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RESEARCH ARTICLE

Strict positivity of Kähler–Einstein currents

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Abstract
Kähler–Einstein currents, also known as singular Kähler–Einstein metrics, have been introduced and constructed a little over a decade ago. These currents live on mildly singular compact Kähler spaces X and their two defining properties are the following: They are genuine Kähler–Einstein metrics on X_{reg}, and they admit local bounded potentials near the singularities of X. In this note, we show that these currents dominate a Kähler form near the singular locus, when either X admits a global smoothing, or when X has isolated smoothable singularities. Our results apply to klt pairs and allow us to show that if X is any compact Kähler space of dimension three with log terminal singularities, then any singular Kähler–Einstein metric of nonpositive curvature dominates a Kähler form.

Contents

Introduction

Introducing the main problem

Einstein metrics are a central object in differential geometry. A Kähler–Einstein metric on a complex manifold is a Kähler metric whose Ricci curvature is proportional to the metric tensor. A foundational result of Yau [Yau78] allows one to construct many examples of these fundamental objects.

In connection with the minimal model program, singular Kähler–Einstein metrics on mildly singular Kähler varieties X have been constructed in [EGZ09, BG14, BBE+19] and further studied by many authors (see [GZ17, Bou18, Don18, Li22] and the references therein). These are Kähler forms $\omega_{\mathrm{KE}} = \omega + dd^c\varphi_{\mathrm{KE}}$ on the regular part X_{reg} of X, where $c_1(X) = \lambda[\omega]$ is proportional to a reference Kähler class $[\omega]$, such that

$$\mathrm{Ric}(\omega_{\mathrm{KE}}) = \lambda\omega_{\mathrm{KE}}$$

and which admit bounded (or mildly unbounded) local potential near the singularities X_{sing}. In particular, ω_{KE} uniquely extends as a positive closed current to X.

One constructs $\omega_{\mathrm{KE}} = \omega + dd^c\varphi_{\mathrm{KE}}$ by solving a complex Monge–Ampère equation

$$(\omega + dd^c\varphi_{\mathrm{KE}})^n = e^{-\lambda\varphi_{\mathrm{KE}}}\mu_X, \tag{KE}$$

where $n = \dim_{\mathbb{C}} X$ and μ_X is an appropriate volume form.

Due to the presence of singularities, the geometry of these *Kähler–Einstein currents* is quite mysterious despite recent important progress [DS14, HS17]. Understanding the asymptotic behavior of φ_{KE} near X_{sing} is a major open problem.

In this note, we partially address the following basic question:

Main Question. Is the Kähler–Einstein metric ω_{KE} solving Equation (KE) a *Kähler current*, that is, does ω_{KE} dominate a Kähler form?

Note that, unless X is smooth, ω_{KE} is never dominated by a Kähler form, cf Corollary 1.9.

Known cases. Although the question sounds much easier than asking for asymptotics of ω_{KE}, it has not yet been addressed in full generality and only a handful of particular cases seem to be understood, which we briefly survey below.

• *Orbifold singularities*. If X has only finite quotient singularities (i.e., X is an orbifold), then ω_{KE} is a smooth orbifold Kähler metric (i.e., it is a Kähler metric in the local smooth uniformizing charts). In particular, ω_{KE} is a Kähler current. Since two-dimensional log terminal singularities are quotient singularities, it follows that the main question admits a positive answer in dimension two for log terminal singularities. More generally, ω_{KE} is orbifold-smooth near any quotient singularity of X at least when X is projective [LT19]; in particular, it is a Kähler current on the orbifold locus of X.

• *Limits of smooth spaces*. Other examples include when X admits a crepant resolution or when X can be suitably obtained as limit of smooth Kähler–Einstein manifolds [RZ11b, DS14, HS17]. The common thread in the situations appearing in *loc. cit.* is that one can embed $\iota : X \hookrightarrow \mathbb{P}^N$ with $\omega_{\mathrm{KE}} = \omega_{\mathrm{FS}}|_X + dd^c\varphi$ and approximate $(X, \omega_{\mathrm{KE}})$ by a sequence of compact Kähler manifolds (X_k, ω_k) such that there are

- embeddings $\iota_k : X_k \hookrightarrow \mathbb{P}^N$ with $\omega_k = \omega_{\mathrm{FS}}|_{\iota_k(X_k)} + dd^c\varphi_k$ and $\|\varphi_k\|_{L^\infty(X_k)} \leqslant C$,
- uniform Ricci lower bounds $\mathrm{Ric}\,\omega_k \geqslant -C\omega_k$

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for some uniform constant $C > 0$. From these estimates, an easy application of Chern–Lu formula yields $\omega_k \geqslant C^{-1}\omega_{\mathrm{FS}}|_{\iota_k(X_k)}$ from which the strict positivity of ω_{KE} follows.

In summary, the ideas above allow to treat the case of singular Kähler–Einstein varieties that are degenerations of *smooth* Kähler manifolds with uniform lower Ricci bound and uniform L^∞ bound for its potential.

Unfortunately, general singular Kähler–Einstein metrics ω_{KE} on a singular space X arise by construction as limits of smooth Kähler metrics ω_ε on a desingularization $\pi : \widehat{X} \to X$ such that $\mathrm{Ric}_{\omega_\varepsilon} \to -\infty$ along any divisor $E \subset \widehat{X}$ with positive discrepancy. Moreover, a general singular variety cannot be obtained as a degeneration of smooth varieties since there might be (even local and topological) obstructions to smoothability. This prevents one from directly applying the above ideas to a general singular Kähler–Einstein space. This paper grew out of an attempt to find the largest possible field of application of the general technique recalled above. More precisely, the goal of the present paper is threefold:

A. Formulate a general framework where the above global strategy applies *mutatis mutandis*, including nonprojective Kähler spaces as well as singular pairs.
B. Provide a local version of the approach that enables to treat isolated smoothable singularities.
C. Develop a systematic use of the technique in order to enlarge the class of singularities (beyond the smoothable or crepant ones) for which one can answer positively the main question above, using a *several step* degeneration process.

So far, most (general) results about singular Kähler–Einstein metrics have been derived by establishing uniform a priori estimates on smooth approximants (space and metric). The novelty of our approach is that, given a singular space/metric $(X, \omega_{\mathrm{KE}})$, we are able to answer the main question positively for $(X, \omega_{\mathrm{KE}})$ as soon as one can suitably approximate our space/metric by a *possibly singular* space/metric on which we can *qualitatively* answer the main question, as long as we have a lower bound on the Ricci curvature and a uniform bound on a suitable potential. We refer to the last paragraph of this introduction or §5 for an explicit application of this principle.

Statement of the results

Let us now get a bit more explicit and expand what we mean by the above-stated goals. Our first main result is as follows:

Theorem A. *Let X be a normal compact Kähler space with log terminal singularities such that $K_X \sim_{\mathbb{Q}} \mathcal{O}_X$, and let α be a Kähler class. If X is smoothable, then the unique singular Ricci-flat metric $\omega_{\mathrm{KE}} \in \alpha$ dominates a Kähler form.*

This strict positivity result actually holds in more general contexts [canonically polarized varieties, $\mathbb{Q}$-Fano varieties, Kawamata log terminal (klt) pairs] as we explain in Theorem 2.8. The existence of a *global* smoothing is a rather restrictive assumption, although it holds, for example, at the boundary of the moduli space of positively curved Kähler–Einstein metrics. The latter has a natural compactification arising from the Gromov–Hausdorff topology, and the geometric meaning of the boundary points was elucidated in [DS14, SSY16].

An isolated singularity is more likely to admit a local smoothing. Shifting perspective, we use the local singular theory developed in [GGZ23] to establish a positivity result of solutions to local Monge–Ampère equations at isolated smoothable singularities. As a consequence, we obtain our second main result.

Theorem B. *Let X be a normal compact Kähler space with log terminal singularities such that $K_X \sim_{\mathbb{Q}} \mathcal{O}_X$, and let α be a Kähler class. The unique singular Ricci-flat metric $\omega_{\mathrm{KE}} \in \alpha$ dominates a Kähler form near any smoothable isolated singularity.*

The result is more general, and we refer the reader to Theorem 4.1 and Corollary 4.6 for more precise results, including the case of klt pairs with isolated singularities.

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As recalled earlier, the basic idea is not new. Deforming X in a smooth Kähler approximant X_t, we would like to use Chern–Lu formula [Che68, Lu68] and establish a uniform lower bound for smooth approximants $\omega_{\mathrm{KE},t} \geqslant C^{-1}\omega_t$ on nearby smooth fibers. This requires to establish uniform a priori bounds for families of degenerate complex Monge–Ampère potentials, a theme which has known important progress in the last decade (see [RZ11a, SSY16, DNGG23]), but which still requires further understanding in order to extend Theorem B to the positively curved setting.

Our last result answers the main question positively in dimension three, in nonpositive curvature, unconditionally to any smoothability assumptions on the singularities.

Theorem C. *Let X be a normal compact Kähler space of dimension three with log terminal singularities such that $K_X \sim_{\mathbb{Q}} \mathcal{O}_X$, and let α be a Kähler class. The unique singular Ricci-flat metric $\omega_{\mathrm{KE}} \in \alpha$ dominates a Kähler form.*

Here again, there is a version of this statement in negative curvature, cf Theorem 5.3, which surprisingly requires significantly more work than the Calabi–Yau case, cf below. The general idea behind the proof of Theorem C is that one can reduce the situation to the smooth case via a *two-step* degeneration. More precisely it goes as follows.

Step 1. One can reduce to the case where X has canonical singularities, up to passing to the index one cover.

Step 2. The first degeneration amounts to considering a terminalization $\pi : \widehat{X} \to X$. Since X is canonical, π is crepant, that is, $K_{\widehat{X}} = \pi^* K_X$ is trivial. Now, one can realize $\pi^*\omega_{\mathrm{KE}}$ as limit of singular Ricci flat metrics $\omega_\varepsilon \in \pi^*\alpha + \varepsilon\beta$, where β is a Kähler class on $\widehat{X}$.

Step 3. Now, $\widehat{X}$ has terminal singularities of index one, hence these are (locally) smoothable by a classification result of Reid and we can apply Theorem B to $(\widehat{X}, \omega_\varepsilon)$ and use Chern–Lu inequality on the singular space $\widehat{X}$ to conclude.

In the case where K_X is ample, the terminalization map is not crepant anymore. Instead, a boundary divisor $\widehat{\Delta}$ arises on $\widehat{X}$ so that $K_{\widehat{X}} + \widehat{\Delta} = \pi^* K_X$ and we are then required to generalize Theorem B to the case of klt pairs. This is not as innocuous as it may sound since, even though $\widehat{X}$ has only isolated singularities, it is not the case for the pair $(\widehat{X}, \widehat{\Delta})$ anymore! Taking care of this difficulty involves an additional (third) degeneration process (cf. Theorem 4.5), highlighting the guiding principle of this article.

Contents

We recall basic facts from analysis on complex spaces in Section 1. We prove Theorem A in Section 2, using uniform a priori estimates from [SSY16, DNGG23]. We study holomorphic families of Dirichlet problems for the complex Monge–Ampère equation in Section 3, using a priori estimates from [GL10, GGZ23], and prove Theorem B in Section 4. We use the previous techniques, together with some classical facts from the Minimal Model Program in dimension three, to establish Theorem C in Section 5.

1. The Monge–Ampère operator on complex spaces

In this section, we let X be a reduced complex analytic space of pure dimension $n \geqslant 1$. We will denote by X_{reg} the complex manifold of regular points of X. The set

$$X_{\mathrm{sing}} := X \setminus X_{\mathrm{reg}}$$

of singular points is an analytic subset of X of complex codimension $\geqslant 1$.

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1.1. Plurisubharmonic functions

By definition, for each point $x_0 \in X$ there exists a neighborhood U of x_0 and a local embedding $j : U \hookrightarrow \mathbb{C}^N$ onto an analytic subset of $\mathbb{C}^N$ for some $N \geqslant 1$.

Using these local embeddings, it is possible to define the spaces of smooth forms of given degree on X. The notion of currents on X is then defined by duality by their action on compactly supported smooth forms on X. The operators ∂ and $\bar{\partial}$, d, d^c and dd^c are then well defined by duality (see [Dem85] for a careful treatment).

In the same way, one can define the analytic notions of holomorphic and plurisubharmonic functions. There are essentially two different notions:

Definition 1.1. Let $u : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ be a given function.

1. We say that u is plurisubharmonic on X if it is locally the restriction of a plurisubharmonic function on a local embedding of X onto an analytic subset of $\mathbb{C}^N$.

2. We say that u is weakly plurisubharmonic on X if u is locally bounded from above on X and its restriction to the complex manifold X_{reg} is plurisubharmonic.

Fornaess and Narasimhan proved in [FN80] that u is plurisubarmonic on X if and only if for any analytic disc $h : \mathbb{D} \longrightarrow X$, the restriction $u \circ h$ is subharmonic on $\mathbb{D}$ or identically $-\infty$.

If u is weakly plurisubharmonic on X, u is plurisubharmonic on X_{reg}, hence upper semicontinuous on X_{reg}. Since no assumption is made on u at singular points, it is natural to extend u to X by the following formula:

$$u^*(x) := \limsup_{X_{\mathrm{reg}} \ni y \to x} u(y), \ x \in X. \tag{1.1}$$

The function u^* is upper semicontinuous, locally integrable on X and satisfies $dd^c u^* \geqslant 0$ in the sense of currents on X. By Demailly [Dem85], the two notions are equivalent when X is locally irreducible. More precisely, we will need the following result:

Theorem 1.2. [Dem85] *Assume that X is a locally irreducible analytic space and $u : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ is a weakly plurisubharmonic function on X, then the function u^* defined by (1.1) is plurisubharmonic on X.*

Observe that since u is plurisubharmonic on X_{reg}, we have $u^* = u$ on X_{reg}. Hence, u^* is the upper semicontinuous extension of $u|_{X_{\mathrm{reg}}}$ to X.

Following [FN80], we say that X is Stein if it admits a $\mathcal{C}^2$-smooth strongly plurisubharmonic exhaustion. We will use the following definition:

Definition 1.3. A domain $\Omega \Subset X$ is strongly pseudoconvex if it admits a negative $\mathcal{C}^2$-smooth strongly plurisubharmonic exhaustion, that is, a function ρ strongly plurisubharmonic in a neighborhood Ω' of $\overline{\Omega}$ such that $\Omega := \{x \in \Omega' ; \rho(x) < 0\}$ and

$$\Omega_c := \{x \in \Omega'; \rho(x) < c\} \Subset \Omega$$

is relatively compact for any $c < 0$.

Our complex spaces will be assumed to be reduced, locally irreducible of dimension $n \geqslant 1$. We denote by $\mathrm{PSH}(X)$ the set of plurisubharmonic functions on X.

1.2. Dirichlet problem on singular complex spaces

The complex Monge–Ampère measure $(dd^c u)^n$ of a smooth psh function in a domain of $\mathbb{C}^n$ is the radon measure

$$(dd^c u)^n = c \det\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) dV_{\mathrm{eucl}},$$

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where $c > 0$ is a normalizing constant. The definition has been extended to any bounded psh function by Bedford–Taylor, who laid down the foundations of *pluripotential theory* in [BT76, BT82].

The Dirichlet problem for the complex Monge–Ampère operator has been studied extensively by many authors (see [GZ17] and the references therein).

The complex Monge–Ampère operator has been defined and studied on complex spaces by Bedford in [Bed82] and Demailly in [Dem85]. It turns out that if $u \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$, the Monge–Ampère measure $(dd^c u)^n$ is well defined on X_{reg} and can be extended to X as a Borel measure with zero mass on X_{sing}. Thus, all standard properties of the complex Monge–Ampère operator acting on $\mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$ extend to this setting (see [Bed82, Dem85]).

The Dirichlet problem has been studied only recently in that context. We recall the following which is a combination of [GGZ23] and [DFS23, Fu23].

Theorem 1.4. *Let X be a Stein space of dimension $n \geqslant 1$, reduced and locally irreducible, with an isolated log terminal singularity $X_{\mathrm{sing}} = \{p\}$. Let $\Omega \subset X$ be a bounded strongly pseudoconvex domain with smooth boundary. Fix a smooth volume form dV on X, $\phi \in \mathcal{C}^\infty(\partial\Omega)$ and $0 < f \in \mathcal{C}^\infty(\overline{\Omega} \setminus \{p\})$ with $f \in L^p(\Omega, dV)$ for some $p > 1$. Fix $\lambda \in \mathbb{R}^+$. Then there exists a unique plurisubharmonic function u in Ω which satisfies the following:*

- *u is continuous on $\overline{\Omega}$ with $u_{|\partial\Omega} = \phi$;*
- *u is smooth in $\Omega \setminus \{p\}$*
- *u satisfies $(dd^c u)^n = e^{\lambda u} f dV$.*

As shown further in [GGZ23], one has a uniform a priori bound on $\|u\|_{L^\infty(\Omega)}$ which depends on n, $p > 1$ and $\|f\|_{L^p(\Omega)}$. This bound only weakly depends on the geometry of $\Omega \hookrightarrow X$, as we indicate in Theorem 3.6 so as to establish a uniform family version of this estimate.

1.3. Canonical measure of a $\mathbb{Q}$-Gorenstein germ

Let (X, x) be a germ of normal complex space of dimension n such that mK_X is Cartier for some integer $m \geqslant 1$. If σ is a trivialization of mK_X over X_{reg}, then the expression

$$i^{n^2}(\sigma \wedge \bar{\sigma})^{\frac{1}{m}}$$

defines a positive measure $\mu_{X,\sigma}$ on X_{reg}; we still denote by $\mu_{X,\sigma}$ its trivial extension to X. If τ is a trivialization of $m'K_X$, then there exists $g \in \mathcal{O}_X(X)^*$ such that $\mu_{X,\tau} = |g|^2\mu_{X,\sigma}$ so that the qualitative behavior of the measure on the singular germ does not depend on the choice of σ. In the following, one will just write μ_X for $\mu_{X,\sigma}$.

Let ω be a smooth Hermitian form on X, restriction of a smooth Hermitian form under an embedding $(X, x) \hookrightarrow \mathbb{C}^N$. We denote by f the density of μ_X with respect to ω^n, that is, $\mu_X = f\omega^n$. In the following, the L^p spaces are considered with respect to ω^n.

We will see below (cf proof of Lemma 1.6) that $-\log f$ is quasi-psh; in particular, $\log f \in L^1$. As a consequence, one can make the following definition

Definition 1.5. Let X, μ_X, ω, f be as above. The Ricci curvature current of ω is defined as

$$\mathrm{Ric}\,\omega := dd^c \log f = -dd^c \log\Big(\frac{\omega^n}{\mu_X}\Big).$$

This expression yields a $(1, 1)$ current with potentials which is independent of the trivialization σ. On X_{reg}, it coincides with the usual Ricci curvature of the Kähler metric ω, but we will see in the lemma below that $\mathrm{Ric}\,\omega$ is not a smooth form unless X is smooth.

One can extend the definition of Ricci curvature for currents that are not necessarily smooth. More precisely, let $T = dd^c\varphi$ be a positive $(1, 1)$ current with potential $\varphi \in L^\infty(X)$. Assume that $T^n = g\omega^n$

has a density g satisfying $\log g \in L^1$. One defines $\mathrm{Ric}\, T := \mathrm{Ric}\,\omega - dd^c \log g$ which yields a $(1,1)$ current with potentials depending only on T (and not on ω or φ). In particular, the construction can be globalized to positive $(1,1)$ currents with local potentials on a normal complex space with $\mathbb{Q}$-Gorenstein singularities.

Recall that (X,x) has log terminal singularities if and only if μ_X has finite mass, that is, $f \in L^1$. It is standard to see that the latter condition is actually equivalent to having $f \in L^p$ for some $p > 1$, cf, for example, [EGZ09, Lemma 6.4].

The following result shows that the measure μ_X has a singular density with respect to a smooth volume form unless (X,x) is smooth, compare [Li23].

Lemma 1.6. *Let (X,x) be a germ of normal complex space of dimension n such that mK_X is Cartier, and let ω be a smooth Kähler metric. Let f be the density of μ_X with respect to ω^n, that is, $\mu_X = f\omega^n$. We have*

$$f \in L^\infty \quad \Longleftrightarrow \quad (X,x) \text{ is smooth}.$$

Moreover, the latter condition is equivalent to the existence of $k \in \mathbb{R}_+$ such that

$$\mathrm{Ric}\,\omega \geqslant -k\omega.$$

Proof. In order to lighten the notation, we will assume that $m = 1$. The proof of the general case is essentially identical. One direction of the lemma is obvious, so we need to show that boundedness of the density implies smoothness of the germ.

We can assume that we have an embedding $X \hookrightarrow \mathbb{C}^N$ such that for all n-tuple $I \subset \{1,\dots,N\}$, the linear projections $p_I : X \to \mathbb{C}^n$ are finite maps. It is not difficult to check that there exists a smooth function G on X such that $\omega^n = e^G \omega^n_{\mathbb{C}^N}|_X$.

Given any I, there exists a holomorphic function ζ_I on X_{reg} such that $dz_I|_{X_{\mathrm{reg}}} = \zeta_I \sigma$. By normality of X, ζ_I extends to a holomorphic function on X. This shows that $f = (e^G \sum_I |\zeta_I|^2)^{-1}$ up to some positive constant. In particular, $-\log f$ is quasi-psh.

Next, if f is bounded on X, then there exists I such that $\zeta_I(x) \neq 0$. In particular, ζ_I is nonvanishing on the germ (X,x), and $dz_I|_{X_{\mathrm{reg}}}$ is a trivialization of $K_{X_{\mathrm{reg}}}$. In other words, the map $p_I : X \to \mathbb{C}^n$ is étale on X_{reg}, hence everywhere by purity of the branch locus. This shows that (X,x) is smooth.

As for the last claim, assume that (X,x) is singular. Then $\psi := \log(\sum_I |\zeta_I|^2)$ is psh with analytic singularities and satisfies $\psi(x) = -\infty$ and by definition, we have $\mathrm{Ric}\,\omega = -dd^c(\psi + G)$. If the Ricci curvature of ω were bounded from below, there would exist $A > 0$ such that $dd^c(A\|z\|^2 - \psi) \geqslant 0$. In particular, $A\|z\|^2 - \psi$ would be psh on X_{reg}, hence bounded above near x, which is absurd. □

Remark 1.7. The Ricci curvature of ω is, however, always bounded *from above*. This follows from the fact that curvature decreases when passing to holomorphic submanifolds.

Remark 1.8. The first statement in Lemma 1.6 extends immediately to the setting of log pairs (X,D). Indeed, if $D \neq \emptyset$, let us write $D = \sum a_i D_i$ with $a_i \neq 0$ and work at a general point $y \in D_i$ where both X and D are smooth so that $D_i = (g_i = 0)$ for some holomorphic function g_i defined near y. In a neighborhood of y, the density of $\mu_{(X,D)}$ looks like $|g_i|^{-2a_i}$ hence it is not bounded.

Corollary 1.9. *Let (X,ω) be a Stein or compact Kähler space with log terminal singularities admitting a Kähler–Einstein metric ω_{KE} in the sense of [GGZ23] or [EGZ09], respectively. Assume that there exists $C > 0$ such that*

$$\omega_{\mathrm{KE}} \leqslant C\omega \quad \text{on } X.$$

Then X is smooth.

Proof. Pick $x \in X$ and choose a neighborhood U of x bearing a trivialization σ of $mK_{U_{\mathrm{reg}}}$ and write $\mu_U := i^{n^2}(\sigma \wedge \bar\sigma)^{\frac{1}{m}}$. Next, one can ensure that there exists $\varphi \in \mathrm{PSH}(U) \cap L^\infty(U)$ such that

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$\omega_{\mathrm{KE}}|_U = dd^c\varphi$. There exist $\lambda \in \mathbb{R}$ and a pluriharmonic (hence smooth) function h on U such that

$$\omega_{\mathrm{KE}}^n = e^{\lambda\varphi+h}\mu_U \quad \text{on } U.$$

In particular, the domination $\omega_{\mathrm{KE}} \leqslant C\omega$ implies that the density of μ_U with respect to ω^n is bounded. The conclusion now follows from Lemma 1.6. □

2. Global smoothing

In this section, we show that singular Kähler–Einstein metrics are Kähler currents when the variety X admits a global smoothing.

2.1. Kähler–Einstein currents

Let X be a Kähler normal compact space. The study of complex Monge–Ampère equations in this context has been initiated in [EGZ09], providing a way of constructing singular Kähler–Einstein metrics and extending Yau's fundamental solution to the Calabi conjecture [Yau78]. More precisely, it is proven there that given a Kähler metric ω on X, a nonnegative number $\lambda \in \{0,1\}$ and a nonnegative function $f \in L^p(X)$ for some $p > 1$ (satisfying $\int_X f\omega^n = \int_X \omega^n$ if $\lambda = 0$), then the equation

$$(\omega + dd^c\varphi)^n = fe^{\lambda\varphi}\cdot\omega^n \tag{2.1}$$

has a unique solution $\varphi \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)$ (with the additional normalization $\sup_X \varphi = 0$ if $\lambda = 0$).

Let us now explain the relation between Equation (2.1) above and the existence of singular Kähler–Einstein metrics.

We choose a pair (X, D) consisting of an n-dimensional compact Kähler variety X and a divisor $D = \sum a_iD_i$ with $a_i \in [0,1]\cap\mathbb{Q}$. We assume that there exists an integer $m \geqslant 1$ such that $m(K_X + D)$ is a line bundle. More precisely, we mean by this that the reflexive hull of the coherent sheaf $\left(\det(\Omega_X^1)\otimes\mathcal{O}_X(D)\right)^{\otimes m}$ is locally free.

Given a Hermitian metric h on $K_X + D$ and the singular metric $e^{-\phi_D}$ on X_{reg} (unique up to a positive multiple), one can construct a measure $\mu_{(X,D),h}$ on X as follows. If U is any open set where $m(K_X + D)$ admits a trivialization σ on U_{reg}, then the expression

$$\frac{(\sigma\wedge\bar{\sigma})^{\frac{1}{m}}}{|\sigma|_{h^{\otimes m}}^{\frac{2}{m}}}e^{-\phi_D}$$

defines a measure on U_{reg} which is independent of m as well as the choice of σ and can thus be patched to a measure on X_{reg}. Its extension by 0 on X_{sing} is by definition $\mu_{(X,D),h}$. We recall the following properties satisfied by the measure $\mu := \mu_{(X,D),h}$, cf [EGZ09, Lemma 6.4].

- The Ricci curvature of μ on X_{reg} is equal to $-i\Theta(h) + [D]$.
- The mass $\int_X d\mu$ is finite if and only if (X, D) has klt singularities.
- If μ has finite mass, then the density f of μ wrt ω^n (i.e., $\mu = f\cdot\omega^n$) satisfies $f \in L^p(X)$ for some $p > 1$.

From now on, we work in the following

Setup 2.1. *Let (X, D) be a pair where X is a compact normal Kähler space and D is an effective $\mathbb{Q}$-divisor. Assume that (X, D) has klt singularities, pick a Kähler metric ω and a Hermitian metric h on $K_X + D$, normalized so that $\int_X d\mu_{(X,D),h} = \int_X \omega^n$. We assume either*

- *$K_X + D$ is ample and $\omega = i\Theta(h)$, or*
- *$K_X + D \equiv 0$ and h satisfies $i\Theta(h) = 0$, or else*
- *$K_X + D$ is antiample and $\omega = -i\Theta(h)$.*

Definition 2.2. In the Setting 2.1 above, a Kähler–Einstein metric $\omega_{\mathrm{KE}} := \omega + dd^c\varphi_{\mathrm{KE}}$ is a solution of the Monge–Ampère equation

$$(\omega + dd^c\varphi_{\mathrm{KE}})^n = e^{\lambda\varphi_{\mathrm{KE}}}\mu_{(X,D),h}, \tag{2.2}$$

where $\lambda = 1, 0$ or -1 according to whether we are in the first, second or third case. It satisfies

$$\mathrm{Ric}(\omega_{\mathrm{KE}}) = -\lambda\omega_{\mathrm{KE}} + [D] \tag{2.3}$$

in the weak sense.

By the results [EGZ09] recalled above, Equation (2.2) admits a unique solution ω_{KE} whenever $\lambda \in \{0, 1\}$. Its potential φ_{KE} is globally bounded on X and ω_{KE} is a honest Kähler–Einstein metric on $X_{\mathrm{reg}} \setminus \mathrm{Supp}(D)$, and it has cone singularities along D generically [Gue13]. For $\lambda = -1$, we refer the reader to [BBE$^+$19, Bou18].

Kähler–Einstein theory in positive curvature is notoriously more complicated than in nonpositive curvature. For that reason, we will make additional assumptions when working in the log Fano case and introduce the following

Setup 2.3. *In the Setup 2.1 and in the case where $-(K_X + D)$ is ample, we assume additionally that*

- *There exists a Kähler–Einstein metric ω_{KE},*
- $\mathrm{Aut}^\circ(X, D) = 0$.

The assumption on the automorphism group is to ensure uniqueness of the KE metric. Actually, assuming existence of a Kähler-Einstein (KE) metric, its uniqueness is equivalent to the discreteness of $\mathrm{Aut}(X, D)$.

In summary, in Setup 2.3 above, there exists in each three cases a unique Kähler–Einstein metric $\omega_{\mathrm{KE}} \in [\omega]$. This is by [EGZ09] if $K_X + D \geqslant 0$ and by [BBE$^+$19, Theorem 5.1] if $K_X + D < 0$.

2.2. Kähler currents

Let us start by recalling the following terminology.

Definition 2.4. Let (X, ω) be a compact Kähler space, and let T be a closed, positive $(1, 1)$-current. We say that T is a *Kähler current* if there exists $\varepsilon > 0$ such that the inequality $T \geqslant \varepsilon\omega$ holds globally on X in the sense of currents.

In our main case of interest, T will have local potentials, that is, there exist a finite open covering $X = \cup_{\alpha\in I} U_\alpha$ and functions $u_\alpha \in \mathrm{PSH}(U_\alpha)$ such that $T|_{U_\alpha} = dd^c u_\alpha$. Up to refining the cover, one can assume that $\omega|_{U_\alpha} = dd^c\varphi_\alpha$ for some strictly psh functions φ_α on U_α. Then T is a Kähler current if and only if there exists $\varepsilon > 0$ such that for all $\alpha \in I$, one has $u_\alpha - \varepsilon\varphi_\alpha \in \mathrm{PSH}(U_\alpha)$.

In the following, we will repeatedly use the following classical result:

Lemma 2.5. *Let X be a normal complex space, and let T be a closed $(1, 1)$- current on X admitting locally bounded potentials. Assume that there exists a closed analytic set $Z \subsetneq X$ such that $T|_{X\setminus Z} \geqslant 0$. Then, $T \geqslant 0$ everywhere on X.*

Proof. The claim is an immediate application of Chern–Levine–Nirenberg inequality that ensures that T puts no mass on pluripolar sets. Alternatively, fix an open neighborhood U of a point $x \in Z$ such that $T|_U = dd^c\varphi$. One can choose φ such that $\varphi|_{U\setminus Z}$ is psh and locally bounded near Z, hence it extends to a psh function ψ on U. Since $\varphi = \psi$ almost everywhere on U, we have $T|_U = dd^c\psi \geqslant 0$. □

The purpose of this section is to study when ω_{KE} is a Kähler current, by using a smoothability assumption. Our results are inspired by a result due to Ruan and Zhang [RZ11b, Lemma 5.2], and the main tool throughout the paper will be Chern–Lu inequality, which we recall below as it can be found in, for example, [Rub14, Proposition 7.2]

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Proposition 2.6 (Chern–Lu inequality). *Let X be a complex manifold endowed with two Kähler metrics $\omega, \widehat{\omega}$. Assume that there are constants $C_1, C_2, C_3 \in \mathbb{R}$ such that*

$$\mathrm{Ric}\,\widehat{\omega} \geqslant -C_1\widehat{\omega} - C_2\omega, \quad \textit{and} \quad \mathrm{Bisec}_\omega \leqslant C_3.$$

Then, we have the following inequality on X

$$\Delta_{\widehat{\omega}} \log \mathrm{tr}_{\widehat{\omega}}\omega \geqslant -C_1 - (C_2 + 2C_3)\,\mathrm{tr}_{\widehat{\omega}}\omega.$$

Next, we introduce the following definition

Definition 2.7. Let (X, D) be a klt pair with X compact, and let ω be a Kähler metric. We say that $(X, D, [\omega])$ admits a $\mathbb{Q}$-Gorenstein smoothing if there exists a triplet $(\mathcal{X}, \mathcal{D}, [\omega_{\mathcal{X}}])$ consisting of a normal complex space $\mathcal{X}$, an effective $\mathbb{Q}$-divisor $\mathcal{D}$, and a smooth, $(1,1)$-form $\omega_{\mathcal{X}}$ on $\mathcal{X}$ admitting a proper, surjective holomorphic map $\pi : \mathcal{X} \to \mathbb{D}$ satisfying:

1. $K_{\mathcal{X}/\mathbb{D}} + \mathcal{D}$ is a $\mathbb{Q}$-line bundle.
2. Every irreducible component of $\mathcal{D}$ surjects onto $\mathbb{D}$.
3. $(\mathcal{X}, \mathcal{D})|_{\pi^{-1}(0)} \simeq (X, D)$.
4. For $t \neq 0$, $(X_t, D_t) = (\mathcal{X}, \mathcal{D})|_{\pi^{-1}(t)}$ is log smooth.
5. For $t \in \mathbb{D}$, $\omega_t := \omega_{\mathcal{X}}|_{X_t}$ is Kähler and $[\omega_0] = [\omega]$.

Let us make few remarks:

1. If $K_X + D$ is ample or antiample, then so is $K_{X_t} + D_t$ for t small. Then, the last condition in the definition above is automatic if $\omega \in \pm c_1(K_X + D)$, by using an embedding of $\mathcal{X}$ into $\mathbb{P}^N \times \mathbb{D}$ via sections of $\pm m(K_{X_t} + D_t)$ for m large.
2. If $K_X + D$ is numerically trivial, then so is $K_{X_t} + D_t$ for t small, cf, for example, [DG18, Lemma 2.12].
3. The pair (X_t, D_t) is automatically klt for any $t \in \mathbb{D}$.

Theorem 2.8. *Let $(X, D, [\omega])$ as in Setup 2.3, and assume that $(X, D, [\omega])$ admits a $\mathbb{Q}$-Gorenstein smoothing. Then, the Kähler–Einstein metric ω_{KE} is a Kähler current. That is, there exists $C > 0$ such that*

$$\omega_{\mathrm{KE}} \geqslant C^{-1}\omega.$$

Remark 2.9. Along the same lines, one can obtain the result above assuming instead that (X, D) admits a crepant resolution, that is, a proper bimeromorphic map $p : \widetilde{X} \to X$ such that $K_{\widetilde{X}} + \widetilde{D} = p^*(K_X + D)$, where $\widetilde{D}$ is the proper transform of D.

Proof. We consider the smoothing $\pi : (\mathcal{X}, \mathcal{D}) \to \mathbb{D}$. Up to shrinking $\mathbb{D}$ slightly and adding $\pi^* dd^c|t|^2$ to $\omega_{\mathcal{X}}$, one can assume that the later form is strictly positive. Since $\omega_{\mathcal{X}}$ is the restriction of a smooth positive $(1,1)$-form under local embedding $\mathcal{X} \underset{\mathrm{loc}}{\hookrightarrow} \mathbb{C}^N$, one can assume that the bisectional curvature of $\omega_{\mathcal{X}}|_{\mathcal{X}^{\mathrm{reg}}}$ is bounded above by a given constant C_1, maybe up to shrinking $\mathbb{D}$ just a little more. This is because the bisectional curvature decreases when passing to holomorphic submanifolds. By the same argument, we see the bisectional curvature of $\omega_t = \omega_{\mathcal{X}}|_{X_t}$ is bounded above by C_1, for any $t \in \mathbb{D}$.

For $t \in \mathbb{D}$, we consider $\widetilde{\omega}_t = \omega_t + dd^c\varphi_t$ the unique Kähler–Einstein metric of the pair (X_t, D_t). For $t \neq 0$, existence and uniqueness of $\widetilde{\omega}_t$ is due to [Koł98] when $\lambda \geqslant 0$ and to [SSY16, LWX19], cf also [PT23] when $\lambda < 0$. In the latter case, we need t to be small enough. The Kähler–Einstein metric solves

$$(\omega_t + dd^c\varphi_t)^n = e^{\lambda\varphi_t} f_t \omega_t^n \tag{2.4}$$

for some nonnegative function f_t on X_t satisfying $\int_{X_t} f_t^p \omega_t^n \leqslant C(p, \omega)$ for some $p > 1$ and some constant $C(p, \omega)$ independent of $t \in \mathbb{D}$. This is proved in [DNGG23, Lemma 4.4] assuming additionally

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X_0 has canonical singularities, but the klt case can be proved with minimal changes, cf also Equation (3.6) in the next section.

If $\lambda = 0$, we normalize φ_t by $\sup_{X_t} \varphi_t = 0$. We claim that up to shrinking $\mathbb{D}$, there exists a constant $C_2 > 0$ such that

$$\|\varphi_t\|_{L^\infty(X_t)} \leqslant C_2 \tag{2.5}$$

for any $t \in \mathbb{D}$.

If $\lambda > 0$, this is a consequence of [DNGG23, Theorem E]. If $\lambda = 0$, this is a consequence of the proof of [DNGG23, Theorem 6.1]. Indeed, in *loc. cit.* the fibers are assumed to be canonical but this is a mostly cosmetic assumption since going from canonical to klt singularities leaves the proof resting on [DNGG23, Lemma 4.4] unchanged, as explained above. Finally, if $\lambda < 0$, the estimate (2.5) is proved in [SSY16, Proposition 2.23], [LWX19, Theorem 1.2 (iii)] when $\mathcal{D}$ is pluriantican onical and in [PT23, Theorem 5.10] in general.

Set $E := \mathrm{Supp}(\mathcal{D})$. We choose a smooth Hermitian metric h on $\mathcal{O}_{\mathcal{X}}(E)$, a section s of that line bundle cutting out E, and we introduce the function $\psi := \log |s|_h^2$. It satisfies $dd^c\psi = -i\Theta_h(E)$ on $\mathcal{X} \setminus E$, hence there exists a constant $C_3 > 0$ such that

$$\psi \leqslant C_3 \quad \text{and} \quad dd^c\psi \geqslant -C_3\omega \quad \text{on } \mathcal{X} \setminus E. \tag{2.6}$$

We introduce a number $\varepsilon > 0$ (meant to go to zero), and we assume wlog that $\varepsilon C_3 \leqslant 1$. We also define $v_t := \log \mathrm{tr}_{\widetilde{\omega}_t} \omega_t$, which is smooth on $X_t \setminus E$ and globally bounded (for $t \neq 0$). We want to quantify that bound.

On $X_t \setminus E$, we have $\mathrm{Ric}\, \widetilde{\omega}_t \geqslant -\widetilde{\omega}_t$ and $\mathrm{Bisec}(X_t, \omega_t) \leqslant C_1$. By Chern–Lu inequality, that is, Proposition 2.6, we get

$$\Delta_{\widetilde{\omega}_t} v_t \geqslant -1 - 2C_1 e^{v_t}$$

Using the identity $\widetilde{\omega}_t = \omega_t + dd^c\varphi_t$ and the inequality (2.6), we get

$$\Delta_{\widetilde{\omega}_t}(v_t - A\varphi_t + \varepsilon\psi) \geqslant e^{v_t} - C_4,$$

where $A = 2(C_1 + 1)$ and $C_4 = An + 1$. The maximum of the term inside the Laplacian is attained on $X_t \setminus E$, and an easy application of the maximum principle shows

$$v_t(x) \leqslant C_5 - \varepsilon\psi(x)$$

for any $x \in X_t \setminus E$ and $\varepsilon > 0$, and where $C_5 = \log C_4 + 2AC_2 + 1$. Passing to the limit when $\varepsilon \to 0$, we find

$$\widetilde{\omega}_t \geqslant C_5^{-1}\omega_t \quad \text{on } X_t \setminus E, \ \text{ for any } t \in \mathbb{D}^*. \tag{2.7}$$

Next, we choose a continuous family of smooth maps $F_t : X_{\mathrm{reg}} \to X_t$ inducing a diffeomorphism onto their image and such that $F_0 = \mathrm{Id}_{X_{\mathrm{reg}}}$. We claim that $F_t^*\widetilde{\omega}_t$ converges locally smoothly to ω_{KE} on $X_{\mathrm{reg}} \setminus E$ when $t \to 0$. Thanks to Equation (2.7), this would imply that

$$\omega_{\mathrm{KE}} \geqslant C_5^{-1}\omega \quad \text{on } X_{\mathrm{reg}} \setminus E,$$

hence everywhere on X by Lemma 2.5.

Set $X^\circ := X_{\mathrm{reg}} \setminus E$. In order to show the convergence, we claim successively:

- The family of functions $F_t^*\varphi_t$ is precompact in the $\mathcal{C}^2_{\mathrm{loc}}(X^\circ)$-topology.
- Each cluster value $\omega_\infty = \omega + dd^c\varphi_\infty$ of $F_t^*\widetilde{\omega}_t$ solves the Monge–Ampère equation

$$(\omega + dd^c\varphi_\infty)^n = e^{\lambda\varphi_\infty} f_0 \omega^n \quad \text{on } X^\circ.$$

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- Each cluster value φ_∞ is globally bounded on X°, hence its unique ω-psh extension to X solves the equation above on X.

By uniqueness of the Kähler–Einstein metric, it would then follow that $\omega_\infty = \omega_{\mathrm{KE}}$, concluding the proof of the convergence. Let us briefly justify each of the items above.

The first point follows from the local Laplacian estimate on X° obtained from Equation (2.7). Indeed, once the Laplacian estimates are obtained, one can invoke Evans–Krylov and Schauder estimates since $F_t^*\omega_t$ (resp. $F_t^*J_t, F_t^*f_t$) converges locally smoothly on X° to ω (resp J, f_0) when $t \to 0$.

The second item is an immediate consequence of the first one. As for the last one, it is a consequence of Equation (2.5). □

3. Isolated singularities

3.1. Families of Monge–Ampère equations in a local setting

Throughout this section, we will work in the following geometric context.

Setup 3.1. *Let $\mathcal{X} \Subset \mathbb{C}^N$ be a bounded normal Stein space of dimension $n+1$ endowed with a surjective holomorphic map $\pi : \mathcal{X} \to \mathbb{D}$ such that*

1. *$\mathcal{X}$ is $\mathbb{Q}$-Gorenstein and $K_{\mathcal{X}/\mathbb{D}} \sim_{\mathbb{Q}} \mathcal{O}_{\mathcal{X}}$.*
2. *For every $t \in \mathbb{D}$, the schematic fiber $X_t = \pi^{-1}(t)$ is irreducible and reduced.*
3. *X_0 has klt singularities.*

In the following, we fix a basepoint $0 \in X_0$.

The first item means that there exists an integer $m \geqslant 1$ such that the reflexive hull of $K_{\mathcal{X}}^{\otimes m}$ is trivial. Once and for all, we pick a trivialization $\Omega \in H^0(\mathcal{X}, mK_{\mathcal{X}})$, that is, $\Omega|_{\mathcal{X}_{\mathrm{reg}}}$ is nonvanishing. We set $\Omega_t := \frac{\Omega}{(d\pi)^{\otimes m}}\big|_{X_t} \in H^0(X_t, mK_{X_t})$; this induces a trivialization of mK_{X_t}, and we define the measure

$$\mu_{X_t} := \mu_{X_t,\Omega_t} = i^{n^2}(\Omega_t \wedge \overline{\Omega}_t)^{\frac{1}{m}} \quad \text{on } X_t,$$

cf §1.3. Recall that X_t has klt singularities if and only if μ_{X_t} has finite mass on each compact subset of X_t. Since X_0 is klt, inversion of adjunction [KM98, Theorem 5.50] (cf also §3.2.1) shows that $\mathcal{X}$ is klt in the neighborhood of X_0, hence for any $\mathcal{X}' \Subset \mathcal{X}$ there exists $\delta > 0$ such that $\mathcal{X}' \cap X_t$ is klt for $|t| < \delta$. We will therefore shrink $\mathcal{X}$ so that each X_t is relatively compact in a klt space for $|t|$ small enough.

Definition 3.2. We will use the following terminology.

1. A holomorphic map $\pi : \mathcal{X} \to \mathbb{D}$ as in Setup 3.1 is a smoothing of X_0 if π is smooth over $\mathbb{D}^*$.
2. A normal Stein space X is smoothable if there exists a family $\pi : \mathcal{X} \to \mathbb{D}$ as in Setup 3.1 such that $X \simeq X_0 = \pi^{-1}(0)$ and $\mathcal{X} \to \mathbb{D}$ is a smoothing of X_0.

In the geometric context provided by Setup 3.1, one can consider a natural family of Monge–Ampère equations which we now describe and whose analysis will take up most of this section. We pick a smooth, strictly psh nonpositive function ρ on $\mathcal{X}$ such that $\partial\mathcal{X} = \{\rho = 0\}$ and set $\omega = dd^c\rho$. The restriction $\omega|_{X_t}$ of ω to the fiber X_t will be denoted by ω_t. Next, we extend F (resp. h) to a smooth function on $\overline{\mathcal{X}}$) (resp. $\partial\mathcal{X}$) which we still denote by F (resp. h). We also choose a (small) neighborhood V of $\partial\mathcal{X}$ and set $V_t = V \cap X_t$. We are interested in the Dirichlet problem, that is, finding a plurisubharmonic function $u_t \in \mathrm{PSH}(X_t) \cap \mathcal{C}^0(\overline{X_t})$ solution of

$$\begin{cases} (dd^c u_t)^n = e^{\lambda u_t + F_t}\mu_{X_t} & \text{on } X_t \\ u_t|_{\partial X_t} = h_t \end{cases}. \tag{MA$_t$}$$

with $\lambda \in \{0, 1\}$. If X_t is smooth, then the existence and uniqueness of u_t is classical and provides a solution which is smooth in $\overline{X_t}$ (see [CKNS85, GL10]). In this more general setting, the existence and

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uniqueness of u_t is provided by [GGZ23, Theorem A] for $\lambda = 0$ since X_t is klt up to the boundary so that the density of μ_{X_t} with respect to ω_t^n belongs to $L^p(X_t, \omega_t^n)$ for some $p > 1$. The case $\lambda = 1$ can be treated along very similar lines.

The main technical contribution of this section is summarized in the following result.

Proposition 3.3. *Let $\pi : \mathcal{X} \to \mathbb{D}$ be a family as in Setup 3.1, and let u_t be the solution of the Monge–Ampère equation (MA$_t$). We have the following.*

1. *There exist constants $\delta, C > 0$ such that for any $|t| < \delta$, we have $\|u_t\|_{L^\infty(X_t)} \leqslant C$.*
2. *If π is smooth outside the basepoint $0 \in X_0$, then for any $t \neq 0$ Equation (MA$_t$) admits a smooth subsolution $\underline{u}_t$ such that*

$$dd^c \underline{u}_t \geqslant \varepsilon\omega_t \text{ on } X_t, \quad \text{and} \quad \|\underline{u}_t\|_{\mathcal{C}^k(\bar{V}_t)} \leqslant C(k)$$

for any $k \in \mathbb{N}$, where $\varepsilon, C = C(k) > 0$ are constant that do not depend on t.

The proof of Proposition 3.3 is quite lengthy and will be provided in §3.2 below.

Building upon Proposition 3.3, one can prove the following local version of Theorem 2.8, which will also be useful for understanding global problems further along in the paper.

Theorem 3.4. *Let $X' \Subset \mathbb{C}^N$ be a normal, connected n-dimensional Stein space with an isolated klt singularity at the origin. Let $X \Subset X'$ be a strongly pseudoconvex domain containing the origin and such that $K_X \sim_{\mathbb{Q}} \mathcal{O}_X$. Consider the solution $u \in \mathrm{PSH}(X) \cap L^\infty(X)$ of*

$$\begin{cases} (dd^c u)^n = e^{\lambda u + F}\mu_X & \text{on } X \\ u|_{\partial X} = h, \end{cases} \tag{MA}$$

where $F \in \mathcal{C}^\infty(\overline{X})$, $h \in \mathcal{C}^\infty(\partial X)$ and $\lambda \in \{0, 1\}$.

If X is smoothable, then $dd^c u$ is a Kähler current. More precisely, there exists $C > 0$ such that $dd^c u \geqslant C^{-1}\omega_{\mathbb{C}^N}|_X$.

The proof of Theorem 3.4 is provided in §3.3. Its flavor is very similar to the proof of its global counterpart, but the a priori estimates in the local setting require significantly more work. The latter are quite classical on a fixed manifold, and one only needs to check that the arguments can be made to work in families, which we have chosen to do carefully in the appendix of the preprint version of this text.

3.2. *Proof of Proposition 3.3*

The proof involves a careful analysis of the behavior of the measure μ_{X_t} which is achieved using the semistable reduction theorem following the lines of [DNGG23, §4]. The two items in the theorem are then proved separately.

3.2.1. Semistable model

By [KKMSD73], one can find a semistable model of π. More precisely, up to shrinking $\mathbb{D}$, there exists a finite cover $\varphi : t \mapsto t^k$ of the disk for some integer $k \geqslant 1$ and a proper, surjective birational morphism $g := \mathcal{Y} \to \mathcal{X}' := (\mathcal{X} \times_\varphi \mathbb{D})^\nu$ from a smooth manifold $\mathcal{Y}$, where ν stands for the normalization, as below

$$\begin{array}{ccccc} & & \overset{f}{\frown} & & \\ \mathcal{Y} & \xrightarrow{g} & \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ & \searrow^{p} & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ & & \mathbb{D} & \xrightarrow{\varphi} & \mathbb{D} \end{array} \tag{3.1}$$

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such that around any point $y \in p^{-1}(0)$, there exists an integer $\ell \leqslant n+1$ and a system of coordinates $(z_0, \dots, z_n)$ centered at y and such that $p(z_0, \dots, z_n) = z_0 \cdots z_\ell$. Moreover, π' is smooth (resp. g is étale) away from $h^{-1}(\mathrm{Sing}(\pi))$, and that set has π'-relative codimension at least two.

Thanks to the generic smoothness theorem, one can shrink $\mathbb{D}$ so that p is smooth away from 0. In particular, the induced morphism $f|_{Y_t} : Y_t \to X_{\varphi(t)}$ is a resolution of singularities for any $t \neq 0$, where $Y_t = p^{-1}(t)$. We want to understand the behavior of the measures (or volume forms) μ_{X_t} when $t \to 0$. This can be achieved quite explicitly on $\mathcal{Y}$ via pull back by f as we now explain.

As we mentioned above, π and π' are smooth in codimension one (even in codimension two); this implies that $\mathcal{X}' \to \mathcal{X} \times_\varphi \mathbb{D}$ is isomorphic in codimension one, hence $X_0' \to (\mathcal{X} \times_\varphi \mathbb{D})_0$ is finite and generically $1-1$. Next, $(\mathcal{X} \times_\varphi \mathbb{D})_0 \to X_0$ is finite and $1-1$ on points. Therefore, $X_0' \to X_0$ is finite and generically $1-1$, hence it is isomorphic since X_0 is normal. In particular, X_0' is irreducible and has klt singularities.

Next, we write

$$K_{\mathcal{Y}} + Y_0 = g^*(K_{\mathcal{X}'} + X_0') + \sum_{i \in I} a_i E_i, \tag{3.2}$$

where the E_i's are g-exceptional divisors and Y_0 is the strict transform of X_0'; it is irreducible. One should observe that some of the divisors E_i's may be irreducible components of $p^{-1}(0)$ so that the inclusion $Y_0 \subsetneq p^{-1}(0)$ is strict in general. The divisors E_i not included in the fiber $p^{-1}(0)$ surject onto $\mathbb{D}$ since we have assumed that $p^{-1}(t)$ is irreducible for $t \neq 0$. In other words, we have

$$\forall i, \quad g(E_i) \cap X_0' \neq \emptyset. \tag{3.3}$$

The divisor $E := \sum_{i \in I} E_i$ is the exceptional locus of g and $E + Y_0$ has simple normal crossing support. Now, restrict Equation (3.2) to each irreducible component of Y_0 and use adjunction to obtain

$$K_{Y_0} = g^* K_{X_0'} + \sum_{i \in I} a_i E_i|_{Y_0}. \tag{3.4}$$

Since $E + Y_0$ is SNC, $g|_{Y_0} : Y_0 \to X_0'$ is a log resolution of X_0' and we find that

$$\forall i, \quad a_i > -1. \tag{3.5}$$

Indeed, since X_0' is klt, the inequality above holds for for any i such that $E_i \cap Y_0 \neq \emptyset$. And that set of indices i coincides with the full set I thanks to Equation (3.3) and the connectedness of the fibers of g.

3.2.2. Analysis of μ_t

Let ω be a fixed Kähler metric on $\mathcal{X}$, and let us define the function γ on $\mathcal{X}_{\mathrm{reg}} = \mathcal{X} \setminus \mathrm{Sing}(\pi)$ by

$$i^{n^2} (\Omega \wedge \overline{\Omega})^{\frac{1}{m}} = e^{-\gamma} \omega^n \wedge d\pi \wedge \overline{d\pi}.$$

The main result in this section is the following:

Lemma 3.5. *Up to shrinking $\mathbb{D}$, there exists $p > 1$ and a constant $C > 0$ such that for any $t \in \mathbb{D}$, one has*

$$\int_{X_t} e^{-p\gamma} \omega_t^n \leqslant C. \tag{3.6}$$

Proof of Lemma 3.5. Equation (3.2) can be reformulated by saying that the form $f^*(\frac{1}{\pi^m}\Omega)$ is a holomorphic section of $mK_{\mathcal{Y}}$ on $\mathcal{Y} \setminus p^{-1}(0)$ with a (possibly negative) vanishing order ma_i along E_i and a pole or order m along Y_0. Given $y \in Y_0$, pick a coordinate chart $(U, \underline{z})$ centered at y such that $p(z_0, \dots, z_n) = z_0 \cdots z_\ell$ for some $\ell \leqslant n+1$. We can relabel the coordinates so that $(z_1, \dots, z_n)$ are a

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system of coordinates on Y_0 and $U \cap Y_0 = (z_0 = 0)$. Note that $(z_1, \dots, z_n)$ remains a system of coordinates on Y_t for $|t|$ small. One can choose an injection $\{1, \dots, \ell\} \longrightarrow I$ such that $U \cap E_i = (z_i = 0)$ for $1 \leqslant i \leqslant \ell$ and $E_i \cap U = \emptyset$ for the other indices $i \in I$.

On $U \setminus p^{-1}(0)$, one can write

$$f^*\left(\frac{1}{\pi^m}\Omega\right) = a(z_0, \dots, z_n)\left(\frac{dz_0}{z_0} \wedge dz_1 \wedge \dots \wedge dz_n\right)^{\otimes m} \tag{3.7}$$

for some holomorphic function a on $U \setminus E$ such that $b(z) := z_1^{ma_1^-} \cdots z_\ell^{ma_\ell^-} a(z)$ extends holomorphically across E, where one defines $x^- := -\min\{0, x\}$ for any $x \in \mathbb{R}$. Since $ma_i > -m$ from Equation (3.5), we get $0 \leqslant ma_i^- < m$.

We have

$$f^* \frac{\Omega}{(d\pi)^{\otimes m}} = \frac{f^*\Omega}{(dp)^{\otimes m}} = \frac{f^*(\frac{1}{\pi^m}\Omega)}{(\frac{dp}{p})^{\otimes m}}$$

as well as $\frac{dz_0}{z_0} \wedge dz_1 \wedge \dots \wedge dz_n = \frac{dp}{p} \wedge dz_1 \wedge \dots \wedge dz_n$ on U. Combining those two identities with Equation (3.7) and recalling that we defined $\Omega_t = \frac{\Omega}{(d\pi)^{\otimes m}}|_{X_t}$, we get on $U \cap Y_t$:

$$f^*\Omega_t = a(z)(dz_1 \wedge \dots \wedge dz_n)^{\otimes m}$$

and therefore there exists $C > 0$ such that

$$f^*\mu_{X_t} \leqslant C \frac{idz_1 \wedge d\bar{z}_1 \wedge \dots \wedge idz_n \wedge d\bar{z}_n}{\prod_{i=1}^{\ell} |z_i|^{2a_i^-}}. \tag{3.8}$$

Arguing as in the proof of [RZ11a, Thm. B.1(i)], we can shrink $\mathcal{X}$ further so that there exist bounded holomorphic functions $(\sigma_1, \cdots, \sigma_r)$ on $\mathcal{X}$ satisfying $V(\sigma_1, \dots, \sigma_r) \subset \mathcal{X}_{\text{sing}}$ and

$$\gamma = \log \sum_j |\sigma_j|^2 + O(1). \tag{3.9}$$

It follows from Equation (3.9) that there exists a constant $A > 0$ such that $f^*\gamma \geqslant A \log |s_E|^2$, where $s_E \in H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(E))$ cuts out the exceptional divisor E and $|\cdot|$ is a smooth Hermitian metric on $\mathcal{O}_{\mathcal{X}}(E)$. Next, since $f : Y_t \longrightarrow X_t$ is generically finite (with degree bounded independently of t), we get for any $p = 1 + \delta$:

$$\int_{X_t} e^{-p\gamma}\omega_t^n \leqslant \int_{Y_t} e^{-\delta f^*\gamma} f^*(i^{n^2}\Omega_t \wedge \overline{\Omega}_t)^{\frac{1}{m}} \leqslant \int_{Y_t} |s_E|^{-2\delta A} d\mu_{X_t}.$$

Now, one can cover Y_t by finitely many open sets $U_t = U \cap Y_t$ as above. On U, the system of coordinates $(z_0, \dots, z_n)$ induces a system of coordinates $(z_1, \dots, z_n)$ on U_t such that we have

$$|s_E|^{-2\delta A}\mu_{X_t} \leqslant C \prod_{i=1}^{\ell} |z_i|^{-2(\delta A + a_i^-)} idz_1 \wedge d\bar{z}_1 \wedge \dots \wedge idz_n \wedge d\bar{z}_n$$

for some uniform constant C thanks to Equation (3.8). Since $U \Subset \mathbb{C}^{n+1}$, the U_t live in a fixed compact subset of $\mathbb{C}^n$ and the lemma follows by taking $\delta < \frac{1-\max a_i^-}{A}$. □

3.2.3. Proof of the uniform estimate

The first item of Proposition 3.3 is a consequence of Equation (3.6) and the following more general statement.

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Theorem 3.6. *Let $\pi : \mathcal{X} \to \mathbb{D}$ be a family as in Setup 3.1. Fix $p > 1$, $\lambda \in \mathbb{R}^+$, $h \in \mathcal{C}^\infty(\partial\mathcal{X})$ and $f_t \in L^p(X_t)$. There exists a unique plurisubharmonic function $u_t \in \mathrm{PSH}(X_t) \cap \mathcal{C}^0(\overline{X_t})$ solution of*

$$\begin{cases}(dd^c u_t)^n = f_t e^{\lambda u_t}\omega_t^n & \text{on } X_t\\ u_t|_{\partial X_t} = h_t.\end{cases}$$

Moreover, $\|u_t\|_{L^\infty(X_t)} \leqslant C\|f_t\|_{L^p(X_t)}^{1/n}$, where C only depends on p, n and $\|h_t\|_{L^\infty(\partial X_t)}$.

Proof. The existence and uniqueness of u_t is proved in [GGZ23, Theorem A] when $\lambda = 0$; the case $\lambda > 0$ can be treated similarly.

The key point here is to show that the solutions u_t are uniformly bounded on X_t when $t \in \mathbb{D}$ varies, as soon as the densities f_t are uniformly bounded in L^p. This follows from the analysis developed in [GGZ23], which is an extension of Kolodziej's technique [Koł98]. Indeed, [GGZ23, Proposition 1.8] (applied with $\nu = 0$) shows that u_t is globally bounded on X_t, while the uniform bound (3.6) allows one to show that this bound is also independent of t. We provide a sketch of the proof as a courtesy to the reader.

Step 1. We claim that there exists $m_1, C_1 \geqslant 1$ such that

$$\int_{X_t} \exp(-2^{-m_1} v_t)\omega_t^n \leqslant C_1, \tag{3.10}$$

for all $t \in \mathbb{D}$ and for all $v_t \in \mathcal{F}_t$, where

$$\mathcal{F}_t := \left\{ w \in \mathrm{PSH}(X_t) \cap L^\infty(X_t),\ w_{|\partial X_t} = 0 \text{ and } \int_{X_t} (dd^c w)^n \leqslant 1 \right\}.$$

Indeed, the family $\mathcal{F}_t$ is relatively compact, and any function w in $\overline{\mathcal{F}_t}$ belongs to the domain of definition of the complex Monge–Ampère operator, with zero boundary values and Monge–Ampère mass less than 1. It follows from Demailly's comparison theorem that the Lelong number $\nu(w, x)$ is less than 1 at a smooth point and less than $\mathrm{mult}(X_t, x)^{-1/n}$ if x is singular.

If $p_t : \widetilde{X_t} \to X_t$ is the blow up of a (single) smooth subvariety, then $\nu(w, p_t(y)) \leqslant \nu(w \circ p_t, y) \leqslant 2\nu(w, p_t(y))$ if y belongs to the exceptional set E_t of p_t, while $\nu(w \circ p_t, y) = \nu(w, p_t(y))$ otherwise. This can be checked by embedding X_t locally in $\mathbb{C}^N$ and using the explicit expression of the blow up of a smooth subvariety in the Euclidean space. We infer that there exists $m_1 \in \mathbb{N}$ such that $\nu(w \circ f, y) \leqslant 2^{m_1}\nu(w, f(y)) \leqslant 2^{m_1}$ for all $y \in Y_t$.

We thus have a uniform control of the Lelong numbers of the compact family $f^*\overline{\mathcal{F}_t}$. Using the subextension trick [GGZ23, Lemma 1.7], we further reduce to controlling

$$\int_{X'_t} \exp(-2^{-m_1} v_t)\omega_t^n = \int_{Y'_t} \exp(-2^{-m_1} v_t \circ f)\omega_t^n$$

on a relatively compact subset $X'_t \Subset X_t$. We finally invoke Skoda's uniform integrability theorem which holds for holomorphic families; see [DNGG23, Theorem 2.9].

Step 2. We claim that for all compact subsets $K \subset X_t$,

$$\mathrm{Vol}_{\omega_t}(K) \leqslant C_1 \exp\left(-\frac{1}{2^{m_1}\mathrm{Cap}(K, X_t)^{1/n}} \right), \tag{3.11}$$

where $\mathrm{Cap}(K, X_t) := \sup\{\int_K (dd^c w)^n,\ w \in \mathrm{PSH}(X_t) \text{ with } 0 \leqslant w \leqslant 1\}$ denotes the Monge–Ampère capacity.

Indeed, set $\lambda = \mathrm{Cap}(K, X_t)^{-1/n}$ and $v_t = \lambda h^*_{K,X_t}$, where h^*_{K,X_t} denotes the relative extremal function of the compact set K (see [GZ17, Definition 4.30]). It follows from [GZ17, Theorem 4.34] that

$\int_{X_t}(dd^c v_t)^n = 1$ and $v_t + \lambda = 0$ a.e. on K, hence

$$\begin{aligned}\mathrm{Vol}_{\omega_t}(K) &= \int_K \exp(-2^{-m}[\lambda + v_t])\omega_t^n \\ &\leqslant \exp(-2^{-m}\lambda)\int_{X_t}\exp(-2^{-m}v_t)\omega_t^n \\ &\leqslant C_1\exp\left(-\frac{1}{2^{m_1}\mathrm{Cap}(K,X_t)^{1/n}}\right),\end{aligned}$$

where the last inequality follows from Equation (3.10).

Step 3. Let Φ denotes the maximal psh extension of h to $\mathcal{X}$: This is the largest psh function in $\mathcal{X}$ which lies below h at the boundary. It is uniformly bounded in $\mathcal{X}$, satisfies $u_t \leqslant \Phi_t$ and coincides with h at the boundary. We claim that for all $s, \delta > 0$

$$\delta^n \mathrm{Cap}\{u_t - \Phi_t < -s - \delta - 1\} \leqslant \frac{c_{n,p}\|f_t\|_{L^p(X_t)}}{2^{m_1 q}}\mathrm{Cap}\{u_t - \Phi_t < -s - 1\}^2, \tag{3.12}$$

where $1/p + 1/q = 1$. It follows indeed from [GKZ08, Lemma 1.3] that

$$\delta^n \mathrm{Cap}\{u_t - \Phi_t < -s - \delta - 1\} \leqslant \int_{\{u_t - \Phi_t < -s-1\}}(dd^c u_t)^n.$$

Since $(dd^c u_t)^n = f_t\omega_t^n$, we can apply Hölder inequality, together with Equation (3.11) and the elementary inequality $\exp(-x^{-1/n}) \leqslant c_{n,p}x^{2q}$, valid for all $x > 0$, to conclude.

Conclusion. It follows from Equation (3.12) that the function $g(s) := \mathrm{Cap}\{u_t - \Phi_t < -s - 1\}^{1/n}$ satisfies $\delta g(s+\delta) \leqslant Bg(s)^2$ for all $s, \delta > 0$, with $B = c_{n,p}^{1/n}\|f_t\|_{L^p(X_t)}^{1/n}2^{-m_1 q/n}$. We can invoke DeGiorgi's lemma (see [GKZ08, Lemma 1.5] with $\tau = 1$) to conclude that $g(s) = 0$ for $s \geqslant 4Bg(0)$. Thus, $u_t \geqslant \Phi_t - 4Bg(0) - 1$, which yields a uniform lower bound on u_t if we can uniformly bound $g(0)$ from above.

To estimate $g(0) = \mathrm{Cap}\{u_t - \Phi_t < -1\}^{1/n}$, we let w_t denote the extremal function of the set $\{u_t - \Phi_t < -1\}$. Recall that $-1 \leqslant w_t \leqslant 0$, hence

$$w_t dd^c(\Phi_t - u_t)^n \leqslant n(-w_t)(\Phi_t - u_t)^{n-1}dd^c u_t \leqslant n(\Phi_t - u_t)^{n-1}dd^c u_t.$$

Using Stokes theorem n times, we thus obtain, following [Bło93],

$$\begin{aligned}\mathrm{Cap}\{u_t - \Phi_t < -1\} &\leqslant \int_{X_t}(\Phi_t - u_t)^n(dd^c w_t)^n \\ &\leqslant n!\int_{X_t}(dd^c u_t)^n \leqslant n!\|f_t\|_{L^p(X_t)}\mathrm{Vol}_{\omega_t}(X_t)^{1/q},\end{aligned}$$

which shows that $g(0)$ is uniformly bounded from above by $c_n'\|f_t\|_{L^p(X_t)}^{1/n}$. □

3.2.4. Existence of a suitable subsolution

The exceptional divisor E of g satisfies that there exist positive rational numbers $(b_i)_{i\in I}$ such that $-\sum_{i\in I} b_i E_i$ is g-ample, hence f-ample as well. On each $\mathcal{O}_{\mathcal{Y}}(E_i)$, one can pick a section s_i cutting out E_i as well as a smooth Hermitian metric h_i such that $\rho' := f^*(A\rho) + \sum_{i\in i} b_i \log|s_i|_{h_i}^2$ is strictly psh on $\mathcal{Y}$ for $A \gg 1$.

From now on, we assume that π is smooth away from our distinguished point $0 \in X_0 \subset \mathcal{X}$. This ensures for all $t \neq 0$, ρ' is bounded on $Y_t \simeq X_{\varphi(t)}$ and that ρ' is smooth on $\partial Y \simeq \partial X$.

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Next, one defines for $\delta > 0$ small enough (to be determined later) the function

$$v = v_\delta := \rho' + \sum_{i \in I} |s_i|_{h_i}^{2\delta}.$$

Up to scaling the metrics h_i, we find that v is strictly psh. More precisely, we can cover $\mathcal{Y}$ with finitely many coordinate charts $(U_\alpha)_\alpha$ such that $U_\alpha \cap E = (z_1 \cdots z_{\ell_\alpha} = 0)$ for some number $\ell_\alpha \leqslant n$ and in these charts we have

$$dd^c v|_{U_\alpha} \geqslant c \left[\omega_{\mathcal{Y}}|_{U_\alpha} + \sum_{k=1}^{\ell_\alpha} \frac{i dz_k \wedge d\bar{z}_k}{|z_k|^{2(1-\delta)}} \right] \tag{3.13}$$

for some $c > 0$ and some fixed Kähler metric $\omega_{\mathcal{Y}}$ on $\mathcal{Y}$. In particular, it follows from Equation (3.13) above and Equation (3.8) that

$$(dd^c v|_{Y_t})^n \geqslant c' f^* \mu_{X_t} \quad \text{on } Y_t$$

for some uniform $c' > 0$. Since v is uniformly bounded above on $\mathcal{Y}$ and F is bounded on $\mathcal{X}$, we can scale up v so that

$$(dd^c v|_{Y_t})^n \geqslant e^{\lambda v + f^* F_t} f^* \mu_{X_t} \quad \text{on } Y_t.$$

Given $t \in \mathbb{D}^*$, there exists s such that $\varphi(s) = t$, and we denote by v_t the function on X_t defined by $v|_{Y_s}$ under the identification $Y_s \simeq X_t$ via f. Clearly, v_t satisfies

$$(dd^c v_t)^n \geqslant e^{\lambda v_t + F_t} \mu_{X_t} \quad \text{on } X_t.$$

At this point, v_t is not a subsolution of Equation ($\mathrm{MA_t}$) because the boundary condition is not satisfied. So we pick a cut-off function χ compactly supported on $\mathcal{X}$ and satisfying $\chi \equiv 1$ near 0. Next, we still denote by h an arbitrary smooth extension of h from $\partial\mathcal{X}$ to $\mathcal{X}$. Finally, we set

$$\underline{u}_t := \chi v_t + B\rho + (1 - \chi) h.$$

One can easily see that for B large enough, $\underline{u}_t$ is a subsolution of Equation ($\mathrm{MA_t}$). Moreover, it is obvious on the shape of $\underline{u}_t$ that the estimates claimed in the second item of Proposition 3.3 are satisfied.

3.3. *Proof of Theorem 3.4*

In this final subsection, we borrow the notation of Setup 3.1 and assume that π is smooth outside of $0 \in X_0$. We consider the solution u_t of Equation ($\mathrm{MA_t}$).

First, we are going to derive higher-order estimates near ∂X_t of u_t of Equation ($\mathrm{MA_t}$). We will then conclude the proof of Theorem 3.4 by using Chern–Lu inequality as in the proof of Theorem 2.8.

Lemma 3.7. *There exists $C_1 > 0$ such that for all $t \neq 0$, $\|u_t\|_{\mathcal{C}^1(\partial X_t)} \leqslant C_1$.*

Proof. Let H be an arbitrary smooth extension of h to $\mathcal{X}$. For A large enough, the function $v := A\rho - H$ is a smooth psh function near $\overline{\mathcal{X}}$ such that $v_t = -h_t$ on ∂X_t. Thus, $u_t + v_t$ is psh in X_t with zero boundary values. It follow from the maximum principle that $u_t + v_t \leqslant 0$ in X_t. Using the subsolution constructed in the previous subsection, we obtain a two-sided bound

$$\underline{u}_t \leqslant u_t \leqslant -v_t,$$

with $\underline{u}_t = u_t = -v_t = h_t$ on ∂X_t. The desired uniform $\mathcal{C}^1$-bound on ∂X_t follows. □

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Once $\|u_t\|_{\mathcal{C}^1(\partial X_t)}$ is under control, one can obtain a global control of $\|u_t\|_{\mathcal{C}^1(V_t)}$ in a neighborhood V_t of ∂X_t that avoids the singular point. The proof of the following proposition is given in the appendix of the preprint version as Proposition 6.1.

Proposition 3.8. *There exists $C_1' > 0$ such that for all $t \neq 0$, $\|u_t\|_{\mathcal{C}^1(V_t)} \leqslant C_1'$.*

In the appendix of the preprint version, we explain how to derive Laplacian estimates near the boundary from lower order ones, cf. Theorem 6.3 there, as a combination of the main results of [CKNS85, GL10] in the setting of holomorphic families. It is then straightforward to obtain the following

Proposition 3.9. *There exists $C_2 > 0$ such that for all $t \neq 0$, we have*

$$\|\Delta u_t\|_{L^\infty(\partial X_t)} \leqslant C_2.$$

Proof. It follows from Proposition 3.3 and Lemma 3.7 that the assumptions of Theorem 6.3 in the preprint version are met and that we have uniform upper bounds on $\|u_t\|_{\mathcal{C}^1(\overline{V_t})}$, $\|v_t\|_{\mathcal{C}^2(\overline{V_t})}$, $\|f_t\|_{\mathcal{C}^1(\overline{V_t})}$, $\|h\|_{\mathcal{C}^4(\partial X_t)}$, ε^{-1}, δ^{-1}. The proposition follows. □

End of the proof of Theorem 3.4. Pick a Kähler metric $\widehat{\omega}$ on $\mathcal{X}$, and set $\widehat{\omega}_t := \widehat{\omega}|_{X_t}$. Since $(dd^c u_t)^n$ is uniformly comparable to $\widehat{\omega}_t^n$ in a small neighborhood of ∂X_t, the uniform bound $\|\Delta u_t\|_{L^\infty(\partial X_t)} \leqslant C_2$ actually yields a uniform constant $c_2 > 0$ such that

$$c_2^{-1}\widehat{\omega}_t \leqslant dd^c u_t \leqslant c_2 \widehat{\omega}_t \quad \text{on } \partial X_t. \tag{3.14}$$

Indeed, let $\sigma := \inf_t \inf_{p\in\partial X_t} \liminf_{z\to p, z\in\partial X_t} \frac{\theta_t^n(z)}{\widehat{\omega}_t^n(z)}$; we have $\sigma > 0$ since π is smooth along $\partial\mathcal{X}$ and $\|u_t\|_{L^\infty(X_t)}$ is uniformly bounded below by Theorem 3.6. Given $p \in \partial X_t$, we have $\limsup_{z\to p, z\in X_t} \operatorname{tr}_{\widehat{\omega}_t}\theta_t(z) \leqslant C$ by our boundary Laplacian estimate, hence

$$\limsup_{z\to p, z\in X_t} \operatorname{tr}_{\theta_t}\widehat{\omega}_t(z) \leqslant \sigma^{-1}C^{n-1},$$

and Equation (3.14) follows.

Arguing as in the proof of Theorem 2.8, we consider $v_t = \log \operatorname{tr}_{\theta_t}(\widehat{\omega}_t)$, where $\theta_t = dd^c u_t$ and deduce from the Chern–Lu formula that

$$\Delta_{\theta_t}(v_t - Au_t) \geqslant e^{v_t} - C_3, \tag{3.15}$$

for uniform constants $A, C_3 > 0$. We infer that $v_t \leqslant C_4$ is uniformly bounded from above. Indeed, either the maximum of the function $v_t - Au_t$ is reached in X_t, and the bound follows from Equation (3.15) and the uniform bound on u_t or the maximum of $v_t - Au_t$ is reached on ∂X_t and we conclude from Equation (3.14).

Therefore, $dd^c u_t \geqslant C_4^{-1}\widehat{\omega}_t$ in X_t for all $t \neq 0$. Similarly to what we have done at the end of the proof of Theorem 2.8, we conclude by letting $t \to 0$ that $dd^c u_0 \geqslant C_4^{-1}\widehat{\omega}_0$, hence $\omega_{\mathrm{KE}} = dd^c u_0$ is a Kähler current. The proof of Theorem 3.4 is complete.

4. Kähler–Einstein currents near isolated smoothable singularities

We now use the previous analysis to establish the strict positivity of singular Kähler–Einstein metrics of nonpositive curvature near smoothable isolated singularities.

4.1. *The case of klt spaces X*

Theorem 4.1. *Let X be a compact Kähler normal space with klt singularities such that either K_X is ample or $K_X \sim_{\mathbb{Q}} \mathcal{O}_X$. Then a Kähler–Einstein metric ω_{KE} in the sense of Definition 2.2 is a Kähler current near an isolated smoothable singularity of X.*

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Remark 4.2. In the case where X is a $\mathbb{Q}$-Fano Kähler–Einstein variety (i.e., $-K_X$ is ample), we expect a similar result to hold as well, but this requires a better understanding of local families of Kähler–Einstein metrics of positive curvature.

Proof. We work near an isolated singular point a. We let B denote a small strictly pseudoconvex neighborhood of a in X, isomorphic to the trace of a ball in some local embedding in $\mathbb{C}^N$, and let ρ denote a local smooth potential for $\omega = dd^c\rho$ in B.

Recall from [EGZ09, Pău08] that the Kähler–Einstein potential φ_{KE} is smooth in $B \setminus \{a\}$. Define $\lambda = 1$ if K_X is ample, and $\lambda = 0$ if $K_X \equiv 0$, and set $F = \lambda\rho$. The local theory recalled in Section 1.2 shows that $\psi = \rho + \varphi_{\mathrm{KE}}$ is the unique solution of the Dirichlet problem

$$(dd^c w)^n = e^{\lambda w + F}\mu_{(X,a),h} \quad \text{in } B, \quad \text{with} \quad w_{|\partial B} = \psi_{|\partial B}.$$

We assume that (X, a) is a smoothable singularity in the sense of Definition 3.2, and we let $\pi : \mathcal{X} \to \mathbb{D}$ denote a smoothing so that $B = \pi^{-1}(0)$ and $X_t = \pi^{-1}(t)$ is smooth for all $t \neq 0$. We let h denote a smooth extension of $(\rho + \varphi_{\mathrm{KE}})_{|\partial B}$ to $\partial\mathcal{X}$ and still denote by F a smooth extension of F to $\mathcal{X}$. It follows from Proposition 3.3 that there exists a unique function $u_t \in \mathrm{PSH}(X_t) \cap \mathcal{C}^0(\overline{X_t})$ such that

$$\begin{cases} (dd^c u_t)^n = e^{\lambda u_t + F_t}\mu_t & \text{on } X_t \\ u_t|_{\partial X_t} = h_t \end{cases}. \tag{4.1}$$

together with a uniform bound $\|u_t\|_{L^\infty(X_t)} \leqslant C_0$. We can thus apply Theorem 3.4 and conclude that $dd^c u_0 = dd^c\psi = \omega_{KE}$ is a Kähler current near a. □

4.2. The case of klt pairs (X, Δ)

We would now like to investigate the strict positivity of Kähler–Einstein currents ω_{KE} associated to compact klt pairs (X, Δ) near a smoothable isolated singularity $x \in X$. There are two possibilities.

Case 1. If $x \notin \mathrm{Supp}(\Delta)$, then one can find a neighborhood U of x such that $\partial U \cap \mathrm{Supp}(\Delta) = \emptyset$ so that ω_{KE} is smooth on ∂U, and the same arguments used in the proof of Theorem 4.1 will carry over mutatis mutandis to show that ω_{KE} is a Kähler current near x.

Case 2. If $x \in \mathrm{Supp}(\Delta)$, then as a singularity of the pair (X, Δ), it is *not isolated* anymore. This reflects on the metric side as well since on the boundary ∂U of a small neighborhood of x, ω_{KE} is not smooth anymore. More precisely, ω_{KE} has conic singularities, to be understood in a generalized sense since Δ may not have simple normal crossings (snc) support near ∂U. Even if Δ were smooth (or snc) away from x, the local analysis developed so far could not be applied directly and one would have to derive boundary Laplacian estimate in this singular conic setting which probably requires a lot of work. Instead, we can regularize the conic singularities *globally* to avoid boundary problems when applying Chern–Lu inequality. This will require us to assume that each component Δ_i of Δ is $\mathbb{Q}$-Cartier and that *any* singularity of X is isolated and smoothable.

We will state the main result of this section with a slightly weaker assumption than local smoothability, which will be useful later when we work with threefolds.

Definition 4.3. We say that a germ of normal complex space (X, x) is $\mathbb{Q}$-smoothable if there exists a finite Galois quasi-étale cover $p : Y \to X$ with Y normal and connected such that for all $y \in p^{-1}(x)$, (Y, y) is smoothable in the sense of Definition 3.2.

In the definition above, one can always shrink Y and assume that $p^{-1}(x)$ is a singleton.

It will be convenient to introduce the following setup.

Setup 4.4. *Let (X, ω_X) be an n-dimensional compact Kähler space endowed with an effective $\mathbb{Q}$-divisor $\Delta = \sum a_i\Delta_i$ such that (X, Δ) has klt singularities. We assume that each component Δ_i of Δ is a $\mathbb{Q}$-Cartier divisor and that X has only $\mathbb{Q}$-smoothable, isolated singularities.*

We are now ready to state the main result.

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Theorem 4.5. *Let* (X, Δ) *be as in Setup 4.4, and consider the unique (normalized) solution* $\varphi \in \mathrm{PSH}(X, \omega_X) \cap L^\infty(X)$ *of the Monge–Ampère equation*

$$(\omega_X + dd^c\varphi)^n = e^{\lambda\varphi + F} d\mu_{(X,\Delta),h}$$

where $F \in \mathcal{C}^\infty(X)$*, h is a smooth Hermitian metric on the* $\mathbb{Q}$*-line bundle* $K_X + \Delta$ *and* $\lambda \in \{0, 1\}$*.*

Then, the current $\omega_\varphi = \omega_X + dd^c\varphi$ *is a Kähler current.*

As an immediate consequence of the theorem above, we get:

Corollary 4.6. *Let* (X, Δ) *be as in Setup 4.4 such that* $K_X + \Delta$ *is ample (resp.* $K_X + \Delta \sim_{\mathbb{Q}} \mathcal{O}_X$*). Then the unique Kähler–Einstein metric* ω_{KE} *(resp.* $\omega_{\mathrm{KE}} \in \{\omega_X\}$*) solving*

$$\mathrm{Ric}\,\omega_{\mathrm{KE}} = -\omega_{\mathrm{KE}} + [\Delta] \quad (resp.\ \mathrm{Ric}\,\omega_{\mathrm{KE}} = [\Delta])$$

is a Kähler current.

Remark 4.7. We would like to add two comments on the result above

- It is conceivable that one could remove the assumption on the components Δ_i of Δ being $\mathbb{Q}$-Cartier by considering a $\mathbb{Q}$-factorialization $Y \longrightarrow X$ of X, but it is not completely clear how the smoothability assumption would lift to Y.
- There is a noticeable difference between the assumptions of Theorem 4.1 and Corollary 4.6, as in the latter one we need to assume that all singularities are isolated. It has to do with the fact that we deal with isolated singular points that may not be isolated as singularity of the pair (X, Δ) as explained in the beginning of §4.2, and this requires a subtle combination of local and global methods. If one is only interested in isolated singular points in $X \setminus \mathrm{Supp}(\Delta)$, then we would not need any global assumptions on the singularities of X elsewhere, cf *ibid*.

Proof of Theorem 4.5. By assumption, one can find an integer $m > 0$ and sections $s_i \in H^0(X, \mathcal{O}_X(m\Delta_i))$ such that $\mathrm{div}(s_i) = m\Delta_i$. We pick Hermitian metrics h_i on the $\mathbb{Q}$-line bundle $O_X(\Delta_i)$ and define $|s_i|^2 := |s_i|^2_{h_i^{\otimes m}}$. Since K_X is $\mathbb{Q}$-Cartier (as a difference $K_X = (K_X + \Delta) - \Delta$) of $\mathbb{Q}$-Cartier divisors), one can find a metric h_X on K_X such that $h = h_X \otimes \bigotimes_i h_i^{\otimes a_i}$. Setting $b_i := \frac{a_i}{m}$, one can rewrite the Monge-Ampère equation solved by φ as

$$(\omega_X + dd^c\varphi)^n = e^{\lambda\varphi + F} \frac{d\mu_{X,h_X}}{\prod_i |s_i|^{2b_i}}.$$

We consider the unique (normalized) solution φ_ε of the regularized equation

$$(\omega_X + dd^c\varphi_\varepsilon)^n = e^{\lambda\varphi_\varepsilon + F} \frac{d\mu_{X,h_X}}{\prod_i (|s_i|^2 + \varepsilon^2)^{b_i}}.$$

and we set $\omega_\varepsilon := \omega_X + dd^c\varphi_\varepsilon$. It follows from [EGZ09, Pău08] that ω_ε is a Kähler form on X_{reg} for any $\varepsilon > 0$, and moreover there is a uniform constant $C_0 > 0$ such that

$$\|\varphi_\varepsilon\|_{L^\infty(X)} \leqslant C_0 \quad \text{and} \quad \varphi_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \varphi \quad \text{in } L^1(X). \tag{4.2}$$

thanks to [GZ12, Theorem C] applied to a desingularization of X. Therefore, we can reduce the proof of the theorem to showing that there exists a constant $C > 0$ independent of ε such that

$$\omega_\varepsilon \geqslant C^{-1}\omega_X \quad \text{on } X. \tag{4.3}$$

Claim 4.8. *The uniform inequality (4.3) holds if for any* $\varepsilon > 0$*, one has the qualitative inequality*

$$\omega_\varepsilon \geqslant C_\varepsilon^{-1}\omega_X \quad on\ X, \tag{4.4}$$

where $C_\varepsilon > 0$ *is a positive constant that may depend on* ε*.*

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Proof of Claim 4.8. Here again, the key tool is Chern–Lu inequality, Proposition 2.6. The bisectional curvature of ω_X is bounded above since ω_X can be extended to a Kähler metric in local embeddings in $\mathbb{C}^N$, and we need to bound $\operatorname{Ric}\omega_\varepsilon$ from below.

A classical computation shows that if s is a holomorphic section cutting out a divisor D and if h_D is a smooth Hermitian metric on $L := \mathcal{O}_X(D)$, then

$$\Theta_{h_D}(L) + dd^c \log(|s|^2_{h_D} + \varepsilon^2) = \frac{\varepsilon^2}{(|s|^2_{h_D} + \varepsilon^2)^2} \cdot |D's \wedge \overline{D's}|^2_{h_D} - \frac{\varepsilon^2}{|s|^2_{h_D} + \varepsilon^2} \cdot \Theta_{h_D}(L).$$

In particular, if one choses a constant $C_D > 0$ such that $\Theta_{h_D}(L) \leqslant C_D\omega_X$, then

$$\Theta_{h_D}(L) + dd^c \log(|s|^2_{h_D} + \varepsilon^2) \geqslant -C_D\omega_X. \tag{4.5}$$

Since

$$\operatorname{Ric}\omega_\varepsilon = -\lambda\omega_\varepsilon + \lambda\omega_X - dd^c F + \sum_i b_i\Big(\Theta_{h_i^{\otimes m}}(\mathcal{O}_X(m\Delta_i)) + dd^c \log(|s_i|^2 + \varepsilon^2)\Big)$$

and F is smooth (so that its Hessian is bounded with respect to ω_X), inequality (4.5) ensures that one can find $C_1 > 0$ such that

$$\operatorname{Ric}\omega_\varepsilon = -\omega_\varepsilon - C_1\omega_X.$$

Finally, let $\psi \in \mathrm{PSH}(X, \omega_X)$ be such that $(\psi = -\infty) = X_{\mathrm{sing}}$ and $\psi|_{X_{\mathrm{reg}}} \in \mathcal{C}^\infty(X_{\mathrm{reg}})$. For any $\delta > 0$, the smooth quantity

$$\log \operatorname{tr}_{\omega_\varepsilon}\omega_X + \delta\psi$$

on X_{reg} tends to $-\infty$ near X_{sing} thanks to our assumption (4.4). In particular, the quantity above achieves its maximum on X_{reg}. Moreover, our curvature estimates coupled with Proposition 2.6 yield a constant $C_2 > 0$ such that

$$\Delta_{\omega_\varepsilon}\big[\log \operatorname{tr}_{\omega_\varepsilon}\omega_X + \delta\psi\big] \geqslant -1 - C_2\operatorname{tr}_{\omega_\varepsilon}\omega_X$$

on X_{reg}. Since $-\Delta_{\omega_\varepsilon}\varphi_\varepsilon \geqslant -n + \operatorname{tr}_{\omega_\varepsilon}\omega_X$, we infer that

$$\Delta_{\omega_\varepsilon}\big[\log \operatorname{tr}_{\omega_\varepsilon}\omega_X + \delta\psi - (C_2+1)\varphi_\varepsilon\big] \geqslant \operatorname{tr}_{\omega_\varepsilon}\omega_X - C_3$$

where $C_3 = -1 + n(C_2 + 1)$. A classical application of the maximum principle shows that

$$\operatorname{tr}_{\omega_\varepsilon}\omega_X \leqslant C_4 e^{\delta(\sup_X \psi - \psi)}$$

with $C_4 = C_3 e^{2C_0(C_2+1)}$. Passing to the limit when $\delta \to 0$, we get

$$\omega_\varepsilon \geqslant C_4^{-1}\omega_X \quad \text{on } X_{\mathrm{reg}},$$

hence everywhere on X by Lemma 2.5. The claim follows. □

In order to prove the theorem, we are left to establishing the qualitative estimate (4.4). In order to achieve that, we use a local deformation argument. From now on, $\varepsilon > 0$ is fixed and all subsequent constants are allowed to depend on ε. Since ω_ε is smooth on X_{reg}, it is enough to work in small neighborhoods of the finitely many singular points in X. From now on, we pick a small Stein neighborhood U' of x admitting a quasi-étale cover $p : V' \to U'$ such that $p^{-1}(x) = \{y\}$ is a singleton and (V', y) admits a smoothing $\mathcal{V} \to \mathbb{D}$ whose fibers V_t satisfy $V_0 \simeq V'$.

One can assume that U' is small enough so that $\omega_X|_{U'} = dd^c\rho$ for some smooth strictly psh function ρ on U'. Next, since p is quasi-étale, we have $K_{V'} = p^*K_{U'}$ and the smooth Hermitian metric $h_{U'} := h_X|_{U'}$ on $K_{U'}$ pulls back to a smooth Hermitian metric $h_{V'}$ on $K_{V'}$ satisfying $p^*\big(d\mu_{U',h_X}\big) = d\mu_{V',h_{V'}}$.

Next, we fix $U \Subset U'$ strongly pseudoconvex and set $V := p^{-1}(U)$. The function $v_\varepsilon := p^*(\rho + \varphi_\varepsilon)$ satisfies

$$\begin{cases}(dd^c v_\varepsilon)^n = e^{\lambda v_\varepsilon + p^*(F-\lambda\rho)}\frac{d\mu_{V,h_V}}{\prod_i(|p^*s_i|^2+\varepsilon^2)^{b_i}} & \text{on } V\\ v_\varepsilon|_{\partial V} = p^*((\rho+\varphi_\varepsilon)|_{\partial U}),\end{cases}$$

hence by Theorem 3.4 the current $dd^c v_\varepsilon$ is a Kähler current. In particular, it dominates a multiple of $p^*(\omega_X)|_U$. Pushing forward, we get that in restriction to U, we have $\omega_\varepsilon \geqslant C_\varepsilon^{-1}\omega_X$ and the theorem is proved. □

5. Kähler–Einstein currents on threefolds

In this final section, we provide two results in dimension three (zero and negative curvature) ensuring that Kähler–Einstein metrics on a compact klt space are Kähler currents without any extra assumption on the singularities. Although the proofs of the two results follow the same lines, we have chosen to write them separately to highlight the nontrivial simplifications occurring in the zero curvature case (or dually to insist on the consequential additional difficulties popping up in the negative curvature case).

The crucial input specific to dimension three is Reid's classification of terminal singularities:

Theorem 5.1. *([Rei80]) Let (X, x) be an (isolated) terminal singularity of dimension three such that $K_X \sim_{\mathbb{Z}} \mathcal{O}_X$. Then (X, x) is a compound du Val singularity. In particular, any terminal singularity of dimension three is $\mathbb{Q}$-smoothable.*

Recall that a compound du Val singularity is a hypersurface singularity isomorphic to $(f + tg = 0) \subset \mathbb{C}^3 \times \mathbb{C}$, where $f, g \in \mathbb{C}[x, y, z]$ are such that $(f = 0) \subset \mathbb{C}^3$ is a du Val surface singularity, cf, for example, [KM98, §4.2 & Definition 5.32]. Such a singularity can be smoothed out, for example, by $(f + tg + s = 0) \subset \mathbb{C}^3 \times \mathbb{C} \times \mathbb{C}$, where the total space is a smooth hypersurface of $\mathbb{C}^5$.

The second statement follows from the first after considering the quasi-étale index one cover $Y \to X$ making K_Y Cartier [KM98, Definition 5.19].

5.1. *Calabi–Yau threefolds with klt singularities*

Theorem 5.2. *Let (X, ω_X) be a normal compact Kähler space of dimension three with klt singularities such that $K_X \sim_{\mathbb{Q}} \mathcal{O}_X$. Then the Kähler–Einstein metric $\omega_{\mathrm{KE}} \in [\omega_X]$ is a Kähler current.*

Proof. Let $p : Y \to X$ be the quasi-étale index one cover of X. Clearly, $p^*\omega_{\mathrm{KE}}$ is the Kähler–Einstein metric in the Kähler class $p^*[\omega_X]$ so that if we prove the statement for Y, it will follow for X by push-forward (since any Kähler form on Y dominates a multiple of $p^*\omega_X$).

From now on, one can assume that $K_X \sim_{\mathbb{Z}} \mathcal{O}_X$ and, in particular, X has canonical singularities. Let us consider a terminalization $\pi : \widehat{X} \to X$ of X, cf [KM98, Theorem 6.23]. The complex space $\widehat{X}$ is Kähler, and it has terminal singularities, hence isolated singularities. Moreover, one has $K_{\widehat{X}} = \pi^*K_X$, that is, π is crepant (but in general, the exceptional locus of π might have codimension one components). This implies that $K_{\widehat{X}} \sim_{\mathbb{Z}} \mathcal{O}_{\widehat{X}}$ so that $\widehat{X}$ has smoothable singularities by Theorem 5.1. Moreover, since π is crepant, it is automatically isomorphic over X_{reg}.

We choose $\widehat{\omega}$ a Kähler metric on $\widehat{X}$, and one considers the singular Ricci flat metric $\omega_\varepsilon \in [\pi^*\omega_X + \varepsilon\widehat{\omega}]$. One can write $\omega_\varepsilon = \pi^*\omega_X + \varepsilon\widehat{\omega} + dd^c\varphi_\varepsilon$ with $\sup_{\widehat{X}}\varphi_\varepsilon = 0$, where φ_ε is a solution of $(\pi^*\omega_X + \varepsilon\widehat{\omega} + dd^c\varphi_\varepsilon)^n = c_\varepsilon\mu_{\widehat{X}} = c_\varepsilon\hat{f}\widehat{\omega}^n$, where $c_\varepsilon = \frac{[\pi^*\omega_X+\varepsilon\widehat{\omega}]^n}{\mu_{\widehat{X}}(\widehat{X})}$ and $\hat{f} \in L^{1+\delta}(\widehat{X}, \widehat{\omega}^n)$ with $\delta > 0$ since $\hat{X}$ has terminal singularities. It follows from the techniques in [EGZ08, DP10] that

$$\|\varphi_\varepsilon\|_{L^\infty(\widehat{X})} \leqslant C_0, \quad \text{and} \quad \omega_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \pi^*\omega_{\mathrm{KE}} \quad \text{weakly}, \tag{5.1}$$

where $C_0 > 0$ is independent of ε. The discerning reader will have noticed that the situation is slightly different from the one in *loc. cit.* since X is singular. To patch this little gap, one could, for example, appeal to [DNGG23, Theorem A] applied to a desingularization of $\widetilde{X} \to \widehat{X}$ with reference form the pull-back of $\pi^*\omega_X + \varepsilon\widehat{\omega}$ to $\widetilde{X}$, and this would yield the uniform estimate. As for the stability statement, it is a classical consequence of the uniform estimate: One first gets higher-order estimates locally on X_{reg} using Tsuji's trick and then one uses uniqueness of the Kähler–Einstein metric to conclude that the relatively compact family $(\varphi_\varepsilon)_{\varepsilon>0}$ has a single cluster value in $L^1(\widehat{X})$ which is nothing but φ_0.

The main point to establish is that there is a constant $C > 0$ independent of ε such that

$$\omega_\varepsilon \geqslant C^{-1}\pi^*\omega_X. \tag{5.2}$$

Indeed, passing to the limit when $\varepsilon \to 0$ would then imply that $\pi^*\omega_{\mathrm{KE}} \geqslant C^{-1}\pi^*\omega_X$ and the theorem would follow by pushing forward by π.

Now, we know that inequality (5.2) holds for any $\varepsilon > 0$ with a constant $C = C_\varepsilon$ depending a priori from ε, as this is the content of Theorem 4.1. As before, we will use Chern–Lu inequality to make this qualitative control quantitative.

Choose a function $\psi \in \mathrm{PSH}(X, \omega_X)$ be such that $(\psi = -\infty) = X_{\mathrm{sing}}$ and $\psi|_{X_{\mathrm{reg}}} \in \mathcal{C}^\infty(X_{\mathrm{reg}})$. For any $\delta > 0$, the smooth quantity

$$\log \mathrm{tr}_{\omega_\varepsilon}\pi^*\omega_X + \delta\psi$$

on $\pi^{-1}(X_{\mathrm{reg}}) \subset \widehat{X} \setminus \mathrm{Exc}(\pi)$ tends to $-\infty$ near $\pi^{-1}(X_{\mathrm{sing}})$ since ω_ε is a Kähler form. In particular, the quantity above achieves its maximum on X_{reg}.

Next, the bisectional curvature of $\pi^*\omega_X$ is well defined and bounded above uniformly on $\pi^{-1}(X_{\mathrm{reg}})$, while $\mathrm{Ric}\,\omega_\varepsilon = 0$. By Chern–Lu inequality (Proposition 2.6), we find a constant $C_2 > 0$ such that

$$\Delta_{\omega_\varepsilon}\left[\log \mathrm{tr}_{\omega_\varepsilon}\pi^*\omega_X + \delta\psi\right] \geqslant -C_2\mathrm{tr}_{\omega_\varepsilon}\pi^*\omega_X$$

on $\pi^{-1}(X_{\mathrm{reg}})$. Since $-dd^c\varphi_\varepsilon = -\omega_\varepsilon + \pi^*\omega_X + \varepsilon\widehat{\omega}$, we have

$$-\Delta_{\omega_\varepsilon}\varphi_\varepsilon \geqslant -n + \mathrm{tr}_{\omega_\varepsilon}\pi^*\omega_X,$$

hence

$$\Delta_{\omega_\varepsilon}\left[\log \mathrm{tr}_{\omega_\varepsilon}\pi^*\omega_X + \delta\psi - (C_2+1)\varphi_\varepsilon\right] \geqslant \mathrm{tr}_{\omega_\varepsilon}\pi^*\omega_X - C_3,$$

where $C_3 = -1 + n(C_2+1)$. Using Equation (5.1), we obtain in the end

$$\mathrm{tr}_{\omega_\varepsilon}\pi^*\omega_X \leqslant C_4 e^{\delta(\sup_X \psi - \psi)}$$

with $C_4 = C_3 e^{C_0(C_2+1)}$, holding on $\pi^{-1}(X_{\mathrm{reg}})$ for any $\delta > 0$. Equivalently, we have $\omega_\varepsilon \geqslant C_4 e^{-\delta(\sup_X \psi - \psi)}\pi^*\omega_X$. After passing to the limit when $\delta \to 0$, one can appeal to Lemma 2.5 to obtain Equation (5.2), hence the theorem. □

5.2. Canonically polarized threefolds with klt singularities

Theorem 5.3. *Let X be a normal projective variety of dimension three with klt singularities such that K_X is ample. Then the Kähler–Einstein metric ω_{KE} is a Kähler current.*

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Proof. The reduction to the terminal case is a bit more involved here. More precisely, we cannot easily assume that X has canonical singularities without resorting to a local argument which would collapse since the singularities of X are not isolated. Instead, we use a $\mathbb{Q}$-factorial terminalization $\pi : \widehat{X} \to X$ whose existence (in any dimension) is guaranteed by [BCHM10, Corollary 1.4.3]. The map π is such that there exists an effective divisor $\widehat{\Delta}$ on $\widehat{X}$ such that $K_{\widehat{X}} + \widehat{\Delta} = \pi^* K_X$ and $(\widehat{X}, \widehat{\Delta})$ is a terminal pair, hence $\widehat{X}$ has isolated $\mathbb{Q}$-smoothable singularities by Theorem 5.1. Moreover, $\widehat{X}$ is $\mathbb{Q}$-factorial.

Next, let $\widehat{\omega}$ be a Kähler metric on $\widehat{X}$ and let us consider the twisted Kähler–Einstein metric $\omega_\varepsilon \in c_1(\pi^* K_X) + \varepsilon[\widehat{\omega}]$, that is, the solution of

$$\operatorname{Ric} \omega_\varepsilon = -\omega_\varepsilon + \varepsilon \widehat{\omega} + [\widehat{\Delta}].$$

If $\omega_X \in c_1(K_X)$ is a Kähler metric, say the curvature of an Hermitian metric h on K_X, one can write $\omega_\varepsilon = \pi^* \omega_X + \varepsilon \widehat{\omega} + dd^c \varphi_\varepsilon$, where φ_ε is a solution of

$$(\pi^* \omega_X + \varepsilon \widehat{\omega} + dd^c \varphi_\varepsilon)^n = e^{\varphi_\varepsilon} \mu_{(\hat{X}, \widehat{\Delta}), \pi^* h} = e^{\varphi_\varepsilon} \hat{f} \widehat{\omega}^n,$$

with $\hat{f} \in L^{1+\delta}(\widehat{X}, \widehat{\omega}^n)$ with $\delta > 0$ since $\hat{X}$ has terminal singularities. As in the Calabi–Yau case, one has

$$\|\varphi_\varepsilon\|_{L^\infty(\widehat{X})} \leqslant C, \quad \text{and} \quad \omega_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \pi^* \omega_{\mathrm{KE}} \quad \text{weakly,} \tag{5.3}$$

where $C > 0$ is independent of ε. The only difference with the Calabi–Yau case is that we first need to show that there exists $C > 0$ independent of ε such that

$$\sup_{\widehat{X}} \varphi_\varepsilon \leqslant C. \tag{5.4}$$

This is certainly classical, but we recall the argument for the reader's convenience. First, one can assume without loss of generality that $\mu := \mu_{(\hat{X}, \widehat{\Delta}), \pi^* h}$ is a probability measure. By Jensen's inequality, we infer that

$$\int_{\widehat{X}} \varphi_\varepsilon d\mu \leqslant \log \int_{\widehat{X}} (\pi^* \omega_X + \varepsilon \widehat{\omega})^n \leqslant C_1$$

for some $C_1 > 0$ independent of ε. Let $p : \widetilde{X} \to \widehat{X}$ be a resolution of singularities, and let $\omega_{\widetilde{X}}$ be a Kähler metric on $\widetilde{X}$. Up to scaling the latter metric, one can assume that $p^*(\pi^* \omega_X + \varepsilon \widehat{\omega}) \leqslant \omega_{\widetilde{X}}$ and $p^* \mu \leqslant \omega_{\widetilde{X}}^n$, where the latter follows since $\widehat{X}$ has terminal (hence canonical) singularities. By the usual compactness properties of $\omega_{\widetilde{X}}$-psh functions, we find a constant C_2 such that

$$\int_{\widetilde{X}} (\sup_{\widetilde{X}} \psi - \psi) \omega_{\widetilde{X}}^n \leqslant C_2$$

for any $\psi \in \operatorname{PSH}(\widetilde{X}, \omega_{\widetilde{X}})$. Now,

$$\begin{aligned} \sup_{\widehat{X}} \varphi_\varepsilon &= \int_{\widehat{X}} (\sup_{\widehat{X}} \varphi_\varepsilon - \varphi_\varepsilon) d\mu + \int_{\widehat{X}} \varphi_\varepsilon d\mu \\ &\leqslant \int_{\widetilde{X}} (\sup_{\widetilde{X}} p^* \varphi_\varepsilon - p^* \varphi_\varepsilon) \omega_{\widetilde{X}}^n + C_1 \\ &\leqslant C_1 + C_2 \end{aligned}$$

since $p^* \varphi_\varepsilon \in \operatorname{PSH}(\widetilde{X}, \omega_{\widetilde{X}})$. Hence, Equation (5.4) follows.

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Coming back to proof of the main result, the remaining point to establish is that there is a constant $C > 0$ independent of ε such that

$$\omega_\varepsilon \geqslant C^{-1}\pi^*\omega_X. \tag{5.5}$$

Thanks to Theorem 4.5, inequality (5.5) holds for any $\varepsilon > 0$ with a constant $C = C_\varepsilon$ depending a priori from ε. In order to make this bound quantitative, the exact same strategy as in the Calabi–Yau case applies. □

Competing interest. The authors have no competing interest to declare.

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