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A MEASURE FOR POLYNOMIALS IN SEVERAL VARIABLES

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ABSTRACT. We define a notion of measure for polynomials in several variables, basing our construction on the geometry of the zero-set of the polynomial. For polynomials in one variable, this measure reduces to the usual one. We begin the development of the theory of this measure along lines parallel to the theory of Mahler's measure, indicating the differences and similarities between the two.

Let P be a polynomial in one variable with complex coefficients, $P(z)=a_0\Pi(z-\alpha_j)$. The measure of P, denoted $M(P)$, is defined by

$$M(P)=|a_0|\,\Pi\max(1,|\alpha_j|). \tag{1}$$

As is well-known, an application of Jensen's formula yields

$$M(P)=\exp\int_0^1\log|P(e^{2\pi i\theta})|\,d\theta. \tag{2}$$

Mahler [8] generalized this concept of measure to polynomials in several variables; if P is a polynomial in n variables then $M(P)$ is defined by

$$M(P)=\exp\int_0^1\cdots\int_0^1\log|P(e^{2\pi i\theta_1},\ldots,e^{2\pi i\theta_n})|\,d\theta_1\cdots d\theta_n. \tag{3}$$

No analogue of (1)—no expression for $M(P)$ in terms of the zero-set of P—has been found for Mahler's measure. It is of course possible that no such analogue exists. We present evidence favoring this proposition, in the form of a family of polynomials with very different zero-sets all having the same measure. We then suggest an alternative measure for polynomials in several variables. Our suggestion is at once a generalization of both (1) and (2).

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To each monomial $m=z_1^{a_1}\cdots z_n^{a_n}$ we associate the point $m^*=(a_1,\ldots,a_n)$ in $\mathbb{R}^n$. Here the a_j, $1\le j\le n$, are integers, negative integers permitted. To a sum $P=\sum_{j=1}^k c_jm_j$ we associate the convex hull $C(P)$ of the set $\{m_1^*,\ldots,m_k^*\}$. In [4],

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$C(P)$ was called the exponent polytope of P; it is a generalization of the familiar Newton polygon.

THEOREM 1. *Let* $P=\sum_{j=1}^{n+1} c_j m_j$, *each* m_j *a monomial in* n *variables. If* $C(P)$ *is* n*-dimensional (i.e., does not lie in any lower-dimensional affine subspace of* $\mathbb{R}^n$*), then* $M(P)=M(c_1z_1+\cdots+c_nz_n+c_{n+1})$.

The proof is an exercise in change of variables, and will be omitted.

We note the following special case of Theorem 1. Let $P(x, y)=x^a y^b+x^c y^d+x^e y^f$. If the points (a, b), (c, d), and (e, f) are not collinear, then $M(P)=\beta$, where β is defined by $\beta=M(x+y+1)$. The first appearance we know of, of the number β in print is in [1], where 1.38135... is given as its value. This constant has appeared in this and other contexts in many subsequent papers. In [12], it was called "C". The maximal volume of a hyperbolic 3-simplex is given on page 20 of [5] (see also [10]); as David Boyd pointed out to the author, this number is in fact $\pi \log \beta$.

Theorem 1 suggests that $M(P)$ is insensitive to the geometry of the zero-set of P. Let us try, then, to arrive at a concept of measure for polynomials in several variables by starting at (1), rather than (2).

First we note that if $P(0)\neq 0$ then (1) is equivalent to

$$M(P)=|P(0)|\prod_{|\alpha_j|\leq 1}|\alpha_j|^{-1} \tag{4}$$

Now for $0\leq t\leq 1$ let $N(t)=\#\{z:|z|<t \text{ and } P(z)=0\}$. Then it is easy to show that (4) is equivalent to

$$M(P)=|P(0)|\exp\int_0^1 N(t)\,dt/t. \tag{5}$$

This expression suggests the following construction. For P a polynomial in n variables with $P(0)\neq 0$, and $0\leq t\leq 1$, let $A(t)=\mu\{z:|z|\leq t \text{ and } P(z)=0\}$, where μ is $2n-2$ dimensional Lebesgue measure normalized so that the unit ball has measure 1; then define the functional Ω by

$$\Omega(P)=|P(0)|\exp\int_0^1 A(t)\,dt/t^{2n-1}. \tag{6}$$

The expression on the right side of (6) was suggested to the author by David Masser (the notation on the left side was suggested by [7], the first paper on our topic). By construction, it is a generalization of (1) to polynomials of several variables. That it is also a generalization of (2) is the content of the following theorem.

THEOREM 2. *Let* $S=\{z \text{ in } \mathbb{C}^n:|z|=1\}$ *be the unit sphere in* $\mathbb{C}^n$, *and let* $d\sigma$ *be the unique rotation-invariant Borel measure on* S, *normalized so that* S *has* *measure one. Then*

$$\Omega(P) = \exp \int_S \log |P| \, d\sigma. \tag{7}$$

In short, where $M(P)$ is the geometric mean of P on the torus, $\Omega(P)$ is the geometric mean of P on the sphere.

Proof. This is essentially formula 17.3.5 (12) of [14] to which we refer the reader.

REMARK. Since the right side of (7) is defined even when $P(0) = 0$, we now take (7) as the definition of $\Omega(P)$.

Mahler's measure has been studied in many recent papers. We now begin a parallel investigation of the Ω-measure.

First, we note that the Ω-measure is sensitive to the number of variables involved in a way that Mahler's measure is not. Let Π be the canonical projection from $\mathbb{C}^n$ to $\mathbb{C}^m$, $m < n$, and let $P = P(z_1, \ldots, z_m)$. Then it is clear that $M(P \circ \Pi) = M(P)$, whereas in general $\Omega(P \circ \Pi) \neq \Omega(P)$ (we will point out an example below). We should perhaps attach a subscript to Ω to indicate the intended dimension, but we expect the intent to be clear from the context.

It is immediate that the formula

$$M(PQ) = M(P)M(Q) \tag{8}$$

has its counterpart in

$$\Omega(PQ) = \Omega(P)\Omega(Q). \tag{9}$$

Next, we evaluate the Ω-measure for some simple polynomials. Our evaluations rely upon formulas 1.4.5 (2) and 1.4.7 (2) of [14], which we quote here as Lemmas 3 and 4.

LEMMA 3. *Let f be a function of one complex variable. Then for any w in S, the unit sphere of $\mathbb{C}^n$, we have*

$$\int_S f(\langle z, w\rangle) \, d\sigma(z) = \frac{n-1}{\pi} \int_0^{2\pi} \int_0^1 (1 - r^2)^{n-2} f(re^{i\theta}) r \, dr \, d\theta, \tag{10}$$

provided the integrals exist. Here $\langle z, w\rangle = \sum z_j \bar{w}_j$.

LEMMA 4. *For $n > 1$, let B_{n-1} be the unit ball in $\mathbb{C}^{n-1}$, and let $d\nu$ be Lebesgue measure normalized so that the ball has measure one. Then*

$$\int_S f \, d\sigma = \int_{B_{n-1}} d\nu(z') \frac{1}{2\pi} \int_0^{2\pi} f(z', e^{i\theta} z_n) \, d\theta, \tag{11}$$

where $z' = (z_1, \ldots, z_{n-1})$, provided $f : S \to \mathbb{C}$ is integrable.

We also need the following special case of Jensen's formula.

LEMMA 5.

$$\frac{1}{2\pi}\int_0^{2\pi} \log|ae^{in\theta}+b|\,d\theta = \max(\log|a|, \log|b|).$$

THEOREM 6. *Let* $P(z)=a_1z_1+\cdots+a_nz_n+b$. *Let* $a=(|a_1|^2+\cdots+|a_n|^2)^{1/2}$. *Then*

$$\Omega(P)=\begin{cases} |b| & \text{if} \quad |b|\geq a,\\ a\exp\displaystyle\int_{|b|/a}^1 ((1-r^2)^{n-1}-1)\,dr/r & \text{if} \quad |b|\leq a.\end{cases}$$

Proof. Apply Lemma 3 to $f(z)=\log|az+b|$, with $w=(\bar{a}_1,\ldots,\bar{a}_n)/a$. The left side of (10) is then $\log\Omega(P)$. On the right side, we find

$$\frac{n-1}{\pi}\int_0^{2\pi}\int_0^1 (1-r^2)^{n-2}\log|are^{i\theta}+b|\,r\,dr\,d\theta$$
$$=2(n-1)\int_0^1 r(1-r^2)^{n-2}\left(\frac{1}{2\pi}\int_0^{2\pi}\log|are^{i\theta}+b|\,d\theta\right)dr.$$

Applying Lemma 5 to the inner integral we find

$$\log\Omega(P)=2(n-1)\int_0^1 r(1-r^2)^{n-2}\max(\log ar, \log|b|)\,dr.$$

If $|b|\geq a$ then $\max(\log ar, \log|b|)=\log|b|$ and it follows that $\Omega(P)=|b|$. This is also immediate from (6) since $|b|\geq a$ implies $A(t)=0$ for $0\leq t<1$. If $|b|\leq a$ then we integrate separately over the ranges $0\leq r\leq|b|/a$ and $|b|/a\leq r\leq 1$ to establish the theorem.

We draw attention to two special cases of Theorem 6.

COROLLARY. (a) $\Omega(z_1+\cdots+z_n)=\exp(-(H_{n-1}-\log n)/2)=e^{-\gamma/2}+O(n^{-1})$, *where* $H_{n-1}=1+\frac{1}{2}+\cdots+1/(n-1)$ *and* $\gamma=0.57721\ldots$ *is Euler's constant.*

(b) *If* $P(z_1,\ldots,z_n)=z_1$ *then* $\Omega(P)=\exp(-H_{n-1}/2)=e^{-\gamma/2}n^{-1/2}+O(n^{-3/2})$.

We note in passing that $M(z_1+\cdots+z_n)=e^{-\gamma/2}\sqrt{n}+O(n^{-1/2}\log n)$, as proved in [13]. Part b) is the example, promised earlier, of the dependence of Ω on the number of variables.

THEOREM 7. *Let* $P(x,y)=ax^m+by^n$, *m and n positive integers*, $ab\neq 0$. *Let* λ, $0<\lambda<1$, *be such that* $|a|\,\lambda^m=|b|\,(1-\lambda^2)^{n/2}$, *and let* $\rho=(1-\lambda^2)^{1/2}$. *Then* $\log\Omega(P)=\lambda^2(\log|b|-\frac{1}{2}n-m\log\lambda)+\rho^2(\log|a|-\frac{1}{2}m-n\log\rho)$.

Proof. Apply Lemma 4 to $f(z_1,z_2)=\log|az_1^m+bz_2^n|$. The left side of (11) is

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$\log \Omega(P)$. On the right side, we have

$$\int_{B_1} d\nu(z_1) \frac{1}{2\pi} \int_0^{2\pi} \log |az_1^m + bz_2^n e^{in\theta}| \, d\theta.$$

Evaluate the inner integral by Lemma 5, then replace $|z_2|$ by $(1-|z_1|^2)^{1/2}$ (since (z_1, z_2) is in S). Noting that B_1 is just the unit disk in $\mathbb{C}$, we go to polar co-ordinates and obtain

$$\log \Omega(P) = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \max(\log |ar^m|, \log |b(1-r^2)^{n/2}|) r \, d\theta \, dr.$$

The integral with respect to θ is trivial. The integral with respect to r is split into the intervals $0 \le r \le \lambda$ and $\lambda \le r \le 1$ and the theorem follows.

THEOREM 8. *Let $P(x, y) = axy + b$. Let $\Delta = |a|^2 - 4|b|^2$. Then*

$$\Omega(P) = \begin{cases} |b| & \text{if} \quad \Delta \le 0, \\ \frac{1}{2}(|a| + \sqrt{\Delta}) e^{-\sqrt{\Delta}/|a|} & \text{if} \quad \Delta \ge 0. \end{cases}$$

Proof. If $\Delta \le 0$ then a simple calculation shows that P has no zeroes inside the unit ball. Then by (6), $\Omega(P) = |b|$.

If $\Delta \ge 0$, apply Lemma 4 to $f(z_1, z_2) = \log |az_1 z_2 + b|$, obtaining

$$\log \Omega(P) = \int_{B_1} d\nu(z_1) \frac{1}{2\pi} \int_0^{2\pi} \log |az_1 z_2 e^{i\theta} + b| \, d\theta.$$

Evaluate the inner integral by Lemma 5, replace $|z_2|$ by $(1-|z_1|^2)^{1/2}$, go to polar co-ordinates and carry out the trivial integration with respect to θ, yielding

$$\log \Omega(P) = 2 \int_0^1 r \max(\log |ar(1-r^2)^{1/2}|, \log |b|) \, dr.$$

Now let

$$r_{\pm} = \left(\frac{1}{2} \pm \frac{1}{2a} \sqrt{\Delta}\right)^{1/2};$$

then

$$\tfrac{1}{2} \log \Omega(P) = \int_0^{r_-} r \log |b| \, dr + \int_{r_-}^{r_+} r \log(|a| \, r(1-r^2)^{1/2}) \, dr + \int_{r_+}^1 r \log |b| \, dr.$$

Evaluation of these integrals establishes the theorem.

The greatest interest in Mahler's measure has been in its restriction to polynomials with integer coefficients. Let us adopt this restriction on our polynomials. It is then evident from (1) that, in the one-variable case, $M(P)$ is an algebraic number. In the many variable case, no value of $M(P)$ has yet been proved transcendental. On the other hand, we know of no reason to think that β (say) is algebraic.

By way of contrast, the values of $\Omega(P)$ arising from Theorems 6, 7 and 8 are in many cases recognizably transcendental, being simply constructed from algebraic numbers and exponentiation. Of course, the examples presented are far too special to justify a conjecture to the effect that $\Omega(P)$ will always have such a simple form.

In the one-variable case, it is clear from (1) that $M(P)\geq 1$, with equality if and only if P is monic and has no roots outside the unit circle. By a theorem of Kronecker, this is equivalent to P being a cyclotomic polynomial (that is, all its roots being roots of unity). In the many-variable case, it is again true that $M(P)\geq 1$, and equality holds if and only if P is a "generalized cyclotomic polynomial"—see [6], [3], or [15].

It is not always the case that $\Omega(P)\geq 1$. From part b) of the corollary to Theorem 6 we see that $\Omega(P)$ can approach zero as the number of variables increases. Even if the number of variables is held at two, part a) of the corollary together with (9) shows that $\Omega((z_1+z_2)^k)=(e/2)^{-k/2}$, and from Theorem 7 we find $\Omega(z_1^k+z_2^k)=(e/2)^{-k/2}$. Nevertheless, we deduce immediately from (6) the following version of Kronecker's theorem (variation on Kronecker's theme?).

THEOREM 9. *Assume* $P(0)\neq 0$. *Then* $\Omega(P)\geq 1$, *with equality if and only if* $P(0)=\pm 1$ *and P has no zeroes inside the unit ball.*

We conclude with some open questions.

1. The examples preceding Theorem 9 suggest that $\Omega(P)$ is small only when the number of variables is large or P vanishes to high order at zero. Can one in fact prove that $\Omega(P)$ is bounded away from zero by some function of the number of variables and the order of vanishing at the origin?

2. Can one characterize all polynomials with integer coefficients not vanishing in the unit ball? Montgomery and Schinzel [11] have done this for the polydisc.

3. Can one find useful inequalities relating $\Omega(P)$ to the coefficients of P? Mahler [8] has done this for $M(P)$; see also [9] and [2].

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