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INVARIANT POLYNOMIALS OF WEYL GROUPS AND APPLICATIONS TO THE CENTRES OF UNIVERSAL ENVELOPING ALGEBRAS

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Introduction. An element in the centre of the universal enveloping algebra of a semisimple Lie algebra was first constructed by Casimir by means of the Killing form. By Schur's lemma, in an irreducible finite-dimensional representation elements in the centre are represented by diagonal matrices of all whose eigenvalues are equal. In section 2 of this paper, we show the existence of a complete set of generators whose eigenvalues in an irreducible representation are closely related to polynomial invariants of the Weyl group W of the Lie algebra (Theorem 1). Such a set of generators and their eigenvalues have found many applications in physics [**5**, p. 1314]. In section 3, we obtain the polynomial invariants of the Weyl groups of the classical Lie algebras as well as those of the five exceptional ones. These invariant polynomials can be used directly to compute eigenvalues of a complete set of generators of $\mathfrak{Z}$. In the last section, we point out a possible way of the explicit constructions of these generators.

1. Preliminaries. All the basic results we need can be found in [**2**, part III]. We use the same notations from that part of [**2**] and will not define them. The dimension of $\mathfrak{h}$ is assumed to be n instead of l as in [**2**].

2. A set of generators of $\mathfrak{Z}$ and their eigenvalues in π_Λ. In this section, we will study the eigenvalues of a set of generators of $\mathfrak{Z}$.

We will first consider some results in [**2**]. The function χ_x [**2**, p. 71] was extended to $\mathfrak{B}$ [**2**, p. 72]. It was also shown [**2**, p. 72] that if β is the isomorphism from $\mathfrak{H}$ to the algebra of polynomials in $x_1, \ldots, x_n$ defined by $\beta(H_i) = x_i$, then every $Z \in \mathfrak{Z}$ has a unique decomposition

$$(2.1)\quad Z = P + H, \quad P \in \mathfrak{P}, H \in \mathfrak{H};$$

furthermore,

$$(2.2)\quad H = \beta^{-1}(\chi_x(Z))$$

and β is an isomorphism of $\mathfrak{Z}$ into the algebra $C[x]$ of polynomials in $x_1, \ldots, x_n$.

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Now if π_Λ is the irreducible representation with highest weight Λ and v_Λ is a vector of weight Λ, then for any $Z \in \mathfrak{Z}$, we have, by (2.1),

(2.3) $\pi_\Lambda(Z)v_\Lambda = \pi_\Lambda(P)v_\Lambda + \pi_\Lambda(H)v_\Lambda = \pi_\Lambda(H)v_\Lambda.$

But $\pi_\Lambda(H_i)v_\Lambda = \Lambda(H_i)v_\Lambda$, $1 \leqq i \leqq n$ and $H = \beta^{-1}(\chi_x(Z))$, thus (2.3) become

(2.4) $\pi_\Lambda(Z)v_\Lambda = \pi_\Lambda(H)v_\Lambda = (\chi_x(Z)|_{xi=\Lambda(H_i)})v_\Lambda$

and the eigenvalue of Z is $\chi_x(Z)|_{xi=\Lambda(H_i)}$.

Example. We apply the above ideas to the Casimir operator. Let $H^1, \ldots, H^n$ be dual basis of $H_1, \ldots, H_n$ with respect to the Killing form and Σ be the set of all roots of $\mathfrak{h}$. If for any $\alpha \in \Sigma$, E_α and $E_{-\alpha}$ satisfy $(E_\alpha, E_{-\alpha}) = 1$ and $[E_\alpha, E_{-\alpha}] = H_\alpha'$, then the Casimir operator Z_c is defined by

$$Z_c = \sum_{i=1}^{n} H_i H^i + \sum_{\alpha\in\Sigma} E_{-\alpha}E_\alpha.$$

Using $[E_\alpha, E_{-\alpha}] = H_\alpha'$, we obtain a decomposition of Z_c (2.1) as

$$Z_c = \left(2 \sum_{\alpha>0} E_{-\alpha}E_\alpha\right) + \left(\sum_{i=1}^{n} H_i H^i + \sum_{\alpha>0} H_\alpha'\right).$$

Thus in π_Λ we have

$$\pi_\Lambda(Z_c)v_\Lambda = \pi_\Lambda \left(\sum_{i=1}^{n} H_i H^i + \sum_{\alpha>0} H_\alpha'\right) v_\Lambda.$$

Now $\sum_1^n \Lambda(H_i)\Lambda(H^i) = (\Lambda, \Lambda)$ and $\sum_{\alpha>0} H_\alpha' = 2\rho$, where $\rho = \frac{1}{2}(\sum_{\alpha>0} H'_\alpha)$ [**1**, p. 246-7]. Thus the eigenvalue of Z_c is $(\Lambda, \Lambda + 2\rho)$.

If $s_{\alpha_i} \in W$ is a fundamental reflection, then $s_{\alpha_i}^{-1} = s_{\alpha_i}$ and from definition [**2**, p. 69], we have

$$\lambda(s_{\alpha_i}H) = s_{\alpha_i}\lambda(H) = \left(\lambda - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i\right)H,$$

thus,

(2.5) $s_{\alpha_i}H = H - \alpha_i(H)H_i.$

From (2.5) it follows that

(2.6) $s_{\alpha_i}x_j = x(s_{\alpha_i}H_j) = x_j - \alpha_i(H_j)x_i,$

where

(2.7) $\alpha_i(H_j) = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}.$

Definition. A polynomial function $f(x_1, \ldots, x_n)$ is said to be invariant under W if it is invariant under the transformation (2.6). We also write $sf(x) = f(x)$, for all $s \in W$.

It is known that each Weyl group W has n basic homogeneous invariant polynomials [**3**].

THEOREM 1. *If* $f_1(x), \ldots, f_n(x)$ *is a basic set of homogeneous invariant polynomials of* W, *then there exists* $Z_1, \ldots, Z_n$ *of* $\mathfrak{Z}$ *such that*

(i) $\chi_x(Z_i) = f_i(x'), 1 \leqq i \leqq n$;

(ii) $Z_1, \ldots, Z_n$ *is a complete set of generators of* $\mathfrak{Z}$.

Proof. Let σ be the unique automorphism of $C[x]$ which maps $x_i \rightarrow x_i + \rho(H_i)$ (i $= 1, \ldots, n$). Let I be the W invariant polynomials of $C[x]$. By [**2**, Lemma 36 p. 72], the function χ_x is an injective homomorphism from $\mathfrak{Z}$ into $C[x]$. It is clear from definition that $\sigma^{-1}\chi_x$ maps $\mathfrak{Z}$ into I. According to [**2**, Lemma 38 and Corollary to Lemma 39], $\sigma^{-1}\chi_x$ maps $\mathfrak{Z}$ onto I; thus $\mathfrak{Z} \simeq I$ (under $\sigma^{-1}\chi_x$).

If $f_1(x), \ldots, f_n(x)$ is a basic set of homogeneous polynomials in I, then the elements $Z_i = \chi_x^{-1}\sigma(f_i(x))(i = 1, \ldots, n)$ clearly satisfies (i) and (ii).

3. Invariant polynomials. In this section, we study polynomials that are invariant under W. Basic sets of homogeneous invariant polynomials will be constructed for the classical Lie algebras as well as the five exceptional ones. A list of the degrees of such a basic set for these algebras can be found in [**4**, p. 780].

If (a_{ij}) denotes the Cartan matrix [**1**, p. 121] of $\mathfrak{L}$ relative to $\mathfrak{h}$, then it follows from (2.6) and (2.7) that the transformation s_{α_i} on $(x_1, \ldots, x_n)$ corresponds to the ith column of (a_{ij}), i.e.,

$$s_{\alpha_i}x_j = x_j - a_{ji}x_i. \tag{3.1}$$

(i) A_n: The Cartan matrix is

$$\begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & -1 \\ & & & & & -1 & 2 \end{bmatrix}.$$

Let s_{α_i} $(1 \leqq i \leqq n)$ be the transformation on $(x_1, \ldots, x_n)$ corresponding to the ith column. Then

$$(x_1, \ldots, x_n) \xrightarrow{s_{\alpha_1}} (-x_1, x_1 + x_2, x_3, \ldots, x_n)$$

$$(x_1, x_2, \ldots, x_n) \xrightarrow{s_{\alpha_2}} (x_1 + x_2, -x_2, x_3 + x_2, x_4, \ldots, x_n)$$

$$\vdots$$

$$(x_1, x_2, \ldots, x_n) \xrightarrow{s_{\alpha_{n-1}}} (x_1, \ldots, x_{n-2} + x_{n-1}, -x_{n-1}, x_n + x_{n-1})$$

$$(x_1, x_2, \ldots, x_n) \xrightarrow{s_{\alpha_n}} (x_1, \ldots, x_{n-2}, x_{n-1} + x_n, -x_n).$$

If

$$\begin{aligned}
y_1 &= nx_1 + (n-1)x_2 + (n-2)x_3 + \ldots + x_n, \\
y_2 &= -x_1 + (n-1)x_2 + (n-2)x_3 + \ldots + x_n, \\
y_3 &= -x_1 - 2x_2 + (n-2)x_3 + \ldots + x_n, \\
y_4 &= -x_1 - 2x_2 - 3x_3 + (n-3)x_4 + \ldots + x_n, \\
&\cdot \\
&\cdot \\
&\cdot \\
&\cdot \\
y_n &= -x_1 - 2x_2 - 3x_3 - \ldots - (n-1)x_{n-1} + x_n \\
y_{n+1} &= -y_1 - \ldots - y_n,
\end{aligned}$$

then s_{α_i} becomes the transposition $(y_i y_{i+1})$ and W becomes the permutation group on $y_1, \ldots, y_{n+1}$. To show that $y_1, \ldots, y_n$ are independent, we notice that the coefficients of x_n in all y'_is $(1 \leqq i \leqq n)$ are 1. Thus by multiplying the last column of the Jacobian $\partial(y_1, \ldots, y_n)/\partial(x_1, \ldots, x_n)$ by i and adding it to the ith column, the Jacobian becomes an upper diagonal matrix whose determinant is $(n+1)^n$. Thus, as basic invariant polynomials, we may choose the symmetric functions (in $y_1, \ldots, y_{n+1}$)

$$\psi_1 = \sum_{i<j} y_i y_j, \quad \psi_2 = \sum_{i<j<k} y_i y_j y_k, \quad \ldots, \quad \psi_n = y_1 \ldots y_{n+1}.$$

(ii) B_n $(n \geqq 2)$: The Cartan matrix is

$$\begin{bmatrix}
2 & -1 & & & & & \\
-1 & 2 & -1 & & & & \\
 & -1 & \cdot & & & & \\
 & & & \cdot & & & \\
 & & & & \cdot & -1 & \\
 & & & & -1 & 2 & -1 \\
 & & & & & -2 & 2
\end{bmatrix}.$$

Let

$$\begin{aligned}
y_1 &= 2x_1 + 2x_2 + \ldots + 2x_{n-1} + x_n, \\
y_2 &= 2x_2 + 2x_3 + \ldots + 2x_n + x_n, \\
&\cdot \\
&\cdot \\
&\cdot \\
&\cdot \\
y_{n-1} &= 2x_{n-1} + x_n, \\
y_n &= x_n.
\end{aligned}$$

Then it can be verified that the transformations $s_{\alpha_i}(1 \leqq i \leqq n-1)$ corresponding to the first $n-1$ columns are $(y_i y_{i+1})$ and that s_{α_n} changes the sign

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of y_n. The Jacobian $\partial(y_1, \ldots, y_n)/\partial(x_1, \ldots, x_n)$ is easily seen to be 2^{n-1}. Thus as basic invariant polynomials, we may choose

$$\psi_1 = \sum_i y_i^2, \quad \psi_2 = \sum_{i<j} y_i^2 y_j^2, \quad \ldots, \quad \psi_n = y_1^2 \ldots y_n^2.$$

(iii) $C_n (n \geqq 3)$: The Cartan matrix is

$$\begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & -1 & \\ & & & & -1 & 2 & -2 \\ & & & & & -1 & 2 \end{bmatrix}.$$

Put

$$\begin{aligned} y_1 &= x_1 + \ldots + x_n \\ y_2 &= x_2 + \ldots + x_n \\ &\cdot \\ &\cdot \\ &\cdot \\ y_n &= x_n. \end{aligned}$$

Then it can be proved that we have a similar case as in B_n. As basic invariants, we may choose

$$\psi_1 = \sum_i y_i^2, \quad \psi_2 = \sum_{i<j} y_i^2 y_j^2, \quad \ldots, \quad \psi_n = y_1^2 \ldots y_n^2.$$

(iv) $D_n (n \geqq 4)$: The Cartan matrix is

$$\begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & \cdot & -1 & & \\ & & & & -1 & 2 & -1 & -1 \\ & & & & & -1 & 2 & 0 \\ & & & & & -1 & 0 & 2 \end{bmatrix}.$$

Let

$$\begin{aligned} y_1 &= 2x_1 + \ldots + 2x_{n-2} + x_{n-1} + x_n, \\ y_2 &= 2x_2 + \ldots + 2x_{n-2} + x_{n-1} + x_n, \\ &\cdot \\ &\cdot \\ &\cdot \\ y_{n-1} &= x_{n-1} + x_n, \\ y_n &= -x_{n-1} + x_n. \end{aligned}$$

It can be verified that the first $n-1$ columns correspond to $(y_i y_{i+1})$ and the last column transposes y_{n-1} and y_n and changes their signs. Also, the Jacobian $\partial(y_1, \ldots, y_n)/\partial(x_1, \ldots, x_n)$ is not zero. Thus as basic invariant polynomials, we may choose

$$\psi_1 = \sum_i y_i^2,\ \psi_2 = \sum_{i<j} y_i^2 y_j^2, \ldots,$$

$$\psi_{n-1} = \sum_{i_1<\ldots<i_{n-1}} y_{i_1}^2 \cdot \ldots \cdot y_{i_{n-1}}^2,\ \psi_n = y_1 \ldots y_n.$$

(v) G_2: The Cartan matrix is

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

Let

$$y_1 = 3x_1 + x_2,\ y_2 = x_2,\ y_3 = -y_1 - y_2.$$

Then the transformation corresponding to the first and second columns are

$$(y_1, y_2, y_3) \to (y_2, y_1, y_3)$$
$$(y_1, y_2, y_3) \to (-y_3, -y_2, -y_1)$$

respectively. Thus a basic set of invariants may be chosen as

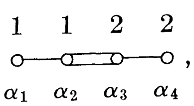
$$\psi_1 = \sum_{i<j} y_i y_j, \quad \psi_2 = (y_1 y_2 y_3)^2.$$

The independence of ψ_1 and ψ_2 can be shown by computing the Jacobian $\partial(\psi_1, \psi_2)/\partial(y_1, y_2)$.

(vi) F_4: A fundamental root system of F_4 can be chosen as

$$\{\tfrac{1}{2}(e_1 - e_2 - e_3 - e_4),\ e_4,\ e_3 - e_4,\ e_2 - e_3\}.$$

The Dynkin diagram is



and the Cartan matrix is

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -2 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}.$$

If $u = \frac{1}{2}(r_1 - r_2 - r_3 - r_4)$, then in R^4 an arbitrary vector (r_1, r_2, r_3, r_4) is transformed by $s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}, s_{\alpha_4}$ to $(r_1 - u, r_2 + u, r_3 + u, r_4 + u)$, $(r_1, r_2, r_3, -r_4)$, (r_1, r_2, r_4, r_3), (r_1, r_3, r_2, r_4) respectively. Thus the 24 linear forms $\pm r_i \pm r_j$ $(i \neq j)$ are permuted under W.

Now let

$$y_1 = 2x_1 + 3x_2 + 4x_3 + 2x_4,$$
$$y_2 = x_2 + 2x_3 + 2x_4,$$
$$y_3 = x_2 + 2x_3,$$
$$y_4 = x_2.$$

Then $\frac{1}{2}(y_1 - y_2 - y_3 - y_4) = x_1$. It can be verified that under the transformations generated by the four columns of the Cartan matrix, the images of (y_1, y_2, y_3, y_4) are $(y_1 - x_1, y_2 + x_1, y_3 + x_1, y_4 + x_1)$, $(y_1, y_2, y_3, -y_4)$, (y_1, y_2, y_4, y_3) and (y_1, y_3, y_2, y_4). For a basic set of invariant polynomials, we may choose the functions

$$\psi_m = \sum_{i<j} \{(y_i + y_j)^m + (y_i - y_j)^m\}, \quad m = 2, 6, 8, 12.$$

To show the independence of $\psi_2, \psi_6, \psi_8, \psi_{12}$, we may either expand the Jacobian and show that it is not identically zero or show that it is not zero for particular values of y_1, y_2, y_3 and y_4 (e.g., $y_1 = 1/\sqrt{2}$, $y_2 = i/\sqrt{2}$, $y_3 = -1/\sqrt{2}$, $y_4 = -i/\sqrt{2}$).

(vii): E_6: The Cartan matrix is

$$\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & -1 \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & \\ & & -1 & & & 2 \end{bmatrix}.$$

This case was studied in [**4**, p. 776-9]. Let

$$y_1 = 5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5$$
$$y_2 = -x_1 + 4x_2 + 3x_3 + 2x_4 + x_5$$
$$y_3 = -x_1 - 2x_2 + 3x_3 + 2x_4 + x_5$$
$$y_4 = -x_1 - 2x_2 - 3x_3 + 2x_4 + x_5$$
$$y_5 = -x_1 - 2x_2 - 3x_3 - 4x_4 + x_5$$
$$y_6 = -y_1 - \dots - y_5$$

and

$$y = -3(x_1 + 2x_2 + 3x_3 + 2x_4 + x_5 + 2x_6).$$

Then the first five columns are the transpositions $(y_1y_2), \dots, (y_5y_6)$ and the last one is

$$(y_1, \dots, y_6; y) \to (y_1 - u, y_2 - u, y_3 - u, y_4 + u, y_5 + u, y_6 + u, y - u)$$

where $u = \frac{1}{2}(y_1 + y_2 + y_3 + y)$. Let

$$a_i = y_i + y, b_i = y_i - y \quad (i = 1, 2, \ldots, 6),$$
$$c_{ij} = -y_i - y_j \quad (i < j).$$

As a basic set of invariant polynomials, we may choose

$$\psi_m = \sum_i a_i^m + \sum_i b_i^m + \sum_{i<j} c_{ij}^m, \quad m = 2, 5, 6, 8, 9, 12.$$

By the method used in (i), it is quite easy to show that $y_1, \ldots, y_5$ are independent. It is then clear that $y_1, \ldots, y_5, y$ are independent. The independence of ψ_n's then follows from [**4**].

(viii) E_7: In the seven dimensional space $\{(r_1, \ldots, r_8) | r_1 + \ldots + r_8 = 0\}$, a fundamental root system can be chosen as

$$\{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_6 = e_6 - e_7, \alpha_7 = \frac{1}{2}e^{(8)} - e_1 - e_2 - e_3 - e_4\}.$$

Where $e^{(8)} = \sum_{i=1}^{8} e_i$, Reflections defined by $\alpha_1, \alpha_2, \ldots, \alpha_6$ are transpositions $(r_1r_2), \ldots, (r_6r_7)$ and the one defined by α_7 is

$$(r_1, \ldots, r_8) \rightarrow (r_1, \ldots, r_8) - \frac{1}{2}(-r_1 - r_2 - r_3 - r_4) \times (-1, -1, -1, -1, 1, 1, 1, 1).$$

The 56 linear forms $r_i + r_j$ and $-r_i - r_j (i \neq j)$ are permuted under W. The Cartan matrix is

$$\begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & -1 \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & \\ & & & -1 & & & 2 \end{bmatrix}.$$

Let

$$y_1 = 3x_1 + 2x_2 + x_3 - x_7$$
$$y_2 = -x_1 + 2x_2 + x_3 - x_7$$
$$y_3 = -x_1 - 2x_2 + x_3 - x_7$$
$$y_4 = -x_1 - 2x_2 - 3x_3 - x_7$$
$$y_5 = -x_1 - 2x_2 - 3x_3 - 4x_4 - x_7$$
$$y_6 = -x_1 - 2x_2 - 3x_3 - 4x_4 - 4x_5 - x_7$$
$$y_7 = -x_1 - 2x_2 - 3x_3 - 4x_4 - 4x_5 - 4x_6 - x_7$$
$$y_8 = -y_1 - y_2 - \ldots - y_7.$$

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Then $(y_1 + y_2 + y_3 + y_4)/2 = -2x_7$. It can be verified that s_i acts as $(y_i y_{i+1})$, $(1 \leqq i \leqq 6)$ and

$$(y_1, \ldots, y_8) \xrightarrow{s_7} (y_1, \ldots, y_8) - \frac{1}{2}(-y_1 - y_2 - y_3 - y_4) \times (-1, -1, -1, -1, 1, 1, 1, 1).$$

Let $a_{ij} = y_i + y_j$; then as invariants we may choose

$$\psi_m = \sum_{i<j} a_{ij}{}^m, \quad m = 2, 6, 8, 10, 12, 14, 18.$$

To show the independence of these functions, we have computed the value of their Jacobian at particular values of the y_i's. For the cases of E_7 and E_8, we have computed the values of the invariants at particular values on an IBM-370 and found them to be non-zero.

(ix) E_8: In R^8, a fundamental root system of this algebra can be chosen as

$$\{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4, \alpha_4 = e_4 - e_5, \alpha_5 = e_5 - e_6, \alpha_6 = e_6 + e_7, \alpha_7 = e_8 - \tfrac{1}{2}e^{(8)}, \alpha_9 = e_6 - e_7\}.$$

Then $s_{\alpha_1} = (r_1 r_2)$, $s_{\alpha_2} = (r_2 r_3)$, $s_{\alpha_3} = (r_3 r_4)$, $s_{\alpha_4} = (r_4 r_5)$, $s_{\alpha_5} = (r_5 r_6)$, $s_{\alpha_8} = (r_6 r_7)$, $s_{\alpha_6} = -(r_6 r_7)$ and

$$(r_1, \ldots, r_8) \xrightarrow{s_{\alpha 7}} (r_1, \ldots, r_8) - \frac{1}{4}(-r_1 - \ldots - r_7 + r_8) \times (-1, \ldots, -1, 1).$$

The 240 linear forms

$$r_i + r_j, r_i - r_j, -r_i - r_j \quad (i \neq j)$$
$$\sum_{i=1}^{8} \epsilon_i r_i, \quad \left(\epsilon_i = \pm 1, \prod_{i=1}^{8} \epsilon_i = -1\right) \quad \text{permute.}$$

The Cartan matrix is

$$\begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{bmatrix}.$$

Let

$$\begin{aligned}
y_1 &= 2x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 + x_6 + x_8\\
y_2 &= 2x_2 + 2x_3 + 2x_4 + 2x_5 + x_6 + x_8\\
y_3 &= 2x_3 + 2x_4 + 2x_5 + x_6 + x_8\\
y_4 &= 2x_4 + 2x_5 + x_6 + x_8\\
y_5 &= 2x_5 + x_6 + x_8\\
y_6 &= x_6 + x_8\\
y_7 &= x_6 + x_8\\
y_8 &= 2x_1 + 4x_2 + 6x_3 + 8x_4 + 10x_5 + 7x_6 + 4x_7 + 5x_8.
\end{aligned}$$

Then it can be verified that $s_1 = (y_1y_2)$, $s_2 = (y_2y_3)$, $s_3 = (y_3y_4)$, $s_4 = (y_4y_5)$, $s_5 = (y_5y_6)$, $s_8 = (y_6y_7)$, $s_6 = -(y_6y_7)$, and

$$(y_1, \ldots, y_8) \xrightarrow{s_7} (y_1, \ldots, y_8) - \frac{1}{4}(-y_1 - y_2 - \ldots - y_7 + y_8) \times (-1, \ldots, -1, 1).$$

Thus we may choose as invariants the functions

$$\psi_m = 2 \sum_{i<j} \{(y_i + y_j)^m + (y_i - y_j)^m\} + \sum_{\pi\epsilon_i=-1,\epsilon_i=\pm 1} \left(\sum \epsilon_i x_i\right)^m,$$
$$m = 2, 8, 12, 14, 18, 20, 24, 30.$$

Independence of these functions can be shown as in the case of E_7.

4. Conclusion. For the explicit construction of complete set of generators of $\mathscr{C}$, we mention a process of lifting invariant polynomials of W to invariants of the adjoint group as outlined in [**6**, p. 63-5]. It follows from a result in [**7**] that invariant polynomials of the adjoint group are closely related to $\mathscr{C}$. In a future paper, we plan to follow these ideas and investigate the constructions of generators arising from invariant polynomials of W for all algebras discussed in this paper.

REFERENCES

1. N. Jacobson, *Lie algebras* (Interscience, New Nork, 1962).

2. Harish-Chandra, *Some applications of the universal enveloping algebra of a semisimple Lie algebra*, Trans. Amer. Math. Soc. *70* (1951), 28–99.

3. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math. *77* (1955), 778–782.

4. H. S. M. Coxeter, *The product of the generators of a finite group generated by reflections*, Duke Math. J. *18* (1951), 765–782.

5. V. S. Poppov and A. M. Perelomov, *Casimir operators for semisimple Lie groups*, Math. USSR-Izv. *2* (1968), 1313–1335.

6. Humphreys, *Modular representations of classical Lie algebras and semisimple groups*, J. Algebra *19* (1971), 51–79.

7. I. M. Gelfand, *The centre of an infinitesimal group algebra*, Mat. Sb. *26* (1950), 103–112.

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