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QUOTIENT BOUNDED ELEMENTS IN LOCALLY CONVEX ALGEBRAS

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Abstract

The quotient bounded and the universally bounded elements in a calibrated locally convex algebra are defined and studied. In the case of a generalized B^*-algebra A, they are shown to form respectively b^* and B^*-algebras, both dense in A. An internal spatial characterization of generalized B^*-algebras is obtained. The concepts are illustrated with the help of examples of algebras of measurable functions and of continuous linear operators on a locally convex space.

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Introduction

Giles, Joseph, Koehler and Sims (1975) have discussed numerical ranges of quotient bounded operators on a calibrated locally convex space. The purpose of this paper is to put their work in the general framework of locally convex algebras. Given a calibrated locally convex algebra (A, P), we introduce the sets B_P and Q_P of elements of A called respectively P-universally bounded and P-quotient bounded. A synthesis of the numerical range theories for locally m-convex algebras due to Giles and Koehler (1973) and for locally convex algebras due to Wood (1977) leads us to the main result of the paper, namely, if (A, P) is a calibrated complete hypocontinuous locally convex GB^*-algebra, then Q_P is a b^*-algebra under a natural topology injected continuously as a dense *subalgebra of A; and conversely this characterizes GB^*-algebras. A similar result holds with B_P as a B^*-algebra. This also generalizes a theorem of Giles and

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Koehler (1973, Theorem 6). Finally the ideas of the paper are illustrated with the help of a few examples.

For numerical ranges in Banach algebras, we refer to Bonsall and Duncan (1977), for generalized B^*-algebras, we refer to Allan (1967) and Dixon (1980). For b^*-algebras, we refer to Apostol (1979) and Giles and Koehler (1973).

2. Prelinimaries: quotient bounded elements

(2.1) DEFINITION. Let A be a locally convex algebra (always assumed with identity 1). Let $P(A)$ denote the collection of all calibrations P on A that determine the topology t of A. Let $P = (p_\alpha)$ be in $P(A)$. An element $a \in A$ is called *left P-quotient bounded* if for each α, there exists a real constant $M_{\alpha,a}$ depending on α and a such that $p_\alpha(ax) \leqslant M_{\alpha,a} p_\alpha(x)$ holds for all $x \in A$. Further, it is called *left P-universally bounded* if the $M_{\alpha,a}$ for all α have an upper bound depending only on a (written M_a).

Let B_P be the set of all left P-universally bounded elements, and Q_P be the set of all left P-quotient bounded elements. The following summarizes the basic properties of these sets. We omit the proof which is a straightforward adaptation of Giles and others (1975).

(2.2) PROPOSITION. *Let (A, t) be a locally convex algebra and let $P = (p_\alpha \mid \alpha \in \Delta)$ be in $P(A)$.*

(a) *The set B_P is a subalgebra of Q_P containing* 1, *and Q_P is a subalgebra of A.*

(b) *For each $a \in Q_P$, $\alpha \in \Delta$, let*

$$\begin{aligned} q_\alpha(a) &= \sup\{p_\alpha(ax) \mid p_\alpha(x) \leqslant 1\} \\ &= \inf\{M_{\alpha,a} \mid p_\alpha(ax) \leqslant M_{\alpha,a} p_\alpha(x) \text{ for all } x \in A\}. \end{aligned}$$

For each $a \in B_P$, let

$$p(a) = \sup_\alpha q_\alpha(a) = \inf\{M_a \mid p_\alpha(ax) \leqslant M_a p_\alpha(x) \text{ for all } x \in A \text{ and for all } \alpha\}.$$

(b_1) *(B_p, p) is a unital normed algebra and*

$$p_\alpha(ax) \leqslant p(a) p_\alpha(x) \text{ for all } \alpha \in \Delta,\ a \in B_P,\ x \in A.$$

(b_2) *Each q_α is a submultiplicative seminorm on Q_P such that $q_\alpha(1) = 1$ and $p_\alpha(ax) \leqslant q_\alpha(a) p_\alpha(x)$ $(x \in A)$ for all $\alpha \in \Delta$, $a \in Q_P$. Further, the family (q_α) defines a Hausdorff locally m-convex (lmc) topology t_P on A such that the identity maps $(B_P, p) \to (Q_P, t_P) \to (A, t)$ are continuous.*

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(c) *If A is complete, then each of (B_P, p) and (Q_P, t_P) is also complete.*

(d) *Let A be complete. Let $S_P = \{a \in B_P \mid p(a) \leq 1\}$. Then S_P is in $\mathfrak{B}$, the collection of all $B \subset A$ such that B is absolutely convex, $1 \in B$, $B^2 \subset B$ and B is closed and bounded.*

Much of the work in Giles and others (1975) can be carried over to this setting, the details of which we omit. Also to fix the notations, we recall the following from Wood (1977, Section 3) and Dixon (1970, Section 2) or Allan (1967, Section 2).

(2.3) DEFINITION. Let E be a locally convex space with a calibration $P = (p_\alpha \mid \alpha \in \Delta)$. Let G be the collection of all bounded subsets of E of the form $B = B_{\{M_\alpha\}} = \{x \in E \mid p_\alpha(x) \leq M_\alpha\}$ where $\{M_\alpha \mid \alpha \in \Delta\}$ is any family of positive real numbers. On the dual E' of E, consider the dual calibration $P' = \{q_B \mid B \in G\}$ where $q_B(f) = \sup\{|f(x)| \mid x \in B\}$ for $f \in E'$. Then P' determines the strong topology β on E'. For each $\alpha \in \Delta$, let

$$\pi_\alpha = \{(x, f) \in E \times E' \mid p_\alpha(x) = f(x) = 1, |f(y)| \leq p_\alpha(y) \text{ for all } y \in E\}.$$

Let T: $E \to E$ be continuous and linear. For each $\alpha \in \Delta$, define $W_\alpha^1(T) = \{f(Tx) \mid (x, f) \in \pi_\alpha\}$, $W_P^1(T) = \bigcup_{\alpha \in \Delta} W_\alpha^1(T)$. Let T': $E' \to E'$ be the adjoint of T. Let $W_P^2(T) = W_{P'}^1(T')$. Then $W_P(T) = W_P^1(T) \cup W_P^2(T)$ is called the *spatial numerical range of T (with respect to P).*

Let A be a locally convex algebra with a calibration P. Let $a \in A$. Then the *numerical range* of a is defined to be $W(a) = W_P(T_a)$ where $T_a x = ax$ $(x \in A)$.

(2.4) DEFINITION. Let A be a locally convex algebra with a continuous involution denoted by *. An element $a \in A$ is called *bounded* if for some $\lambda \neq 0$, $\{(\lambda^{-1}a)^n \mid n = 1, 2, \ldots\}$ is bounded. The algebra A is called *symmetric* if for each $x \in A$, $(1 + x^*x)^{-1}$ exists and is bounded. Let $\mathfrak{B}^* = \{B \in \mathfrak{B} \mid B^* = B\}$. Then A is called a locally convex *GB^*-algebra* if

(i) A is symmetric,

(ii) the collection $\mathfrak{B}^*$ has greatest member B_0, called the unit ball of A, and

(iii) the *subalgebra $A(B_0) = \{\lambda x \mid \lambda \in \mathbf{C},\ x \in B_0\}$ is a Banach algebra with the Minkowski functional $\|\cdot\|_{B_0}$ of B_0 in $A(B_0)$ as the norm.

A locally convex algebra A is called *hypocontinuous* if for every bounded set B and every neighborhood U, there is a neighborhood V such that $BV \subset U$ and $VB \subset U$.

By Wood (1977, Theorem 8.15), if A is a complete hypocontinuous locally convex GB^*-algebra, then there exists a calibration P in $P(A)$ such that $A = H(A, P) + iH(A, P)$ where $H(A, P) = \{a \in A \mid W(a) \subset \mathbf{R}\}$, the Hermitian elements of A. We call such a P a GB^*-calibration on A.

3. Main results

(3.1) THEOREM. *Let (A, t) be a complete hypocontinuous locally convex GB^*-algebra with a GB^*-calibration $P = (p_\alpha)$. Then the following hold.*

(1) *Q_P is a *subalgebra of A and is a b^*-algebra with b^*-calibration $\{q_\alpha\}$.*

(2) *B_P is a *subalgebra of Q_P which is a B^*-algebra with the B^*-norm p; and is isometrically isomorphic with $(A(B_0), \|\cdot\|_{B_0})$.*

(3) *$(B, p) \to (Q_P, t_P) \to (A, t)$ are sequentially dense continuous injections.*

For the proof of the theorem, we shall need to compare, for elements of Q_P, the algebra numerical range $V(Q_P, \{q_\alpha\}, a)$ due to Giles and Koehler (1973) and the spatial numerical range due to Wood (Definition 2.3). This is given in the following lemma, the idea of the proof of which is borrowed from Giles and others (1975).

(3.2) LEMMA. *Let A be a complete locally convex algebra and let $P = (p_\alpha)$ be in $P(A)$. Then for each $a \in Q_P$*

$$W_P^1(a) \subset V(Q_P, \{q_\alpha\}, a) \subset \overline{co}\, W_P^1(a).$$

PROOF. For each α, let $N_{(\alpha)} = \{x \in A \mid p_\alpha(x) = 0\}$, $X_{(\alpha)} = A/N_{(\alpha)}$ a linear space with norm $\tilde{p}_\alpha(x_{(\alpha)}) = p_\alpha(x)(x_{(\alpha)} = x + N_{(\alpha)}$ for $x \in A)$. Let $\tilde{X}_{(\alpha)}$ be its Banach space completion. Since $a \in Q_P$, it defines a continuous linear operator T_a^α on $\tilde{X}_{(\alpha)}$ by $T_a^\alpha x_{(\alpha)} = (ax)_{(\alpha)}$ for $x \in A$.

Again for each α, let $N_\alpha = \{a \in Q_P \mid q_\alpha(a) = 0\}$, a two sided ideal of Q_P. Let $((Q_P)\tilde{}_\alpha, \tilde{q}_\alpha)$ be the Banach algebra obtained upon completing the normed algebra $(Q_P)_\alpha = Q_P/N_\alpha$ with the norm $\tilde{q}_\alpha(x_\alpha) = q_\alpha(x)$ where $x_\alpha = x + N_\alpha$ $(x \in Q_P)$. Then

$$V(Q_P, \{q_\alpha\}, a) = \bigcup_\alpha V\big((Q_P)\tilde{}_\alpha, \tilde{q}_\alpha, a_\alpha\big), \tag{1}$$

a union of Banach algebra numerical ranges.

Now consider the Banach algebra $B(\tilde{X}_{(\alpha)})$ of all bounded linear operators on $\tilde{X}_{(\alpha)}$ with the operator norm $\|\cdot\|_\alpha$. The mapping $\varnothing_\alpha$: $(Q_P)\tilde{}_\alpha \to B(\tilde{X}_{(\alpha)})$ defined as $\varnothing(x_\alpha) = T_x^\alpha$ on $(Q_P)_\alpha$ and extended continuously to $(Q_P)\tilde{}_\alpha$ embeds $(Q_P)\tilde{}_\alpha$

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isometrically onto a unital subalgebra of $B(\tilde{X}_{(\alpha)})$. Hence by Bonsall and Duncan (1971, Theorems 2.4 and 9.4) it follows that for each α, $V((Q_P)\tilde{}_\alpha, \tilde{q}_\alpha, a_\alpha) = V(B(\tilde{X}_{(\alpha)}), |\cdot|_\alpha, T_a^\alpha) = \overline{co}\, V(T_a^\alpha)$. (Here $V(T_a^\alpha)$ denotes the spatial numerical range of the Banach space operator T_a^α.)

Further, for each α, let $B_\alpha = \{x \in A \mid p_\alpha(x) \leq 1\}$ and let $A'(\alpha) = \{f \in A' \mid f$ is bounded on $B_\alpha\}$. Then the linear subspace $A'(\alpha)$ of A' is canonically isomorphic to the dual $\tilde{X}'_{(\alpha)}$ of $\tilde{X}_{(\alpha)}$ under the map $f \to f_\alpha$ where f_α on $\tilde{X}_{(\alpha)}$ is defined as $f_\alpha(\tilde{x}_{(\alpha)}) = f(x)$ $(x \in A)$. This with the natural map $x \to x_{(\alpha)}$ embeds the set π_α of Definition (2.3) onto a subset K of the set

$$\pi_{(\alpha)} = \left\{(z, \varphi) \in \tilde{X}_{(\alpha)} \times \tilde{X}'_{(\alpha)} \mid \varphi(z) = \tilde{p}_\alpha(z) = 1,\ |\varphi(y)| \leq \tilde{p}_\alpha(y) \text{ for } y \in \tilde{X}_{(\alpha)}\right\}$$

in such a way that the set $R = \{z \in \tilde{X}_{(\alpha)} \mid (z, \varphi) \in K$ for some $\varphi \in \tilde{X}_{(\alpha)}\}$ is dense in $S(\tilde{X}_{(\alpha)}) = \{z \in \tilde{X}_{(\alpha)} \mid \tilde{p}_\alpha(z) = 1\}$. Indeed, given $z \in S(\tilde{X}_{(\alpha)})$, there is a sequence $\{x^{(n)} \mid n = 1, 2 \ldots\}$ in A such that $x^{(n)}_{(\alpha)} \to z$. Hence $p_\alpha(x^{(n)}) \to \tilde{p}_\alpha(z) = 1$. For each n, let $y^{(n)} = x^{(n)}/p_\alpha(x^{(n)})$ (which can be assumed to be well defined). Then $y^{(n)}_{(\alpha)} \to z$, $\tilde{p}_\alpha(y^{(n)}_{(\alpha)}) = 1$. Also, for each n, the Hahn-Banach theorem gives an $f^{(n)} \in A'(\alpha)$ such that $\| f^{(n)}_\alpha \| = f^{(n)}_\alpha(y^{(n)}_\alpha) = 1$. Then $(y^{(n)}, f^{(n)}) \in \pi_\alpha$, $y^{(n)}_{(\alpha)} \in R$. Thus R is dense in $S(\tilde{X}_{(\alpha)})$. This with Bonsall and Duncan (1971, Theorem 9.3) gives $W^1_\alpha(a) \subset V(T^\alpha_a)$,

$$\overline{co}\, V(T_a^\alpha) = V\left(B\left(\tilde{X}_{(\alpha)}\right), |\cdot|_\alpha, T_a^\alpha\right) = \overline{co}\, W^1_\alpha(a). \tag{2}$$

Now (1) gives $W^1_P(a) \subset V(Q_P, \{q_\alpha\}, a)$, whereas additionally (2) and Bonsall and Duncan (1971, Theorem 9.4) give $V(Q_P, \{q_\alpha\}, a) \subset \overline{co}\ W^1_P(a)$. Hence the lemma follows.

It follows from the above lemma and Wood (1977, Propositions 5.4 and 3.9) that for $a \in Q_P$, $\overline{co}\ V(Q_P, \{q_\alpha\}, a) = \overline{co}\ W^1_P(a)$, $H(B_P, p) = H(A, P) \cap B_P$ and $Q_P \cap H(A, P) \subset H(Q_P, \{q_\alpha\})$. Here $H(B_P, p) = \{x \in B_P \mid V(B_P, p, x) \subset \mathbf{R}\}$ and $H(Q_P, \{q_\alpha\}) = \{x \in Q_P \mid V(Q_P, \{q_\alpha\}, x) \subset \mathbf{R}\}$. We conjecture that $\overline{co}\ W_P(a) = \overline{co}\ V(Q_P, \{q_\alpha\}, a)$ and $Q_P \cap H(A, P) = H(Q_P, \{q_\alpha\})$. In the course of the proof of the theorem, these will be established under an additional hypothesis.

We shall also need the following version of a result due to the author (1980). It is a non-commutative extension of a result due to Allan (1967, Lemma 3.2).

(3.3) Lemma. *Let A be a locally convex GB^*-algebra with unit ball B_0. Let $x \in A$ and for each $n = 1, 2, \ldots$, $x_n = x(1 + \frac{1}{n}x^*x)^{-1}$. Then $x_n \in A(B_0)$ and $x_n \to x$.*

Proof. $x_n = \sqrt{n}\,(x/\sqrt{n})(1 + (x/\sqrt{n})^*(x/\sqrt{n}))^{-1}$ which is easily seen to be in $A(B_0)$ by applying a result in Rudin (1974, Theorem 13.13) via the representation theorem due to Dixon (1970, Theorem 7.11). Also,

$$x - x_n = \frac{1}{\sqrt{n}}(xx^*)\left(\frac{x}{\sqrt{n}}\right)\left(1 + \left(\frac{x}{\sqrt{n}}\right)^*\left(\frac{x}{\sqrt{n}}\right)\right) \in \frac{1}{\sqrt{n}}xx^*B_0.$$

Now by the separate continuity of multiplication in A, for each o-neighbourhood V in A, there is a o-neighbourhood U such that $xx^*U \subset V$. As B_0 is bounded, $\sqrt{r}\,B_0 \subset U$ for sufficiently small $r > 0$. Hence, for sufficiently large n, $x - x_n \in V$ and $x_n \to x$. Hence the lemma follows.

PROOF OF THE THEOREM. Since $A = H(A, P) + iH(A, P)$, a result due to Wood (1977, Theorem 8.15) implies that $H(A, P) = \operatorname{sym} A = \{x \in A \mid x = x^*\}$ and $B_0 = S_P$. Hence the B^*-algebra $(A(B_0), \|\cdot\|_{B_0})$ is isometrically isomorphic to (B_P, p). The Vidav-Palmer theorem for B^*-algebras gives

(3) $$H(B_P, p) = \operatorname{sym} A(B_0) = (\operatorname{sym} A) \cap A(B_0).$$

Now by Giles and Koehler (1973, Theorem 3) and (3) above, $A(B_0) = \{a \in Q_P \mid V(Q_P, \{q_\alpha\}, a) \text{ is bounded}\}$. We aim to prove that $Q_P = H(Q_P, \{q_\alpha\}) + iH(Q_P, \{q_\alpha\})$. Let $a \in Q_P$ and for each n, $a_n = a(1 + \frac{1}{n}a^*a)^{-1}$. By Lemma (3.3), $a_n \in B_P \subset Q_P$ and $a_n \to a$ in the relative topology from that of A. We first show that (a_n) is bounded in the topology t_P on Q_P. For that, consider

$$b_n = a - a_n = a\frac{a^*a}{n}\left(1 + \frac{1}{n}a^*a\right)^{-1} = ak(1 + k)^{-1}$$

where $k = a^*a/n \geqslant 0$ in A. By taking a Gelfand representation (Dixon, 1971, Theorem 4.6), $k(1 + k)^{-1} \in B_0 = S_P$ so $b_n \in aB_0 \subset Q_P$ for all n. Then for each α and for each n,

$$\begin{aligned} q_\alpha(b_n) &= q_\alpha\big(ak(1 + k)^{-1}\big) \\ &= q_\alpha(a)q_\alpha\big(k(1 + k)^{-1}\big) \\ &\leqslant q_\alpha(a)p\big(k(1 + k)^{-1}\big) = q_\alpha(a) \quad \text{as } k(1 + k)^{-1} \in S_P. \end{aligned}$$

It follows that (b_n) and so (a_n) is t_p-bounded, say for each α,

(4) $$q_\alpha(a_n) \leqslant r_\alpha \text{ for all } n, \text{ and some } r_\alpha.$$

Further a can be written as $a = h + ik$ with h, k in $H(A, P)$. Similarly for each n, $a_n = h_n + ik_n$ with h_n, k_n in $H(A, P)$. As $H(A, P) = \operatorname{Sym} A$, $a^* = h - ik$, $a_n^* = h_n - ik_n$. This, on one hand, implies, by the continuity of the involution, that $h_n \to h$, $k_n \to k$; on the other hand, since B_P is a *subalgebra of A, each of h_n and k_n are in $A(B_0) \cap H(A, P) = H(B_P, p)$. Hence h_n and k_n are in $H(Q_P, \{q_\alpha\})$. But then for each α and for each n, in the notations of the proof of Lemma 1,

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$(h_n)_\alpha$ and $(k_n)_\alpha$ are in $H((Q_P)\tilde{}_\alpha, \tilde{q}_\alpha)$, and $(a_n)_\alpha = (h_n)_\alpha + i(k_n)_\alpha$. By the inequality (1) in (Bonsall and Duncan, 1971, Lemma 5.8, page 50)

(5) $q_\alpha(h_n) = \tilde{q}_\alpha((h_n)_\alpha) \leqslant e\tilde{q}_\alpha((h_n)_\alpha + i(k_n)_\alpha) = e\tilde{q}_\alpha((a_n)_\alpha) = eq_\alpha(a_n).$

Let $x \in A$ be arbitrary. Then for each α,

$$\begin{aligned} p_\alpha(hx) &= p_\alpha\Big(\lim_n h_n x\Big) = \lim_n p_\alpha(h_n x) \\ &\leqslant \limsup_n q_\alpha(h_n) p_\alpha(x) \quad \text{as } h_n \in Q_P \\ &\leqslant \limsup_n e q_\alpha(a_n) p_\alpha(x) \quad \text{by (5)} \\ &\leqslant e r_\alpha p_\alpha(x) \end{aligned}$$

which shows that $h \in Q_P$. Similarly $k \in Q_P$. Thus $h, k \in Q_P \cap H(A, P) \subset H(Q_P, \{q_\alpha\})$. It follows that $Q_P = H(Q_P, \{q_\alpha\}) + iH(Q_P, \{q_\alpha\})$. Note that this with Giles and Koehler (1973, Corollary 1) also prove that

$$H(Q_P, \{q_\alpha\}) = Q_P \cap H(A, P). \tag{6}$$

Now the Vidav-Palmer theorem for b^*-algebras (Giles and Koehler, 1973, Theorem 6) implies that Q_P is a b^*-algebra with the involution determined by $\operatorname{sym} Q_P = H(Q_P, \{q_\alpha\})$. It follows from above (6) that Q_P is a *subalgebra of A, the induced involution from A agreeing with the b^*-involution determined by the $\{q_\alpha\}$ Hermitian decomposition. That the q_α satisfy $q_\alpha(x^*x) = q_\alpha(x)^2$ $(x \in Q_P)$ is easily seen by representing Q_P, as in Michael (1952, Theorem 5.1), as the projective limit of $((Q_P)\tilde{}_\alpha, \tilde{a}_\alpha)$ and applying the Vidav-Palmer theorem (Bonsall and Duncan, 1971, Theorem 6.9) to each of these factor algebras.

Finally it follows from Giles and Koehler (1973, Theorem 6) that B_P is dense in (Q_P, t_P); in fact sequentially dense by Apostol (1971, Theorem 2.3); whereas by a result due to the author (1980), B_P, and so Q_P, is sequentially dense in A. This completes the proof of the theorem.

The following theorem is a converse of Theorem (3.1) and is a partial generalization to the GB^*-setting of a b^*-algebra result by Giles and Koehler (1973, Theorem 6 (v) $\Rightarrow$ (i)).

(3.4) THEOREM. *Let A be a complete hypocontinuous locally convex *algebra with a continuous involution. The following are equivalent.*

(a) *A is a GB^*-algebra.*

(b) *There exists a calibration $P = (p_\alpha)$ on A such that*

(i) *Q_P is a *subalgebra of A,*

(ii) *(Q_P, t_P) is a b^*-algebra with (q_α) as a b^*-calibration,*

(iii) $Q_P \to A$ *is a sequentially dense continuous injection.*

(c) *There exists a calibration* $P = (p_\alpha)$ *on* A *such that*

(i) B_P *is a *subalgebra of* A,

(ii) (B_P, p) *is a* B^**-algebra,*

(iii) $B_P \to A$ *is a sequentially dense continuous injection.*

PROOF. That (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is contained in the proof of Theorem (3.1). We show that (c) $\Rightarrow$ (a). By the Vidav-Palmer theorem for B^*-algebras (Bonsall and Duncan, 1971, Theorem 6.7), the involution of the B^*-algebra B_P is given by sym $B_P = H(B_P, p)$. Further by the lower semi-continuity of $x \to W_P(x)$ (Wood, 1977, Proposition 4.4) in A, $H(A, P)$ is closed and so complete. Let $a \in A$. Then there exist h_n, k_n in $H(B_P, p) \subset H(A, P)$ such that $h_n + ik_n = a_n \to a$. Hence $a_n^* = h_n - ik_n$ is Cauchy in A, and so are (h_n) and (k_n). It follows that $h_n \to h \in H(A, P)$, $k_n \to k \in H(A, P)$, $a = h + ik$. The extended Vidav-Palmer theorem of Wood (1977, Theorem 8.15) gives (a).

4. Examples

(4.1) *Quotient bounded operators on locally convex spaces.* Let X be a separated locally convex space with a calibration $\Gamma = (p_\alpha \mid \alpha \in \Delta)$. Let $L(X)$ be the algebra of all continuous linear operators on X. As defined in Giles, Koehler, Joseph and Sims (1975), let $B(X, \Gamma)$ be the subalgebra of all universally bounded operators on X; and let $Q(X, \Gamma)$ be the subalgebra of all quotient bounded operators on X. The seminorms $q_\alpha(T) = \sup\{p_\alpha(Tx) \mid p_\alpha(x) \leqslant 1\}$ define a Hausdorff lmc topology t_Γ on $Q(X, \Gamma)$ and the norm $p_\Gamma(T) = \sup\{p_\alpha(Tx) \mid p_\alpha(x) \leqslant 1 \text{ for all } \alpha\}$ defines a norm topology t_p on $B(X, \Gamma)$. In case (X, Γ) is complete, each of $B(X, \Gamma)$ and $Q(X, \Gamma)$ is also complete. These algebras depend only on the calibration Γ on X and are independent of a specific topology on $L(X)$. We show that they form respectively the quotient bounded and the universally bounded elements of $L(X)$ with respect to any of the standard topologies with the natural calibration.

Let G be a family of bounded subsets of X covering X and satisfying the defining conditions of Treves (1967, Chapter 32). Let $\mathfrak{U}$ be a o-neighbourhood base for X. Then, on $L(X)$, the topology τ_G of uniform convergence on members of G is defined by taking a o-neighbourhood base consisting of sets of the form $U(B, V) = \{T \in L(X) \mid T(B) \subset V\}$ $(B \in G, V \in \mathfrak{U})$. It is also determined by the calibration $P_{\Gamma,G} = \{p_{\alpha,B} \mid \alpha \in \Delta, B \in G\}$ where

$$p_{\alpha,B}(T) = \sup\{p_\alpha(Tx) \mid x \in B\}.$$

The cases of interest are G to be

$$G_1\text{: all finite subsets on } X,$$
$$G_2\text{: all compact convex subsets of } X,$$
$$G_3\text{: all compact subsets of } X,$$
$$G_4\text{: all bounded subsets of } X,$$

yielding on $L(X)$ respectively the topologies τ_σ of pointwise convergence, τ_γ of compact convex convergence, τ_c of compact convergence and τ_β of bounded convergence. Throughout by G we mean any one of these families. Then $(L(X), \tau_G)$ is a locally convex topological algebra. We denote the sets of the quotient bounded elements and the universally bounded elements of the calibrated algebra $(L(X), P_{\Gamma,G})$ by $Q_{P_{\Gamma,G}}$ and $B_{P_{\Gamma,G}}$ respectively. As in Section 1, the natural lmc topology $t_{P_{\Gamma,G}}$ on $Q_{P_{\Gamma,G}}$ is determined by the seminorms

$$q_{\alpha,B}(T) = \sup\{p_{\alpha,B}(TS) \mid p_{\alpha,B}(S) \leqslant 1\},$$

whereas the natural norm on $B_{P_{\Gamma,G}}$ is $p_{P_{\Gamma,G}}(T) = \sup_\alpha q_{\alpha,B}(T)$.

ASSERTIONS. (a) $Q_{P_{\Gamma,G}}$ *is topologically isomorphic to* $(Q(X,\Gamma), t_\Gamma)$.
(b) $B_{P_{\Gamma,G}}$ *is isometrically isomorphic to* $(B(X,\Gamma), p_\Gamma)$.

PROOF. Let $T \in Q(X,\Gamma)$. Then for each α and for each B, $p_{\alpha,B}(TS) \leqslant q_\alpha(T)p_{\alpha,B}(S)$ $(S \in L(X))$ and so $T \in Q_{P_{\Gamma,G}}$ with $q_{\alpha,B}(T) \leqslant q_\alpha(T)$. Hence

$$t_{P_{\Gamma,G}} \subseteq t. \tag{7}$$

Conversely let $T \in Q_{P_{\Gamma,G}}$. Then $p_{\alpha,B}(TS) \leqslant q_{\alpha,B}(T)p_{\alpha,B}(S)$ for all $S \in L(X)$, $\alpha \in \Delta$. Therefore $\sup\{p_\alpha(TSx) \mid x \in B\} \leqslant q_{\alpha,B}(T)\sup\{p_\alpha(Sx) \mid x \in B\}$. Let $x_0 \in X$ be arbitrarily fixed. Take $B = \{x_0\} \in G$. Then

$$p_\alpha(TSx_0) \leqslant q_{\alpha,\{x_0\}}(T)p_\alpha(Sx_0) \qquad (S \in L(X)). \tag{8}$$

Let $y \in X$. By the Hahn Banach theorem, we can choose $f \in X'$, the dual of X, such that $f(x_0) = 1$. Define $f \otimes y \in L(X)$ by $(f \otimes y)(x) = f(x)y$ $(x \in X)$. Then $(f \otimes y)(x_0) = y$. Thus given $y \in X$, we can choose $S_y \in L(X)$ such that $S_y(x_0) = y$. This in (7) gives $p_\alpha(Ty) \leqslant q_{\alpha,\{x_0\}}(T)p_\alpha(y)$ $(y \in X)$. Thus $T \in Q(X,\Gamma)$ and

$$q_\alpha(T) \leqslant q_{\alpha,\{x\}}(T) \text{ for all } x \in X. \tag{9}$$

Hence using (7), $Q(X,\Gamma) = Q_{P_{\Gamma,G}} = A$ (say) and $t_{P_{\Gamma,G}} \subseteq t_\Gamma \subseteq t_{P_{\Gamma,G}}$ with $q_\alpha(T) = q_{\alpha,B}(T)$ $(T \in A)$ with $B = \{x\}$. So $t_\Gamma = t_{P_{\Gamma,G_1}}$. It only remains to show that $t_{P_{\Gamma,G}} \supseteq t_\Gamma$ for $G = G_2, G_3, G_4$; and this follows from $t_{P_{\Gamma,G_1}} \subseteq t_{P_{\Gamma,G}}$. Indeed, the defining seminorms for $t_{P_{\Gamma,G}}$ are

$$q_{\alpha,B}(T) = \sup\{p_{\alpha,B}(TS) \mid p_{\alpha,B}(S) \leqslant 1, S \in L(X)\} \qquad (T \in A).$$

Hence, since given $F \in G_1$, by taking closed convex hull, if necessary, we can choose $B(F) \in G_i$ $(i = 2, 3, 4)$ such that $\{q_{\alpha,F} \mid \alpha \in \Delta,\ F \subset X \text{ finite}\} \subset \{q_{\alpha,B} \mid \alpha \in \Delta,\ B \in G_i\}$. Thus $t_{P_{\Gamma, G_1}} \subseteq t_{P_{\Gamma, G_i}}$ $(i = 2, 3, 4)$.

The proof of (b) is similar.

(4.2) *Arens' algebra $L^\omega(X)$ of a finite measure space.* The b^*-algebra Q_P of quotient bounded elements of a complete hypocontinuous locally convex GB^*-algebra A may be trivial. Here is an example of a GB^*-algebra A with the property that A does not contain any b^*-subalgebra properly containing the B^*-algebra $A(B_0)$.

Let (X, Σ, μ) be a finite measure space. Let $L^\omega(X) = \bigcap_{1 \leq p < \infty} L^p(X)$. As in Arens (1946), $L^\omega(X)$ is a *algebra under pointwise operations containing $L^\infty(X)$. Let τ^ω be the locally convex topology on $L^\omega(X)$ defined by the family $P = \{\|\cdot\|_p \mid 1 \leq p < \infty\}$ of norms $f \to \|f\|_p = (\int_X |f|^p \, d\mu)^{1/p}$. As the measure space is finite, it is also determined by $P = \{\|\cdot\|_n \mid n = 1, 2, \ldots\}$ or equivalently by the metric

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_n}{1 + \|f - g\|_n}.$$

Then as in Allan (1967, Example 4), $L^\omega(X)$ is a complete metrizable (and hence hypocontinuous) locally convex GB^*-algebra with unit all $B_0 = \{f \in L^\infty(X) \mid \|f\|_\infty \leq 1\}$ and the underlying B^*-algebra $(L^\infty(X), \|\cdot\|_\infty)$.

ASSERTIONS. (a) *P is a GB^*-calibration on $L^\omega(X)$ and $B_P = Q_P = L^\infty(X)$.*

(b) *There does not exist a *subalgebra Q of A admitting a b^*-topology such that $L^\infty(X) \subsetneq Q \subsetneq L^\omega(X)$.*

PROOF. (a) We show that $Q_P \subset L^\infty(X)$, since it is clear that $L^\infty(X) \subset B_P$. Let $f \in Q_P$. Then for all $g \in L^\omega(X)$, $\|fg\|_n < M_n \|g\|_n$ $(n \in N)$ for some constants M_n. In particular, $|\int_X fg \, d\mu| \leq \int_X |fg| \, d\mu \leq M_1 \int_X |g| \, d\mu$ which shows that $g \to \int_M f \, d\mu$ is a $\|\cdot\|_1$ continuous linear functional on $L^\omega(X)$; and hence, since $L^\omega(X)$ is dense in $L^1(X)$, it extends uniquely to an element $\varphi \in (L^1)' = L^\infty$ given by $\varphi(g) = \int_X hg \, d\mu$ $(g \in L^\omega(X))$ for a unique $h \in L^\infty(X)$. Hence $f = h$ a.e. and so $Q_P \subset L^\infty(X)$. Thus $Q_P = B_P = L^\infty(X)$. Further, as P is countable, (Q_P, t_P) is a metrizable b^*-algebra. The open mapping theorem shows that Q_P and B_P are topologically identical, which in turn are identical to $(L^\infty(X), \|\cdot\|_\infty)$ by the uniqueness of norm on a B^*-algebra.

(b) Let M be the carrier space of $L^\infty(X)$. With the Gelfand topology, it is a hyperstonian compact Hausdorff space. Let $\varphi = L^\infty(X) \to C(M)$ be the Gelfand representation $\varphi(f) = \hat{f}$: $M \to \mathbf{C}$ by $\hat{f}(\varphi) = \varphi(f)$ $(\varphi \in M)$ mapping $L^\infty(X)$

isomorphically onto $C(M)$, the B^*-algebra of all continuous complex valued functions on M. Further, the Riesz representation theorem gives a positive finite regular Borel measure $\hat{\mu}$ on M such that $\int_X f\,d\mu = \int_M \varphi(f)\,d\hat{\mu}$ ($f \in L^\infty(X)$) with $\operatorname{supp}\hat{\mu} = M$. This with Lusin's Theorem gives $C(M) = L^\infty(M)$. Then by Dixon (1967, Theorem 4.6) φ extends to a *isomorphism φ' of $L^\omega(X)$ onto a *algebra of functions (Dixon (1957), Definition 4.5) on M.

Now suppose that there is a *subalgebra Q of $L^\omega(X)$ carrying a b^*-topology τ such that $L^\infty(X) \subsetneq Q \subsetneq L^\omega(X)$, each a continuous injection. Let, as in Apostol (1971, Section 3), Z be the carrier space of Q. It is a real-compact T_2 space with its Stone-Čech compactification $\beta Z = M$, such that φ', in a suitable sense, establishes the *isomorphisms $L^\infty(X) \approx C_b(Z) \approx C(M)$, $Q \approx C(Z)$ and $L^\omega(X)$ to a *algebra of extended complex valued functions on Z containing $C(Z)$. Let $\varphi(g) = \int_X g\,d\mu$ ($g \in L^\omega(X)$). Let $F(\varphi'(g)) = \varphi(g)$ ($g \in L^\omega(X)$). Under restriction, F defines a positive functional on $C(Z)$ which, by Feldman and Porter (1975, Section 2, Theorem B), is given by $F(\varphi'(g)) = \int_Z \varphi'(g)\,d\beta$ ($g \in Q$) for some positive Borel measure β on Z with compact support $\operatorname{supp}\beta = K$. Hence for each $g \in L^\infty(X)$,

$$\int_X g\,d\mu = \int_Z \varphi(g)\,d\beta = \int_K \varphi(g)\,d\beta.$$

But $\int_X g\,d\mu = \int_M \varphi'(g)\,d\hat{\mu}$. Hence $\hat{\mu} = \beta$. That $\operatorname{supp}\beta = K \neq M = \operatorname{supp}\hat{\mu}$ provides the desired contradiction.

(4.3) *Arens' algebra* (*σ-finite measure space*). Let (X, Σ, μ) be a σ-finite measure space. Let $X = \bigcup_1^\infty X_n$ with, for each n, $X_n \in \Sigma$, $X_n \subset X_{n+1}$ and $\mu(X_n) < \infty$. For $1 \leqslant p < \infty$, let $L^p_{\mathrm{loc}}(X)$ be the vector space of all those complex valued measurable functions (modulo equality a.e.) which are locally L^p in the sense that for each $F \in \Sigma$ with $\mu(F) < \infty$, $\int_F |f|^p\,d\mu < \infty$. Let L^∞_{loc} be the algebra under pointwise operations of all locally L^∞-functions (defined similarly) on X. Let $L^\omega_{\mathrm{loc}}(X) = \bigcap_{1 \leqslant p < \infty} L^p_{\mathrm{loc}}(X)$. Then $L^\omega_{\mathrm{loc}}(X)$ is a *algebra under pointwise operations. On the vector space $L^p_{\mathrm{loc}}(X)$, a complete metrizable locally convex topology τ^p_{loc} is defined by the collection $P_p = \{\|\cdot\|_{k,p} \mid k = 1, 2, \ldots\}$ of seminorms, where $\|f\|_{k,p} = (\int_{X_k} |f|^p\,d\mu)^{1/p}$. This induces a locally convex metrizable *algebra topology $\tau^\omega_{\mathrm{loc}}$ on L^ω_{loc} which is defined by the calibration $F = \{\|\cdot\|_{k,p} \mid k, p \in \mathbf{N}\}$. The topology $\tau^\infty_{\mathrm{loc}}$ on L^∞_{loc} is defined by the seminorms $\|f\|_{k,\infty} = \operatorname{ess.\,sup}_{t \in X_k} |f(t)|$. Then the following assertions are easily verified.

(1) $(L^\omega_{\mathrm{loc}}(X), \tau^\omega_{\mathrm{loc}})$ is a GB^*-algebra with GB^*-calibration P.

(2) Q_P is topologically *isomorphic to $(L^\infty_{\mathrm{loc}}(X), \tau^\infty_{\mathrm{loc}})$.

(3) B_P is isometrically *isomorphic to $(L^\infty(X), \|\cdot\|_\infty)$.

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