shall show some relations between $E(K/k)$, $E'(K/k)$ and other arithmetical invariants for K/k (for example, central class number, genus number etc.), which, at the same time, give direct and simple proofs of the formulae of $E(K/k)$ and $E'(K/k)$.

In [9], R. Sasaki obtained another proof of the formula of $E(K/k)$.

§2, Notation and terminology

Let A be a multiplicative group and B be a subgroup of finite index. We denote the index by $[A:B]$ and abbreviate $[A:\{1\}]$ to $[A]$. Let k be an algebraic number field of finite degree over the rational field Q and T be an algebraic torus defined over k. We denote a Galois splitting field of T by K and the Galois group $\mathrm{Gal}(K/k)$ by G. $\hat{T}$ denotes the character module $\mathrm{Hom}(T, G_m)$ and $\hat{T}_0 = \mathrm{Hom}(\hat{T}, Z)$ denotes the integral dual of $\hat{T}$. Here G_m denotes the multiplicative group of the universal domain Ω. We consider the torus G_m is defined over k. Let $T(K)$ denote the group of K-rational points of T. Then $T(K)$ is isomorphic to $\hat{T}_0 \otimes K^\times$ as G-module, where $K^\times$ is the multiplicative group of K. For any place

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P of K, K_P denotes the P-completion of K. Then $T(K_P)$ the group of K_P-rational points of T is isomorphic to $\hat{T}_0 \otimes K_P^\times$. $T(O_P)$ denotes the maximal compact subgroup of $T(K_P)$, which is isomorphic to $\hat{T}_0 \otimes O_P^\times$, where $O_P^\times$ is the unit group of $K_P^\times$. Let us denote the K-adelization of T by $T(K_A)$. Then $T(K_A)$ is isomorphic to $\hat{T}_0 \otimes K_P^\times$, where $K_A^\times$ is the idele group of K. We define the unit group of $T(K_A)$ by putting

$$T(U_K) = \prod_{P:\text{finite}} T(O_P) \times \prod_{P:\text{infinite}} T(K_P)\,.$$

Then $T(U_K)$ is isomorphic to $\hat{T}_0 \otimes U_K$ as G-module, where U_K is the unit group of $K_A^\times$. Let us denote the k-adelization of T by $T(k_A)$, k-rational points of T by $T(k)$ and the unit group of $T(k_A)$ by $T(U_k)$. Then these are isomorphic to $T(K)^G$, $T(K_A)^G$ and $T(U_K)^G$. Here, for a G-module X, X^G denotes the submodule of X consisting of all the G-invariant elements of X. We define the class group of T by putting

$$C(T) = T(k_A)/T(U_k)\cdot T(k).$$

We define the class number of T by $[C(T)]$ and denote it by $h(T)$. We note here that the class group $C(G_m)$ is the class group of the algebraic number field k and $h(G_m) = h_k$ is the class number of k.

§ 3. The formula for $E(K/k)$

In this section, we shall investigate the relation between $E(K/k)$ and other arithmetical invariants for K/k, for the case when K is a normal extension of k. First, consider the following exact sequence of algebraic tori defined over k

$$(1) \qquad 0 \longrightarrow R^{(1)}_{K/k}(G_m) \longrightarrow R_{K/k}(G_m) \xrightarrow{N} G_m \longrightarrow 0\,,$$

where $R_{K/k}$ is the Weil functor and N is the norm map. It is known that K is a common Galois splitting field of $R^{(1)}_{K/k}(G_m)$, $R_{K/k}(G_m)$ and G_m. We denote Gal(K/k) by G. For the sake of simplicity, we shall denote $R^{(1)}_{K/k}(G_m)$, $R_{K/k}(G_m)$ and G_m by T', T and T''. The "Euler number" $E(K/k)$ is defined by putting $E(K/k) = h(T)/(h(T')\cdot h(T''))$. Let us denote $Z[G]/Zs$ ($s = \sum_{\sigma\in G}\sigma$) by $J[G]$. Then we have $\hat{T}' \cong J[G]$, $\hat{T} \cong Z[G]$, $\hat{T}'' \cong Z$. Hence the following sequence of the character modules is exact

$$(1)' \qquad 0 \longrightarrow Z \xrightarrow{\delta} Z[G] \longrightarrow J[G] \longrightarrow 0,$$

where δ is defined by $\delta(1) = s$. Then the integral dual of $(1)'$ is

$$(1)'' \qquad 0 \longrightarrow I[G] \longrightarrow Z[G] \xrightarrow{\varepsilon} Z \longrightarrow 0\,,$$

where ε is defined by $\varepsilon(\sigma) = 1$, for every $\sigma \in G$, and $I[G]$ is the kernel of this surjective homomorphism ε. From § 2, we have $T'(K_A) \cong I[G] \otimes K_A^\times$, $T(K_A) \cong Z[G] \otimes K_A^\times$ and $T''(K_A) \cong K_A^\times$. Hence we have the exact sequence of G-modules

$$(2) \qquad 0 \longrightarrow I[G] \otimes K_A^\times \longrightarrow Z[G] \otimes K_A^\times \longrightarrow K_A^\times \longrightarrow 0\,.$$

From the long exact sequence derived from (2), we have

$$0 \longrightarrow (I[G] \otimes K_A^\times)^G \longrightarrow K_A^\times \xrightarrow{N} k_A^\times \longrightarrow H^1(G, I[G] \otimes K_A^\times) \longrightarrow 0\,.$$

We denote $\{x \in K_A^\times \mid N_{K/k}(x) = 1\}$ by $N^{-1}(1)$. Then from the above exact sequence, we have $T'(k_A) \cong (I[G] \otimes K_A^\times)^G \cong N^{-1}(1)$. In the same way as above, we have $T'(k) \cong ([I[G] \otimes K^\times)^G \cong N^{-1}(1) \cap K^\times$, and $T'(U_k) \cong (I[G] \otimes U_K)^G \cong N^{-1}(1) \cap U_K$. Consider a natural homomorphism α: $C(T') \to C(T)$. Then, from the fact that $C(T) \cong K_A^\times / U_K \cdot K^\times$ and $C(T') \cong N^{-1}(1)/(N^{-1}(1) \cap U_K) \cdot (N^{-1}(1) \cap K^\times)$, it is easy to show $\operatorname{Cok} \alpha$ is isomorphic to $K_A^\times / N^{-1}(1) \cdot U_K \cdot K^\times$. It is known that $K_A^\times / N^{-1}(1) \cdot U_K \cdot K^\times$ is isomorphic to the central class group of K/k when K is a normal extension of k. We denote the central class number $[K_A^\times : N^{-1}(1) \cdot U_K \cdot K^\times]$ by $Z(K/k)$. On the other hand, we have

$$\begin{aligned} \operatorname{Ker} \alpha &\cong N^{-1}(1) \cap (U_K \cdot K^\times)/(N^{-1}(1) \cap U_K) \cdot (N^{-1}(1) \cap K^\times) \\ &\overset{f}{\cong} O_k^\times \cap N_{K/k} K^\times / N_{K/k} O_K^\times\,, \end{aligned}$$

where $O_k^\times$ and $O_K^\times$ are the global unit groups of k and K.

The mapping f is defined by putting

$$f(x) = N_{K/k}(u) \,(\operatorname{mod} N_{K/k} O_K^\times) \qquad \text{for any } x = u \cdot y \in N^{-1}(1) \cap (U_K \cdot K^\times)\,,$$

where $u \in U_K$ and $y \in K^\times$.

First, we shall verify that f is well defined. If $x = v \cdot z$ ($v \in U_K$ and $z \in K^\times$), then $v = u \cdot w^{-1}$ and $z = w \cdot y$ ($w \in O_K^\times$). Hence $N_{K/k}(v) = N_{K/k}(u) \cdot N_{K/k}(w^{-1}) = N_{K/k}(u) \,(\operatorname{mod} N_{K/k} O_K^\times)$. Therefore the map f is well defined. Now, it is easy to show that the map f is a homomorphism.

In the next, we shall examine that this homomorphism f is injective. For $x = u \cdot y \in N^{-1}(1) \cap (U_K \cdot K^\times)$ ($u \in U_K, y \in K^\times$), $f(x) = 1$, if and only if $N_{K/k}(u) = N_{K/k}(w)$ for some $w \in O_K^\times$. If we put $x = u \cdot w^{-1} \cdot w \cdot y$, we see $u \cdot w^{-1} \in N^{-1}(1) \cap U_K$ and $w \cdot y \in N^{-1}(1) \cap K^\times$. Hence $x \in (N^{-1}(1) \cap U_K) \cdot (N^{-1}(1) \cap K^\times)$. Therefore f is injective.

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Finally, we shall show that f is surjective. Let $N_{K/k}(z)$ $(z \in K^\times)$ be an element of $O_k^\times \cap N_{K/k}K^\times$. Then, from the fact that $U_k \cap N_{K/k}K_A^\times = N_{K/k}U_K$, there exists an element $u \in U_K$ such that $N_{K/k}(z) = N_{K/k}(u)$. Hence $x = u \cdot z^{-1} \in N^{-1}(1) \cap (U_K \cdot K^\times)$ and $f(x) = N_{K/k}(u) = N_{K/k}(z) \pmod{N_{K/k}O_K^\times}$. Therefore f is surjective.

THEOREM 1. *With the notation as above, the following sequence of finite abelian groups is exact*

$$0 \longrightarrow O_k^\times \cap N_{K/k}K^\times/N_{K/k}O_K^\times \longrightarrow C(T') \xrightarrow{\alpha} C(T) \longrightarrow K_A^\times/N^{-1}(1)\cdot U_K \cdot K^\times \longrightarrow 0,$$

where the last group $K_A^\times/N^{-1}(1)\cdot U_K\cdot K^\times$ is isomorphic to the central class group of K/k.

Let us denote the class number of $R_{K/k}^{(1)}(G_m)$ by $h_{K/k}$. Then, from the above theorem, we have

COROLLARY 1. *The following equation holds for any finite normal extension K/k*

$$h_{K/k}\cdot Z(K/k) = h_K \cdot [O_k^\times \cap N_{K/k}K^\times : N_{K/k}O_K^\times]\,.$$

It is easy to show the following equation

$$Z(K/k) = \frac{h_k \cdot i(K/k)\cdot [U_k : N_{K/k}U_K]}{[K_0 : k]\cdot [O_k^\times : O_k^\times \cap N_{K/k}K^\times]}\,,$$

where K_0 is the maximal abelian extension of k contained in K and $i(K/k)$ is the order of the number knot group $k^\times \cap N_{K/k}K_A^\times/N_{K/k}K^\times$. From Corollary 1, the following equation holds

$$\begin{aligned} E(K/k) = \frac{h_K}{h_k \cdot h_{K/k}} &= \frac{Z(K/k)}{h_k\cdot [O_k^\times \cap N_{K/k}K^\times : N_{K/k}O_K^\times]} \\ &= \frac{i(K/k)\cdot [U_k : N_{K/k}U_K]}{[K_0 : k]\cdot [O_k^\times : N_{K/k}O_K^\times]} = \frac{i(K/k)\cdot [H^0(G, U_K)]}{[K_0 : k]\cdot [H^0(G, O_K^\times)]}\,. \end{aligned}$$

Let us denote the genus number of K/k by $g(K/k)$. Then

$$\begin{aligned} Z(K/k)/g(K/k) &= [N^{-1}(k^\times)\cdot U_K \cdot K^\times : N^{-1}(1)\cdot U_K \cdot K^\times] \\ &= [(k^\times \cap N_{K/k}K_A^\times)\cdot (N_{K/k}U_K)\cdot N_{K/k}K^\times : (N_{K/k}U_K)\cdot (N_{K/k}K^\times)] \\ &= [k^\times \cap N_{K/k}K_A^\times : (k^\times \cap N_{K/k}U_K)\cdot N_{K/k}K^\times] \\ &= \frac{[k^\times \cap N_{K/k}K^\times : N_{K/k}K^\times]}{[k^\times \cap N_{K/k}U_K : N_{K/k}U_K \cap N_{K/k}K^\times]}\,. \end{aligned}$$

From the fact that $k^\times \cap N_{K/k}U_K = O_k^\times \cap N_{K/k}K_A^\times$ and $N_{K/k}U_K \cap N_{K/k}K^\times = O_k^\times \cap N_{K/k}K^\times$, we see

$$Z(K/k) = \frac{g(K/k)\cdot i(K/k)\cdot [O_k^\times \cap N_{K/k}K^\times : N_{K/k}O_K^\times]}{[O_k^\times \cap N_{K/k}K_A^\times : N_{K/k}O_K^\times]}\,.$$

Hence we have another equation

$$E(K/k) = \frac{i(K/k)\cdot g(K/k)}{h_k\cdot [O_k^\times \cap N_{K/k}K_A^\times : N_{K/k}O_K^\times]}\,.$$

§ 4. The formula for $E'(K/k)$

Consider the following exact sequence of algebraic tori defined over k

(3) $$0 \longrightarrow G_m \longrightarrow R_{K/k}(G_m) \longrightarrow R_{K/k}(G_m)/G_m \longrightarrow 0\,.$$

In the following, we shall abbreviate G_m, $R_{K/k}(G_m)$ and $R_{K/k}(G_m)/G_m$ to T', T and T'', respectively. The number $E'(K/k)$ is defined by putting

$$E'(K/k) = \frac{h(T)}{h(T')\cdot h(T'')} = \frac{h_K}{h_k\cdot h'_{K/k}}\,,$$

where $h'_{K/k}$ is the class number of the torus T''. The character modules $\hat{T}'$, $\hat{T}$, $\hat{T}''$ are isomorphic to Z, $Z[G]$, $I[G]$. Hence $\hat{T}'_0 \cong Z$, $\hat{T}_0 \cong Z[G]$, $\hat{T}''_0 \cong J[G]$. Therefore we have $T'(K_A) \cong K_A^\times$, $T(K_A) \cong Z[G] \otimes K_A^\times$, $T''(K_A) \cong J[G] \otimes K_A^\times$. In the same way as § 3, we see $C(T) \cong K_A^\times/U_K\cdot K^\times$ and

$$C(T'') \cong (J[G] \otimes K_A^\times)^G/(J[G] \otimes U_K)^G\cdot(J[G] \otimes K^\times)^G\,.$$

Consider a homomorphism $\beta\colon C(T) \to C(T'')$. By using Hilbert Theorem 90, we get a short exact sequence derived from (3)

(4) $$0 \longrightarrow k_A^\times \longrightarrow K_A^\times \xrightarrow{g} (J[G] \otimes K_A^\times)^G \longrightarrow 0\,.$$

From this exact sequence, the homomorphism β is obviously surjective. In the followng, we shall examine that $\operatorname{Ker}\beta \cong I_K^G/P_K^G$, where I_K is the ideal group of K and P_K is the principal ideal group of K.

Consider the following exact sequences

$$0 \longrightarrow Z \longrightarrow Z[G] \longrightarrow J[G] \longrightarrow 0\,,$$
$$0 \longrightarrow U_K \longrightarrow K_A^\times \longrightarrow I_K \longrightarrow 0\,.$$

From these, we have the following commutative diagram with exact rows and columns

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$$
(5) \qquad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & U_k \arrow[r] \arrow[d] & U_K \arrow[r, "g"] \arrow[d] & (J[G] \otimes U_K)^G \arrow[r] \arrow[d] & H^1(G, U_K) \arrow[r] & 0 \\
0 \arrow[r] & k_A^\times \arrow[r] \arrow[d] & K_A^\times \arrow[r, "g"] \arrow[d] & (J[G] \otimes K_A^\times)^G \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & I_K^G \arrow[r] & I_K \arrow[r, "\bar{g}"] \arrow[d] & (J[G] \otimes I_K)^G \arrow[r] & 0\ , & \\
 & & 0 & & &
\end{tikzcd}
$$

where $\bar{g}$ is surjective because of the fact $H^1(G, I_K) = 0$. From Hilbert Theorem 90, we see $g(K^\times) = (J[G] \otimes K^\times)^G$. Therefore

$$
\begin{aligned}
\operatorname{Ker} \beta &= \{x \in K_A^\times \,|\, g(x) \in (J[G] \otimes U_K)^G \cdot (J[G] \otimes K^\times)^G\} / U_K \cdot K^\times \\
&= \{x \in K_A^\times \,|\, g(x) \in (J[G] \otimes U_K)^G\} \cdot K^\times / U_K \cdot K^\times .
\end{aligned}
$$

From diagram (5), we have $g(x) \in (J[G] \otimes U_K)^G$ if and only if $\bar{g}(\bar{x}) = 0$ in $(J[G] \otimes I_K)^G$. Here $\bar{x}$ is the ideal corresponding to x. Hence $\bar{x} \in I_K^G$, that is $x^{\sigma-1} \in U_K$ for every $\sigma \in G$. Combining these, we have

$$
\begin{aligned}
\operatorname{Ker} \beta &\cong (\{x \in K_A^\times \,|\, x^{\sigma-1} \in U_K \text{ for every } \sigma \in G\} \cdot K^\times / U_K) / (U_K \cdot K^\times / U_K) \\
&\cong I_K^G \cdot P_K / P_K \cong I_K^G / P_K^G ,
\end{aligned}
$$

where $I_K^G \cdot P_K / P_K$ is isomorphic to the group of all the ideal classes represented by ambiguous ideals in K/k. Consider the exact sequences of Galois modules

$$
\begin{aligned}
0 \longrightarrow O_K^\times \longrightarrow K^\times \longrightarrow P_K \longrightarrow 0 , \\
0 \longrightarrow U_K \longrightarrow K_A^\times \longrightarrow I_K \longrightarrow 0 .
\end{aligned}
$$

From long exact sequences derived from these sequences and Hilbert Theorem 90, we have

$$
P_K^G / P_k \cong H^1(G, O_K^\times) , \qquad I_K^G / I_k \cong H^1(G, U_K) .
$$

Hence

$$
\begin{aligned}
[\operatorname{Ker} \beta] = [I_K^G : P_K^G] &= \frac{[I_K^G : P_k]}{[P_K^G : P_k]} = \frac{[I_K^G : I_k] \cdot [I_k : P_k]}{[P_K^G : P_k]} \\
&= \frac{h_k \cdot [H^1(G, U_K)]}{[H^1(G, O_K^\times)]} .
\end{aligned}
$$

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THEOREM 2. *The following sequence is exact*

$$0 \longrightarrow I_K^G/P_K^G \longrightarrow C(T) \xrightarrow{\beta} C(T'') \longrightarrow 0\,,$$

where I_K^G/P_K^G is isomorphic to the group of all the ideal classes represented by ambiguous ideals in K/k.

COROLLARY 2. *Let $a_{K/k}^0$ be the order of the group I_K^G/P_K^G. Then $h_K = h'_{K/k}\cdot a_{K/k}^0$.*

From this corollary, we have the equation

$$E'(K/k) = \frac{h_K}{h_k\cdot h'_{K/k}} = \frac{a_{K/k}^0}{h_k} = \frac{[H^1(G,\,U_K)]}{[H^1(G,\,O_K^\times)]}\,.$$

§ 5. Relation between $E(K/k)$ and $E'(K/k)$

We shall show that $E(K/k) = E'(K/k)$, when K/k is cyclic. From the definitions of $E(K/k)$, $E'(K/k)$, we see $E(K/k) = E'(K/k)$ if and only if $h(R_{K/k}^{(1)}(G_m)) = h(R_{K/k}(G_m)/G_m)$. The character modules of $R_{K/k}^{(1)}(G_m)$ and $R_{K/k}(G_m)/G_m$ are isomorphic to $J[G]$ and $I[G]$. Let σ be a generator of G and n be the order of σ, that is $G = \langle\sigma\rangle$ and $\sigma^n = 1$.

Let $\gamma\colon J[G] = Z[G]/Zs \to I[G]$ be an isomorphism of Z-modules defined by

$$\gamma(\sigma^i \bmod Zs) = \sigma^{i+1} - \sigma^i \qquad (1 \leqq i \leqq n-1)\,.$$

Then, for $1 \leqq i \leqq n-2$,

$$\sigma(\gamma(\sigma^i \bmod Zs)) = \sigma^{i+2} - \sigma^{i+1} = \gamma(\sigma(\sigma^i \bmod Zs))\,.$$

For $i = n-1$,

$$\begin{aligned}\gamma(\sigma(\sigma^{n-1} \bmod Zs)) &= \gamma(1 \bmod Zs)\\ &= \gamma\Big(\Big(\sum_{i=1}^{n-1} - \sigma^i\Big) \bmod Zs\Big)\\ &= -\sum_{i=1}^{n-1}(\sigma^{i+1} - \sigma^i) = -(1-\sigma) = \sigma - 1\\ &= \sigma(\gamma(\sigma^{n-1} \bmod Zs))\,.\end{aligned}$$

Therefore, we see γ is an isomorphism as G-modules, and it is a sufficient condition for the equality

$$h(R_{K/k}^{(1)}(G_m)) = h(R_{K/k}(G_m)/G_m)\,.$$

Remark. When K/k is cyclic, it is well known that Hasse norm principle holds for K/k, that is $i(K/k) = 1$. Therefore, when K/k is cyclic, we have

$$E(K/k) = \frac{Z(K/k)}{h_k \cdot [O_k^\times \cap N_{K/k}K^\times : N_{K/k}O_K^\times]} = \frac{g(K/k)}{h_k \cdot [O_k^\times \cap N_{K/k}K^\times : N_{K/k}O_K^\times]}$$
$$= E'(K/k) = \frac{a_{K/k}^0}{h_k} = \frac{a_{K/k}^0 \cdot [O_k^\times \cap N_{K/k}K^\times : N_{K/k}O_K^\times]}{h_k \cdot [O_k^\times \cap N_{K/k}K^\times : N_{K/k}O_K^\times]}.$$

Let C_K be the class group of K. Then the ambiguous class number $[C_K^G]$ satisfies the following equation

$$\begin{aligned}[C_K^G] &= [I_K^G : P_K^G] \cdot [H^1(G, P_K)] = a_{K/k}^0 \cdot [\mathrm{Ker}\,(H^2(G, O_K^\times) \longrightarrow H^2(G, K^\times))] \\ &= a_{K/k}^0 \cdot [\mathrm{Ker}\,(H^0(G, O_K^\times) \longrightarrow H^0(G, K^\times))] \\ &= a_{K/k}^0 \cdot [O_k^\times \cap N_{K/k}K^\times : N_{K/k}O_K^\times].\end{aligned}$$

Therefore we have proved the well known equation $Z(K/k) = g(K/k) = [C_K^G]$ for the case when K/k is cyclic.

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