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INDEX THEORY FOR PERTURBATIONS OF DIRECT SUMS OF NORMAL OPERATORS AND WEIGHTED SHIFTS

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In this paper we consider the construction of a norm-continuous index theory for unitary equivalence up to commutative normality of operators on a separable Hilbert space, and establish some essentially best possible results in this direction for operators which can be written as direct sums of weighted shifts and normal operators. Indeed, for such operators we establish both a norm-continuous analogue and a norm-continuous extension of the elegant results of L. Brown, R. Douglas and P. Fillmore [**4**].

Our key result (Theorem 3) is that for an operator T which is a direct sum of a weighted shift and a normal operator and which T has no λ in its spectrum such that $T - \lambda$ is of a well-defined Fredholm index other than 0 ("T has only zero indices", for short), the operator norm of the self-commutator provides a norm estimate of the distance from T to the closed set of normal operators. That is, for $\epsilon > 0$ there exists $\delta > 0$ such that if T is as described above and if $||T^*T - TT^*|| < \delta$ then there is a normal operator N such that $||T - N|| < \epsilon$.

If, in addition, the self-commutator $T^*T - TT^*$ is compact then $T - N$ is a compact operator still satisfying $||T - N|| < \epsilon$.

Once we have these results we can then establish in Theorem 5 analogous results yielding appropriate equivalence between any two operators which are direct sums of normal operators and weighted shifts and which have the same Fredholm indices everywhere.

Let us describe the problem and its origin.

In [**2**] we showed that a normal operator on a separable Hilbert space could be written as the direct sum of an arbitrarily small compact operator and a diagonal operator.

From this it follows (see P. Halmos [**6**]) that if N_1 and N_2 are normal operators then for $\epsilon > 0$ there exist compact operators K_1 and K_2 each of norm less than ϵ and a unitary V such that $N_1 + K_1$ and $V(N_2 + K_2)V^{-1}$ commute. If, in addition, N_1 and N_2 have the same essential spectrum then we can make the normal operator $(N_1 + K_1) - V(N_2 + K_2)V^{-1}$ compact. If N_1 and N_2 have only essential spectrum then we can have $N + K_1 = V(N_2 + K_2)V^{-1}$.

This immediately opens two lines of investigation.

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If we consider operators with compact self-commutators what are the appropriate unitary equivalence classes? If we ask for unitary equivalence modulo the ideal of compact operators then the result of Brown, Douglas and Fillmore, preceded by the important special case due to Deddens and Stampfli [**4**], provides an answer. Indeed, Brown, Douglas and Fillmore showed that the essential spectrum and the Fredholm indices of the points in the spectrum provide a complete set of unitary invariants for equivalence modulo compact operators. We will refer to this paper and result as *BDF*.

The second line of investigation and its coalescence with the first line is our principal concern in this paper.

That is, if we consider operators with self-commutators of small norm, then for what appropriate equivalences can we hope? It is clear that we must look for unitary equivalence modulo a commuting normal operator under a perturbation of small norm. We should not look for as neat an algebraic formulation as that of *BDF*. After all, the normal operators are not even a linear space. Conditions on the essential spectrum, as such, are useless because the essential spectrum is very unstable under perturbations of small norm, and the reader will convince himself that the requirement that the self-commutator be small in norm adds no stability. However, the Fredholm indices of the operator do supply some stability under perturbations of small norm.

Just as in considering only compactness of perturbation as in *BDF* here also we should postulate that the operators in question have the same Fredholm indices. For, just as the Fredholm indices are invariant under compact perturbations so also are they invariant under perturbations of small norm, where, moreover, the bound on such perturbations does not depend on the norm of the self-commutator. (A weighted shift with weights gently varying from 1 at $-\infty$ to 0 at ∞ and hence of non-zero index under any perturbation of norm less than $\frac{1}{2}$ despite the small compact self-commutator provides a useful example here.)

We present the following conjectures as natural. We first give our conjecture in the case of only zero indices because of its immediate appeal. The general index case may not look as accessible. Since it is a conjecture, there seems no point in inventing special awkward definitions to obtain a neat statement.

CONJECTURE (zero-index case). *Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that if T is of norm at most* 1 *and satisfies:*

1) $||T^*T - TT^*|| < \delta$,

2) Index $T - \lambda = 0$ *for all λ for which $T - \lambda$ has a well defined index, then there exists a normal operator N such that $||T - N|| < \epsilon$.*

If, in addition, $T^*T - TT^*$ is compact, then we can have $T - N$ compact and $||T - N|| < \epsilon$.

Our conjecture in the case of non-zero indices is not as natural because it is not sharp.

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CONJECTURE (general case). *Let* $\epsilon > 0$ *be given. Then there exists* $\delta > 0$ *such that if* T_1 *and* T_2 *are of norm at most* 1 *and*

1) $||T_iT_i^* - T_iT_i^*|| < \delta$ *for* $i = 1, 2$,

2) Index $T_1 - \lambda$ = Index $T_2 - \lambda$ *for all* λ *for which these indices are well-defined, then there exist unitary* U *and perturbations* Q_1, Q_2 *and* Q_3 *such that* $N = (T_1 + Q_1) - U(T_2 - Q_2)U^{-1} + Q_3$ *is normal,* $T_1 + Q_1$ *commutes with* $U(T_2 + Q_2)U^{-1}$, *and* $||Q_i|| < \epsilon$.

If, in addition, $T_1^*T_1 - T_1T_1^*$ *and* $T_2^*T_2 - T_2T_2^*$ *are compact then* Q_1, Q_2 *and* Q_3 *may be chosen compact.*

If we require the essential spectrum of T_1 *to be that of* T_2 *then* N *can be taken compact as well as normal.*

The obvious imperfection in the form of the conjecture lies in the lack of conclusions about the norm of N, but our hypotheses, even augmented with spectral conditions, do not allow such conclusions.

We deal here with those operators which are direct sums of weighted shifts and normal operators. The equivalences which we actually establish in § 2 provide more information than the conjecture requests in the general index case. However since we cannot establish the conjecture as it stands, it seems pointless to sharpen it so as to include the best results we have. We also establish an extension of the conjecture appropriate to the case of a unilateral shift as a direct summand, in which case the hypothesis of self-commutator of small norm is not reasonable.

We produced in [**3**] and describe here in our Theorem 1 a result establishing our conjecture for weighted shifts with only zero indices. We address ourselves in this paper to direct sums of general weighted shifts with normal operators.

In general we do not have the best possible numerical bounds on the perturbations required. In the case, however, where our bilateral shift has constant weights on two successive levels we indicate in Observation 4 that our results are numerically best possible.

We point out that the addition of the normal direct summands in the hypotheses of our theorems is essential to our results. Indeed, the addition of normal operators may wipe out the distinct Fredholm indices which make two operators fail to fit the hypotheses of our theorem. For example, if S is a weighted shift of small self-commutator with weights tapering from 1 at $-\infty$ to 0 at ∞, say, and N is a normal operator whose spectrum includes the spectrum of S then $N \oplus S$ never has a well defined index other than 0 and so is close in norm to a normal operator even though the indices of S prevent S itself from ever being closer than $\frac{1}{2}$ to a normal operator.

We have mentioned the important work of Brown, Douglas and Fillmore. Our approach, however, is more in the spirit of Deddens and Stampfli [**5**] and some earlier work of C. Pearcy and N. Salinas [**7**]. We must also mention work of Bastian and Harrison [**1**] and Stampfli [**8**] on subnormal weighted shifts and recent work of P. Halmos [**6**] in this general area. Lemma 1, in particular, owes

a debt to Bastian and Harrison [**1**]. We acknowledge with pleasure valuable conversations with L. Brown and P. Fillmore.

1. Notation and preliminary results. In this article we introduce our notation, which is rather standard, the useful technical device of "gradual exchange" to which we direct the reader's attention and our already described Theorem 1.

We assume all operators are defined on l_2. If we write $T_1 = T_2 \oplus T_3$ we will simply be identifying $l_2 \oplus l_2$ with l_2 and say no more about it. If we have an operator T defined on l_2 and we want to add an operator S so as to have $T \oplus S$ without redefining the space we will simply move T onto a subspace.

We will denote unitary equivalence by $\simeq$.

We will denote the canonical basis for l_2 under the unilateral shift S_1 as $\phi_0, \phi_1, \ldots$. That is, $S_1(\phi_i) = \phi_{i+1}$ for $i \geqq 0$.

We will denote the basis for the bilateral shift, $S_{1,1}$, as $\ldots, \phi_{-1}, \phi_0, \phi_1, \ldots$. That is $S_{1,1}(\phi_i) = \phi_{i+1}$ for $i > -\infty$.

When we say that the weighted shift S has *weight* a_n *at the* nth *place* we mean $S(\phi_n) = a_n \phi_{n+1}$.

If $i(j)$ is an increasing subsequence we will call $\lim_j a_{i(j)}$, if it exists, a *weight of* S *at* ∞ and define a weight at $-\infty$ similarly. In general, the weights at $\pm\infty$ are not unique, even if $S^*S - SS^*$ is compact. (Because the shift with weights $\{a_i\}$ is unitarily equivalent to the shift with weights $\{|a_i|\}$ we will often casually assume we are dealing with positive weights without making it explicit.)

We will use the same notation when we have a shift operating on a direct summand of l_2.

If S is a weighted shift the self-commutator of S has operator norm $||SS^* - S^*S|| = \sup ||a_i|^2 - |a_{i+1}|^2|$. We find

$$\sup_{||x||=1} \big| \, ||S(x)|| - ||S^*(x)|| \, \big| = \sup \big||a_i| - |a_{i+1}|\big|$$

more convenient to our needs. If $||S|| \leqq 1$ it is clear that these are equivalent as far as approach to 0 is concerned.

We will say that an operator T has *only zero indices* if for each complex λ either $T - \lambda$ has Fredholm index 0, that is, deficiency $T - \lambda =$ nullity $T - \lambda$, or $T - \lambda$ has no well-defined Fredholm index. A normal operator is an example of an operator with only zero indices.

When we refer to two operators T_1 and T_2 as having the same indices "everywhere" we mean that if λ is complex then $T_1 - \lambda$ and $T_2 - \lambda$ have the same index.

We now describe a useful procedure which we will refer to as the procedure of "gradual exchange".

Let us suppose we have an operator T which on some large finite set of basis vectors is the direct sum of two shifts. That is, for some set of basis vectors $\{\phi_i\} \cup \{\psi_i\}$

$$T(\phi_i) = \phi_{i+1}, \quad i = 0, \ldots, n$$
$$T(\psi_i) = \psi_{i+1}, \quad i = 0, \ldots, n.$$

We would like to find K, a compact perturbation of small norm, such that $\tilde{T} = T + K$ satisfies:

$$\tilde{T}^n(\phi_0) = \psi_n,$$
$$\tilde{T}^n(\psi_0) = \phi_n.$$
$$\tilde{T} = T \text{ on the space orthogonal to the span of } \phi_0, \ldots, \phi_n, \psi_0, \ldots, \psi_n.$$

That is, we will exchange the ϕ_0 orbit for the ψ_0 orbit. Now if we took the bull by the horns and simply defined $\tilde{T}(\phi_0) = \psi_1$ and $\tilde{T}(\psi_0) = \phi_1$ and let $\tilde{T} = T$ elsewhere we would have our desired exchange, but $\|\tilde{T} - T\| = 1$ and so we would have a large perturbation. However, we may accomplish the exchange gradually over a large subspace by shifting through a sequence of basis vectors of the form $a_i\phi_i + b_i\psi_i$ in which the ψ_i components gradually become predominant and a corresponding orthogonal sequence in which the ϕ_i components gradually predominate.

For the sake of formal propriety we give a lemma embodying this idea, but the reader is requested to consider the technique, not the statement of the lemma, as the object of contemplation. In particular, it is the fact that agreement on a long stretch of indices allows exchange with a small compact perturbation that is of interest, not our estimate on the length required.

LEMMA (gradual exchange lemma). *Let H have the basis $\{\phi_j\} \cup \{\psi_j\}$, $-\infty < j < \infty$. Let T of norm 1 be the direct sum of two bilateral shifts defined by*

$$S(\phi_j) = a_j\phi_{j+1},$$
$$S(\psi_j) = b_j\psi_{j+1}.$$

Let $\epsilon > 0$ be given. Let $n \geqq 30/\epsilon$ be an integer. Suppose that, after an appropriate re-indexing, we have

$$|a_i - b_j| < \epsilon/3 \quad \text{for } i, j = 0, 1, \ldots, n.$$

(That is, our two shifts have almost the same constant weight on a stretch of n indices.) Then there exists a $2n$ dimensional operator K such that $\tilde{T} = T + K$ satisfies

1. $\|K\| < \epsilon$,
2. *$\tilde{T}$ is the direct sum of two bilateral shifts,*
3. $\tilde{T}(\phi_i) = T(\phi_i)$ *for* $i \neq 1, \ldots, n$,
   $\tilde{T}(\psi_i) = T(\psi_i)$ *for* $i \neq 1, \ldots, n$,
   $\tilde{T}^n\phi_0 = \lambda_1\psi_{n+1}$, *and* $\tilde{T}^n\psi_0 = \lambda_2\phi_{n+1}$, *for some scalars* λ_1, λ_2.

That is, for $i \leqq 0$, $\tilde{T}$ exchanges the orbit of ϕ_i with that of ψ_i after $n - i$ repetitions.

Proof. We first set both the a_i and the b_i with indices between 0 and n equal to a_1, thereby incurring a $2n$ dimensional perturbation of norm at most $2\epsilon/3$. Because $|a_1| \leqq 1$ we will not decrease the perturbation by assuming $a_1 = 1$, and therefore all weights in the stretch at issue are 1.

We let $\alpha_j = \cos j\pi/2n$ and let $\beta_j = \sin j\pi/2n$ for $0 \leqq j \leqq n$. We now define

$$\xi_j = \alpha_j\phi_j + \beta_j\psi_j$$
$$\eta_j = (-\beta_j\phi_j + \alpha_j\psi_j)e^{j\pi i/n}.$$

The idea here is that $\{\xi_j\}$ slowly tapers from ϕ_0 to ψ_n and $\{\eta_j\}$ tapers from ψ_0 to ϕ_n. It is clear that $\xi_0, \ldots, \xi_n, \eta_0, \ldots, \eta_n$ is an orthonormal basis for the span of $\phi_0, \ldots, \phi_n, \psi_0, \ldots, \psi_n$.

We now define $\tilde{T}$ by

$$\tilde{T}(\xi_j) = \xi_{j+1} \quad \text{for } 0 \leqq j < n,$$
$$\tilde{T}(\xi_n) = \psi_{n+1},$$
$$\tilde{T}(\eta_j) = \eta_{j+1} \quad \text{for } 0 \leqq j < n,$$
$$\tilde{T}(\eta_n) = \phi_{n+1},$$
$$\tilde{T} = T, \quad \text{elsewhere.}$$

It is clear that

$$||T(\eta_j) - \tilde{T}(\eta_j)|| = |(-\beta_j\phi_{j+1} + \alpha_j\psi_{j+1})e^{j\pi i/n} - (-\beta_{j+1}\phi_{j+1} + \alpha_{j+1}\psi_{j+1})e^{(j+1)\pi i/n}|$$

approaches 0 as $n \to \infty$.

Similarly,

$$\lim_{n\to\infty} ||T(\xi_j) - \tilde{T}(\xi_j)|| = 0.$$

Finally we note that

$$||T - \tilde{T}|| \leqq \max_{0\leqq j\leqq n} |T(\eta_j) - \tilde{T}(\eta_j)| + |T(\xi_j) - \tilde{T}(\xi_j)|$$

and so $\lim_{n\to\infty} ||T - \tilde{T}|| = 0$. Further, a crude estimate yields $||T - \tilde{T}|| \leqq 5\pi/2n < 10/n \leqq \epsilon/3$. This completes the proof of the exchange lemma.

An elaboration of the exchange lemma, with computations, is given in [**3**].

By exploiting the technique of gradual exchange we have the following theorem, which is a conflation of our Theorem 2 and 3 of [**3**].

THEOREM 1. *Let $\{\phi_i\}$, $-\infty < i < \infty$, be a basis for H. Let $\{a_i\}$ be a set of complex scalars such that* $\sup_i |a_i| = 1$. *Let S be a weighted bilateral right shift with weights a_i defined by*

$$S(\phi_i) = a_i\phi_{i+1}.$$

Suppose $\{a_i\}$ *satisfies the following two conditions:*

Condition 1. $\liminf_{i\to-\infty} |a_i| \leqq \limsup_{i\to\infty} |a_i|,$

$$\liminf_{i\to\infty} |a_i| \leqq \limsup_{i\to-\infty} |a_i|.$$

Condition 2. $\sup ||a_i| - |a_{i+1}|| < \eta$ *where* $\eta < 1/256$. *Then there exists a normal operator N such that* $||N - S|| < 100\sqrt{\eta}$.

Let, in addition to the preceding hypotheses, $\{a_i\}$ *satisfy:*

Condition 3. $\lim_{|i|\to\infty} |a_i| - |a_{i+1}| = 0,$

that is, $S^*S - SS^*$ *is compact. Then there exists a normal operator N such that* $N - S$ *is compact and* $||N - S|| < 100\sqrt{\eta}$.

Gloss on Theorem 1. Because Theorem 1 is proved in [**3**] we shall not repeat the proof. However, a hint at the construction, using as it does the idea of gradual exchange, may be in order. First let us note that Condition 1 is actually the condition that S has no non-zero Fredholm indices. That is, for all complex λ either $S - \lambda$ is of index 0 or else is not semi-Fredholm. Condition 2 is the condition that the self-commutator of S is small. Condition 3 causes S to meet the hypotheses of *BDF*.

Now for the sketch of the proof assuming Conditions 1 and 2. We are confronted in Theorem 1 by a gently varying weighted shift, all of whose weights we can assume to be positive, and with these weights occasionally getting close to a given weight ρ at both ∞ and $-\infty$. That is, ρ is a weight of S at both $-\infty$ and ∞. By our procedure of gradual exchange we exchange the initial segment of a hump in the weights with a terminal segment and pinch off a section with almost constant weights to leave a direct summand which is a scalar multiple of a unitary operator and leave the shift with one hump slightly flattened. We continue until we have a constant shift of modulus ρ and hence a direct sum of normal operators. If Condition 3 holds, then the exchange can be made more gradual towards $-\infty$ and ∞ thus allowing the perturbations to be made compact. This completes our outline of the proof of Theorem 1.

2. The zero-index case.

In this article we show that the direct sum of any weighted bilateral shift and a normal operator with a small self-commutator is almost normal provided that this direct sum has no non-zero index.

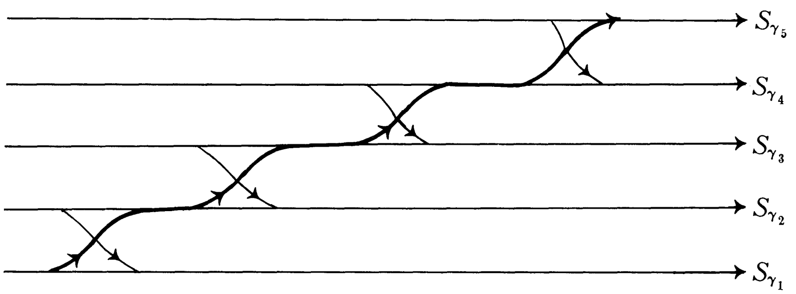
THEOREM 2. *Let N be a normal operator whose spectrum includes the* α, β *annulus (the annulus centered at the origin of inner radius* α *and outer radius* β). *Let* $\alpha \leqq \rho_1 \leqq \beta$ *and* $\alpha \leqq \rho_2 \leqq \beta$ *be given. Let* $\epsilon > 0$ *be given. Then there exists S, a bilateral shift with weight* ρ_1 *at* $-\infty$ *and weight* ρ_2 *at* ∞, *and K, a compact operator satisfying* $||K|| < \epsilon$, *such that*

$$N + K \simeq N \oplus S.$$

Proof. We can assume, without loss of generality, that the spectrum of N is the α, β annulus and we can assume that $\rho_1 = \alpha$ and $\rho_2 = \beta$. If we let $\{\gamma_i\}$ be a dense sequence in the interval $[\alpha, \beta]$ and if we denote by S_{γ_i} the bilateral shift of constant weight γ_i we see that $T = \sum \oplus S_{\gamma_i}$ has the α, β annulus as spectrum. In like manner $T \oplus T \oplus T \oplus \ldots$ has the same α, β annulus as spectrum and so, because we are only dealing with unitary equivalence up to a small compact operator, we can assume that $N = T \oplus T \oplus T \oplus \ldots$.

We now choose a finite set of weights $\alpha = \gamma_1 < \gamma_2 < \ldots < \gamma_n = \beta$ such that $|\gamma_{i+1} - \gamma_i| < \epsilon/2$. We can now use our gradual exchange procedure to exchange S_{γ_1} with S_{γ_2} incurring a finite dimensional perturbation of norm less than ϵ. (We pick up $\epsilon/2$ of this bound because of the difference in weights of S_{γ_1} and S_{γ_2}; we pick up the remaining $\epsilon/2$ to allow for the exchange procedure itself.) We then exchange S_{γ_2} with S_{γ_3} on a set of basis vectors of high enough index that the S_{γ_1} exchange with S_{γ_2} was already complete by these indices, once again incurring a finite dimensional perturbation of norm bounded by ϵ in a space orthogonal to that affected by our first perturbation. Continue in this manner up to S_{γ_n}. We have now, at the expense of a compact perturbation K such that $||K|| < \epsilon$, written $T + K = S_1 \oplus Q$ where S_1 is a shift with weight α at $-\infty$ and weight β at ∞ and Q is the remaining direct summand.

The following sketch for the case $n = 5$ may be of value.



The heavy line represents S_1.

We observe from the preceding construction that for any $\epsilon' > 0$ there exists δ such that if the weights of S_i are monotonely increasing in a finite number of steps from α at $-\infty$ to β at ∞ and the difference between successive weights is bounded by δ, then we have

$$T + K_i \simeq S_i \oplus Q_i \quad \text{where } ||K_i|| < \epsilon'.$$

We now construct a sequence of such decompositions $T + K_n \simeq S_n \oplus Q_n$ satisfying

1) $||K_n|| \underset{n}{\rightarrow} 0$,
2) $||K_n|| < \epsilon$,
3) $||S_n - S_{n+1}|| \underset{n}{\rightarrow} 0$.

We achieve this construction by starting with S_1 such that $T + K_1 \simeq S_1 \oplus Q_1$ where $||K_1|| < \epsilon_1 = \epsilon$, and suppose successive weights of S_1 differ by at most some η_1. In successive copies of T we stretch out the graph of S_1 by one index at a time, incurring an error $||S_i - S_{i+1}|| \leqq \eta_1$. We still have $||K_i|| < \epsilon_1$. After the graph is sufficiently stretched, at stage S_m, say, we may make a finite perturbation bounded by $\eta_1/2$ to obtain S_{m+1} with successive weights differing by at most $\eta_1/2$. Continuing this process we see that $||K_i|| \to 0$ and $||S_i - S_{i+1}|| \to 0$ as desired.

Now we assemble the pieces and complete our proof. Because $||K_n|| \to_n 0$ we see that $K = \sum_n \oplus K_n$ is a compact operator such that $||K|| < \epsilon$. Now we write

$$N + K \simeq \sum \oplus S_n \oplus Q_n \simeq S_1 \oplus \sum \oplus (S_{n+1} \oplus Q_n).$$

But recalling that $\sum \oplus (S_{n+1} - S_n)$ is compact and of norm less than ϵ we see that there exists compact K' such that $||K'|| < 2\epsilon$ and such that

$$N + K' \simeq S_1 \oplus \sum \oplus S_n \oplus Q_n.$$

Hence $K' + K$ is compact and $||K' + K|| < 3\epsilon$ and furthermore

$$N + K' + K \simeq S_1 \oplus N.$$

This completes the proof of Theorem 2.

We will state here, without proof, a lemma of ours which figured in an earlier version of Theorem 2 and which we consider interesting. The reader will notice that this lemma produces a set of S_i which could replace the S_i of our proof. It is this Lemma 1 whose proof uses an idea similar to one of Bastian and Harrison [**1**], which allows a proof of Theorem 3 somewhat in the spirit of Deddens, Stampfli [**5**] and which itself jibes with Stampfli [**8**].

LEMMA 1. *Let $z(\alpha, \beta)$ denote multiplication by z on L^2 of the annulus of inner radius $\alpha \geqq 0$ and outer radius β. Then there exist subspaces $\mathscr{H}_1, \mathscr{H}_2, \ldots$ such that $z(\alpha, \beta)$ takes $\mathscr{H}_i$ into $\mathscr{H}_i$ as a bilateral weighted shift, S_i, with weights running from α at $-\infty$ to β at ∞ and satisfying*

1) $||S_i - S_{i+1}|| \to 0$,
2) $||S_iS_i^* - S_i^*S_i|| \to 0$, ($S_iS_i^* - S_i^*S_i$ *is necessarily compact.*)
3) $S_i(\delta_0) \to ((\alpha + \beta)/2)\delta_1$.

That is, there is a sequence of subnormal S_i with the desired properties.

THEOREM 3. *Let N be a normal operator whose spectrum includes the α, β annulus (the annulus centered at the origin of inner radius α and outer radius β). Let S be the bilateral shift given by $S(\phi_i) = a_i\phi_{i+1}$ and satisfying:*

1) sup $|a_i| \leqq 1$,
2) sup $||a_i| - |a_{i+1}|| < \eta$, *for some* $\eta < 1/256$,
3) *S has a weight at $-\infty$ in the α, β annulus and a weight at ∞ in the α, β annulus.*

Then there exists a normal operator $\tilde{N}$ such that $||N \oplus S - \tilde{N}|| < 200\sqrt{\eta}$.

If we have the additional hypothesis

4) $\lim_{|i|\to\infty} |a_i| - |a_{i+1}| = 0$

(*that is, S has a compact self-commutator*), *then we get the additional conclusion, $N \oplus S - \tilde{N}$ is compact.*

Finally, if for all a_i we have $\alpha \leqq |a_i| \leqq \beta$, then we may choose $\tilde{N}$ so that $\tilde{N} \simeq N$.

Proof. We may assume $a_i \geqq 0$ because these weights could be realized by a unitary transformation on S. Choose γ_1 and γ_2 to be weights of S at $-\infty$ and $+\infty$ respectively.

Now, for $\epsilon > 0$ we can, by virtue of Theorem 2, produce a compact operator such that $||K|| < \epsilon$ and a bilateral shift T with weight γ_1 at $+\infty$ and weight γ_2 at $-\infty$ such that

$$N \simeq (N \oplus T) + K.$$

We now perform a gradual exchange of S and T incurring a compact perturbation K_1 such that $||K_1|| < 100\sqrt{\eta}$. (The $\sqrt{\eta}$ factor results from choosing a block of about $1/\sqrt{\eta}$ in the exchange itself. Over a block of length $1/\sqrt{\eta}$ the weights may vary by $\sqrt{\eta}$, requiring a similar perturbation.)

We now have $S \oplus T + K_1 \simeq S_1 \oplus S_2$ where S_1 and S_2 are bilateral shifts with slowly varying weights and with no non-zero Fredholm indices. That is, if we denote the weights of S_1 by a_{i1} we have

$$\limsup_{i\to-\infty} a_{i1} \geqq \liminf_{i\to\infty} a_{i1}$$

$$\limsup_{i\to\infty} a_{i1} \geqq \liminf_{i\to-\infty} a_{i1}$$

and similarly for S_2.

Hence S_1 and S_2 satisfy the hypotheses of our Theorem 1, and so there exist normal N_1 and N_2 such that $||S_1 - N_1|| < 100\sqrt{\eta}$ and $||S_2 - N_2|| < 100\sqrt{\eta}$. Summarizing we have

$$N \oplus S \simeq N \oplus T \oplus S + K \simeq N \oplus S_1 \oplus S_2 + K_2 \simeq N \oplus N_1 \oplus N_2 + K_2 + Q$$

where K_2 is compact, $||K_2|| < 100\sqrt{\eta}$ and $Q = (S_1 - N_1) \oplus (S_2 - N_2)$ satisfies $||Q|| < 100\sqrt{\eta}$. Thus, $N \oplus N_1 \oplus N_2$ is our desired operator $\tilde{N}$. This completes our proof of Theorem 3 assuming only hypothesis 1, 2, and 3.

If we have hypothesis 4, then S_1 and S_2 have compact self-commutators and our Theorem 1 yields the compactness of Q.

Finally, if $\alpha \leqq |a_i| \leqq \beta$ for all a_i, then the normal operators N_1 and N_2 each have spectrum in the α, β annulus and so $N_1 \oplus N_2 \oplus N \simeq N + K$ for an arbitrarily small compact K, which is then absorbed in the original error estimate.

This completes the proof of Theorem 3.

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For those readers trying to reconcile our result with BDF where Hypothesis 4 yields the existence of a compact K such that $(N \oplus S) + K \simeq N$ without requiring all the coefficients of S to fall in the annulus, we point out that if hypothesis 4 holds then N_1 and N_2 themselves each differ from a normal operator all of whose spectrum lies in the α, β annulus by a compact normal operator. By incorporating these compact normal operators, over whose norms we have no control, into our compact K we would obtain unitary equivalence with N but at the expense of our norm estimates.

Hypothesis 3 guaranteed the non-existence of any non-zero indices of $N \oplus S$. It appeared overtly in allowing the exchange with the shift T so as to produce a new shift with no non-zero indices and so, by Theorem 1, close to normal.

If S has a long succession of weights a_i, $N_1 \leqq i \leqq N_2$, with moduli between α and β such that $||a_i| - |a_{i+1}|| < \eta_1$ for some η_1 much smaller than η (recalling $\eta > \sup ||a_i| - |a_{i+1}||$) then the error caused by exchange as opposed to the error involved in applying Theorem 2, can be accordingly reduced. As an extreme case of special interest we have the following situation in which, contrary to the usual state of affairs, our numerical estimate itself is best possible.

Observation 4. Let S be a bilateral shift with constant positive weights α, say, up to the ϕ_0 coordinate and constant positive weights β afterwards and N be a normal operator whose spectrum includes the α, β annulus. Then for any $\epsilon > 0$ we can find a compact K such that

$$N + K \simeq N \oplus S \quad \text{and } ||K|| < |(\alpha - \beta)/2| + \epsilon.$$

We attain this estimate on K by reducing the exchange error to less than ϵ, but it is not hard to see that a compact K such that $N + K \simeq N \oplus S$ must satisfy $||K|| > |(\alpha - \beta)/2|$ and so our result is best possible.

Indeed, setting $||K|| = |(\alpha - \beta)/2|$ and satisfying the necessary condition for normality

$$||(N \oplus S + K)\phi_i|| = ||(N \oplus S + K)^*\phi_i||$$

will produce a K such that $||K(\phi_i)|| = |(\alpha - \beta)/2|$ for each ϕ_i and so non-compact.

3. Non-zero index equivalences. In this article we establish the promised equivalences of operators with the same Fredholm indices. The reader will notice a plethora of normal operators, compact and otherwise, as direct summands in our conclusions. The problem is essentially that a weighted shift with spectrum 0 and an arbitrarily small self-commutator may have as its best normal approximant a normal operator of norm, and hence of spectral radius, 1. A glance back at our Theorem 1 will remind the reader of the provenance of these normal operators. As long as we insist on keeping estimates on our norms we cannot avoid these summands.

THEOREM 5. *Let S_1 and S_2 be bilateral shifts of norm at most* 1 *with weights*

$\{a_i\}$ and $\{b_i\}$ respectively. Suppose

$$\sup ||a_i| - |a_{i+1}|| < \eta < 1/512,$$
$$\sup ||b_i| - |b_{i+1}|| < \eta < 1/512.$$

Let a_∞ and $a_{-\infty}$ (not necessarily unique) be weights of S_1 at ∞ and $-\infty$ respectively and let b_∞ and $b_{-\infty}$ be similar weights for S_2.

Let N_1 be a normal operator whose spectrum includes the $a_{-\infty}$, $b_{-\infty}$ annulus and the a_∞, b_∞ annulus. Let N_2 be normal (or absent). Then there exists a normal operator N_3, a unitary operator U and error operators Q_1 and Q_3 such that

$$||Q_i|| < 300\sqrt{\eta},\ U(S_1 \oplus N_1 + Q_1)U^{-1} - (S_2 \oplus N_2) = N_3 + Q_3,$$

and $S_2 \oplus N_2$ commutes with $U(S_1 \oplus N_1 + Q_1)U^{-1}$.

If, in addition, S_1 and S_2 have compact self-commutators then we may take Q_1 and Q_3 compact with the same norm bound.

Finally, if all the a_i and b_i lie on one annulus in the spectrum of N_1 and $N_2 \simeq N_1$, then we can achieve

$$N \oplus S_1 + Q_1 \simeq N_2 \oplus S_2.$$

Proof. We first invoke Theorem 2 to write $N_1 \oplus S_1$ as $N_1 \oplus \tilde{S}_1 \oplus S_1$ where $\tilde{S}_1$ is a bilateral shift with weights running from $b_{-\infty}$ at $-\infty$ to $a_{-\infty}$ at ∞ and as small a compact self-commutator as desired. We then exchange $\tilde{S}_1$ with S_1 on some stretch of weights on which $\tilde{S}_1$ is close to S_1 and obtain

$$N_1 \oplus S_1 \simeq N_1 \oplus S_3 \oplus S_4 + R_1$$

where S_3 is a bilateral shift with weights running from $b_{-\infty}$ to a_∞, S_4 has weights running from $a_{-\infty}$ to $a_{-\infty}$ and $||R_1|| < 100\sqrt{\eta}$. Moreover S_3 and S_4 have the same η bound on their weight differences as S_1 and S_2.

We repeat the process with S_3 playing the part of S_1, observing S_3 has the same η bound on successive differences, and we get a bilateral shift S_5 with weights running from $b_{-\infty}$ to b_∞ and S_6 with weights running from a_∞ to a_∞.
We now have

$$N_1 \oplus S_1 \simeq N_1 \oplus S_4 \oplus S_5 \oplus S_6 + R_1 + R_2.$$

Theorem 1 now implies that, at the expense of yet another perturbation, R_3, we may absorb S_4 and S_6 into a normal M and have

$$Q_1 + N_1 \oplus S_1 \simeq M \oplus S_5 \quad \text{where } ||Q_1|| < 300\sqrt{\eta}.$$

Moreover, at the expense of an arbitrarily small compact perturbation which we can absorb in our estimates we can assume that M commutes with N_2. This leaves us with

$$U(N_1 \oplus S_1 + Q_1)U^{-1} - N_2 \oplus S_2 = (M \oplus S_5) - (N_2 \oplus S_2).$$

Now we apply Theorem 1 once again to $S_5 - S_2$, a shift with 0 weights at both

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$\pm\infty$ and successive differences 2η, and so we have

$$U(N_1 \oplus S_1 + Q_1)U^{-1} - N_2 \oplus S_2 = N + Q_2$$

where the operators commute as desired.

Finally, if N_1 and N_2 are unitarily equivalent and all the a_i and b_i lie in one annulus in the spectrum of N_1 the situation is altogether simpler. For here we use Theorem 3 directly to get $N_1 \oplus S_1$ as $N_1 \oplus S_1 \oplus S_2 + R_1$ and serve $N_2 \oplus S_2$ similarly. This leaves us $N_1 \oplus S_1 + Q_1 \simeq N_2 \oplus S_2$ with the same estimates on Q_1 as before.

This completes the proof of Theorem 5.

If we ask that S_1 and S_2 both have compact self-commutators, the R_i and Q_i produced will be compact, again by Theorems 1 and 3.

We remark that the problem we contemplate must allow the trivial case where $a_\infty = b_\infty$ and $a_{-\infty} = b_{-\infty}$ allowing N_1 to be finite dimensional or lacking entirely. In this case the analysis proceeds as from the middle of the previous proof where we had obtained S_5 and we get

$$U(S_1 + Q_1)U^{-1} - (S_2 \oplus N_2) = N + Q_2,$$

with the same estimate as before. The unitary transformation was devoted to extracting a normal direct summand from S_1, if necessary, to match with N_2.

In Theorem 5 we were able to exchange shifts on any stretch where the weights were almost constant and so we could perturb a bilateral shift in a neighborhood of $\pm\infty$ alone with a bound on the norm which depended on the self-commutator for large indices only. This observation acquires importance when we consider perturbations of unilateral shifts in which case the leading weight may cause a large self-commutator but the non-trivial perturbations are those at ∞.

We give an appropriate version of Theorem 5 with a sketch of the modifications required in the proof.

THEOREM 5a (Unilateral shift version). *Assume S_1 and S_2 in Theorem 6 are unilateral right shifts with weights $\{a_i | i = 0, 1, 2, \ldots\}$ and $\{b_i | i = 0, 1, 2, \ldots\}$ respectively. Let N_1 include the a_∞, b_∞ annulus.*

Then we may achieve the same results as in Theorem 5 but with the error estimates

$$\|Q_1\| < |a_0 - b_0| + 300\sqrt{\eta} \quad \text{and} \quad \|Q_3\| < 300\sqrt{\eta}.$$

Proof. We can use the same techniques as in Theorem 5 to exchange a bilateral shift with weight a_∞ at $-\infty$ and weight b_∞ at ∞ with S_1 so as to get another bilateral shift with equal weights a_∞ at $\pm\infty$ and a unilateral shift S_3 which agrees with S_1 on the early indices but which has weight b_∞ at ∞. Now S_2 can be matched with S_3 up to a normal in a neighborhood of ∞ just as in Theorem 6 but the error owing to the difference in initial terms remains. This completes the sketch of the required modifications.

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