# INTEGRALS INVOLVING PRODUCTS OF BESSEL FUNCTIONS

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§ 1. *Introductory.* The first formula to be proved is

$$\int_0^\infty \lambda^{k-1} K_{2m}(z/\surd\lambda) K_n(\lambda)\, d\lambda$$

$$= \sum_{m,-m} \Gamma\left(\frac{k+m+n}{2}\right) \Gamma\left(\frac{k+m-n}{2}\right) \Gamma(2m) 2^{k+3m-3} z^{-2m}$$

$$\times F\left( ; \tfrac{1}{2}, 1-\frac{k+m+n}{2}, 1-\frac{k+m-n}{2}, \tfrac{1}{2}-m, 1-m ; 4^{-5}z^4\right)$$

$$- \sum_{m,-m} \Gamma\left(\frac{k+m+n-1}{2}\right) \Gamma\left(\frac{k+m-n-1}{2}\right) \Gamma(2m-1) 2^{k+3m-6} z^{2-2m}$$

$$\times F\left( ; \tfrac{3}{2}, \frac{3-k-m-n}{2}, \frac{3-k-m+n}{2}, 1-m, \tfrac{3}{2}-m ; 4^{-5}z^4\right)$$

$$+ \sum_{n,-n} \Gamma(-n)\Gamma(-k-m-n)\Gamma(-k+m-n)\, 2^{-2k-3n-2} z^{2k+2n}$$

$$\times F\left( ; 1+n, 1+\frac{k+m+n}{2}, \frac{1+k+m+n}{2}, 1+\frac{k-m+n}{2}, \frac{1+k-m+n}{2} ; 4^{-5}z^4\right), \quad (1)$$

where $R(z)>0$.

The proof is given in § 2, and a second formula is established in § 3.

§ 2. *Proof of the formula.* In (1), formula (1), replace $k$ by $k-m$ and $z$ by $4/x^2$ and take $p=0$, $q=1$, $\rho_1=2m+1$ ; then, if $x$ is real and positive and $R(k\pm n)>-\frac{3}{4}$,

$$\int_0^\infty \lambda^{k-1} J_{2m}(x/\surd\lambda)\, K_n(\lambda)\, d\lambda$$

$$= \frac{2^{k-3m-2}\pi^3 (x/2)^{2m}}{\sin\left(\frac{k-m+n}{2}\pi\right)\sin\left(\frac{k-m-n}{2}\pi\right)}$$

$$\times E\left( : \tfrac{1}{2}, 1-\frac{k-m+n}{2}, 1-\frac{k-m-n}{2}, m+\tfrac{1}{2}, m+1 : e^{\pm i\pi} 4^5/x^4\right)$$

$$- \frac{2^{k-3m-5}\pi^3 (x/2)^{2m+2}}{\cos\left(\frac{k-m+n}{2}\pi\right)\cos\left(\frac{k-m-n}{2}\pi\right)}$$

$$\times E\left( : \tfrac{3}{2}, \frac{3-k+m-n}{2}, \frac{3-k+m+n}{2}, m+1, m+\tfrac{3}{2} : e^{\pm i\pi} 4^5/x^4\right)$$

$$+ \sum_{n,-n} \frac{2^{-2k-3n-1}\pi^3 (x/2)^{2k+2n}}{\sin n\pi \sin (k-m+n)\pi}$$

$$\times E\left( : 1+n, \frac{k-m+n+1}{2}, \frac{k-m+n+2}{2}, \frac{k+m+n+1}{2}, \frac{k+m+n+2}{2} : e^{\pm i\pi} 4^5/x^4\right).$$

............(2)

Now $$i^{2m} G_{2m}(z) = \tfrac{1}{2}\pi \sum_{m,\,-m} i^{2m} J_{-2m}(z) \operatorname{cosec}(2m\pi). \qquad \text{......(3)}$$

Hence, if $0 \leqq \operatorname{amp} z \leqq \pi$, $R(k \pm n) > -\tfrac{3}{4}$,

$$i^{2m}\int_0^\infty \lambda^{k-1} G_{2m}(z/\sqrt{\lambda}) K_n(\lambda)\, d\lambda$$

$$= \sum_{m,\,-m} \frac{2^{k+3m-3}\pi^4 (2i/z)^{2m}}{\sin(2m\pi)\sin\left(\frac{k+m+n}{2}\pi\right)\sin\left(\frac{k+m-n}{2}\pi\right)}$$

$$\times E\left( \ : \tfrac{1}{2},\, 1-\frac{k+m+n}{2},\, 1-\frac{k+m-n}{2},\, \tfrac{1}{2}-m,\, 1-m : e^{\pm i\pi}4^5/z^4\right)$$

$$+ \sum_{m,\,-m} \frac{2^{k+3m-6}\pi^4 (2i/z)^{2m-2}}{\sin(2m\pi)\cos\left(\frac{k+m+n}{2}\pi\right)\cos\left(\frac{k+m-n}{2}\pi\right)}$$

$$\times E\left( \ : \tfrac{3}{2},\, \frac{3-k-m+n}{2},\, \frac{3-k-m-n}{2},\, 1-m,\, \tfrac{3}{2}-m : e^{\pm i\pi}4^5/z^4\right)$$

$$+ \sum_{n,\,-n} \frac{2^{k-2}\pi^4 (z^2/32)^{k+n}}{\sin n\pi \sin(2m\pi)\sin(k+m+n)\pi \sin(k-m+n)\pi}$$

$$\times E\left( : 1+n,\, \frac{k+m+n+1}{2},\, \frac{k+m+n+2}{2},\, \frac{k-m+n+1}{2},\, \frac{k-m+n+2}{2} : e^{\pm i\pi}4^5/z^4\right)$$

$$\times \{ i^{2m}\sin(k-m+n)\pi - i^{-2m}\sin(k+m+n)\pi \}.$$

But the last line is equal to $-\sin(2m\pi)\, i^{-2k-2n}$; hence, on replacing $z$ by $iz$ and applying the formula

$$i^{2m} G_{2m}(iz) = K_{2m}(z), \qquad \text{......(4)}$$

formula (1) is obtained.

§ 3. *A second Bessel Function Integral.* The formula to be proved is

$$\int_0^\infty \lambda^{k-1} J_{2m}(x/\sqrt{\lambda})\, J_n(\lambda)\, d\lambda$$

$$= \frac{2^{k-3m-1}\Gamma\left(\frac{k-m+n}{2}\right)}{\Gamma\left(1-\frac{k-m-n}{2}\right)\Gamma(1+2m)} x^{2m} F\left( \ ; \tfrac{1}{2},\, 1-\frac{k-m+n}{2},\, 1-\frac{k-m-n}{2},\, \tfrac{1}{2}+m,\, 1+m;\ -4^{-5}x^4\right)$$

$$- \frac{2^{k-3m-4}\Gamma\left(\frac{k-m+n-1}{2}\right)}{\Gamma\left(\frac{3-k+m+n}{2}\right)\Gamma(2+2m)} x^{2m+2}$$

$$\times F\left( \ ; \tfrac{3}{2},\, \frac{3-k+m-n}{2},\, \frac{3-k+m+n}{2},\, 1+m,\, \tfrac{3}{2}+m;\ -4^{-5}x^4\right)$$

$$+ \frac{2^{-2k-3n}\Gamma(m-k-n)}{\Gamma(k+m+n+1)\Gamma(1+n)} x^{2k+2n}$$

$$\times F\left( \ ; 1+\frac{k-m+n}{2},\, \frac{1+k-m+n}{2},\, 1+\frac{k+m+n}{2},\, \frac{1+k+m+n}{2},\, 1+n;\ -4^{-5}x^4\right), \quad (5)$$



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where $x$ is real and positive and $R(m-k) > -\frac{3}{2}$, $R(k+n) > -\frac{3}{4}$.

Now the L.H.S. of (2) can be written

$$x^{2k} \int_0^\infty \lambda^{k-1} J_{2m}(1/\surd\lambda) K_n(\lambda x^2)\, d\lambda.$$

Here let amp $x$ decrease by $\frac{1}{4}\pi$, so that $x^2$ becomes $x^2/i$ ; then, since

$$K_n(z) = i^n G_n(iz),$$

the L.H.S. becomes

$$i^{n-k} x^{2k} \int_0^\infty \lambda^{k-1} J_{2m}(1/\surd\lambda) G_n(\lambda x^2)\, d\lambda = i^{n-k} \int_0^\infty \lambda^{k-1} J_{2m}(x/\surd\lambda) G_n(\lambda)\, d\lambda,$$

where $x$ is real and positive and $R(k \pm n) > -\frac{3}{4}$, $R(m-k) > -\frac{3}{2}$.

Similarly, if amp $x$ increases by $\frac{1}{4}\pi$, the L.H.S. of (2) becomes

$$i^{n+k} \int_0^\infty \lambda^{k-1} J_{2m}(x/\surd\lambda)\, G_n(e^{i\pi}\lambda)\, d\lambda,$$

where $x$ is real and positive and $R(k \pm n) > -\frac{3}{4}$, $R(m-k) > -\frac{3}{2}$.

Hence, on applying the formula

$$\pi i J_n(\lambda) = G_n(\lambda) - i^{2n} G_n(e^{i\pi}\lambda), \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

formula (5) is obtained.

## REFERENCE

(1) MacRoberts, T. M., *Integrals involving a modified Bessel function of the second kind and an E-function*, Proc. Glasg. Math. Ass. **2** (1956), 93–96.

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