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BOUNDS FOR THE DISTANCE TO FINITE-DIMENSIONAL SUBSPACES

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We establish upper bounds for the distance to finite-dimensional subspaces in inner product spaces and improve some generalisations of Bessel's inequality obtained by Boas, Bellman and Bombieri. Refinements of the Hadamard inequality for Gram determinants are also given.

1. INTRODUCTION

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $\{y_1,\dots,y_n\}$ a subset of H and $G(y_1,\dots,y_n)$ the *gram matrix* of $\{y_1,\dots,y_n\}$ where $(i,j)-$entry is $\langle y_i,y_j\rangle$. The determinant of $G(y_1,\dots,y_n)$ is called the *Gram determinant* of $\{y_1,\dots,y_n\}$ and is denoted by $\Gamma(y_1,\dots,y_n)$. Thus,

$$\Gamma(y_1,\dots,y_n)=\begin{vmatrix}\langle y_1,y_1\rangle & \langle y_1,y_2\rangle & \cdots & \langle y_1,y_n\rangle\\ \langle y_2,y_1\rangle & \langle y_2,y_2\rangle & \cdots & \langle y_2,y_n\rangle\\ \cdots & \cdots & \cdots & \cdots\\ \langle y_n,y_1\rangle & \langle y_n,y_2\rangle & \cdots & \langle y_n,y_n\rangle\end{vmatrix}.$$

Following [4, p. 129–133], we state here some general results for the Gram determinant that will be used in the sequel.

(1) Let $\{x_1,\dots,x_n\}\subset H$. Then $\Gamma(x_1,\dots,x_n)\neq 0$ if and only if $\{x_1,\dots,x_n\}$ is linearly independent;

(2) Let $M=\operatorname{span}\{x_1,\dots,x_n\}$ be $n-$dimensional in H, that is, $\{x_1,\dots,x_n\}$ is linearly independent. Then for each $x\in H$, the distance $d(x,M)$ from x to the linear subspace H has the representations

$$d^2(x,M)=\frac{\Gamma(x_1,\dots,x_n,x)}{\Gamma(x_1,\dots,x_n)} \tag{1.1}$$

and

$$d^2(x,M)=\|x\|^2-\beta^T G^{-1}\beta, \tag{1.2}$$

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where $G = G(x_1, \dots, x_n)$, G^{-1} is the inverse matrix of G and

$$\beta^T = \big(\langle x, x_1\rangle,\ \langle x, x_2\rangle, \dots, \langle x, x_n\rangle\big),$$

denotes the transpose of the column vector β.

Moreover, one has the simpler representation

$$d^2(x, M) = \begin{cases} \|x\|^2 - \dfrac{\left(\sum_{i=1}^n |\langle x, x_i\rangle|^2\right)^2}{\left\|\sum_{i=1}^n \langle x, x_i\rangle x_i\right\|^2} & \text{if } x \notin M^\perp, \\ \|x\|^2 & \text{if } x \in M^\perp, \end{cases} \tag{1.3}$$

where $M^\perp$ denotes the orthogonal complement of M.

(3) Let $\{x_1, \dots, x_n\}$ be a set of nonzero vectors in H. Then

$$0 \leqslant \Gamma(x_1, \dots, x_n) \leqslant \|x_1\|^2 \|x_2\|^2 \cdots \|x_n\|^2. \tag{1.4}$$

The equality holds on the left (respectively right) side of (1.4) if and only if $\{x_1, \dots, x_n\}$ is linearly dependent (respectively orthogonal). The first inequality in (1.4) is known in the literature as *Gram's inequality* while the second one is known as *Hadamard's inequality.*

(4) If $\{x_1, \dots, x_n\}$ is an orthonormal set in H, that is, $\langle x_i, x_j\rangle = \delta_{ij}$, $i, j \in \{1, \dots, n\}$, where δ_{ij} is Kronecker's delta, then

$$d^2(x, M) = \|x\|^2 - \sum_{i=1}^n \big|\langle x, x_i\rangle\big|^2. \tag{1.5}$$

The following inequalities which involve Gram determinants may be stated as well [**9**, p. 597]:

$$\frac{\Gamma(x_1, \dots, x_n)}{\Gamma(x_1, \dots, x_k)} \leqslant \frac{\Gamma(x_2, \dots, x_n)}{\Gamma(x_1, \dots, x_k)} \leqslant \cdots \leqslant \Gamma(x_{k+1}, \dots, x_n), \tag{1.6}$$

$$\Gamma(x_1, \dots, x_n) \leqslant \Gamma(x_1, \dots, x_k)\Gamma(x_{k+1}, \dots, x_n) \tag{1.7}$$

and

$$\Gamma^{1/2}(x_1 + y_1, x_2, \dots, x_n) \leqslant \Gamma^{1/2}(x_1, x_2, \dots, x_n) + \Gamma^{1/2}(y_1, x_2, \dots, x_n). \tag{1.8}$$

The main aim of this paper is to point out some upper bounds for the distance $d(x, M)$ in terms of the linearly independent vectors $\{x_1, \dots, x_n\}$ that span M and $x \notin M^\perp$, where $M^\perp$ is the orthogonal complement of M in the inner product space $(H; \langle\cdot,\cdot\rangle)$.

As a by-product of this endeavour, some refinements of the generalisations for Bessel's inequality due to several authors including: Boas, Bellman and Bombieri are obtained. Refinements for the well known Hadamard's inequality for Gram determinants are also derived.

2. Upper Bounds for $d(x, M)$

The following result may be stated.

Theorem 1. *Let $\{x_1, \dots, x_n\}$ be a linearly independent system of vectors in H and $M := \operatorname{span}\{x_1, \dots, x_n\}$. If $x \notin M^\perp$, then*

$$d^2(x, M) < \frac{\|x\|^2 \sum_{i=1}^n \|x_i\|^2 - \sum_{i=1}^n |\langle x, x_i\rangle|^2}{\sum_{i=1}^n \|x_i\|^2} \tag{2.1}$$

or, equivalently,

$$\Gamma(x_1, \dots, x_n, x) < \frac{\|x\|^2 \sum_{i=1}^n \|x_i\|^2 - \sum_{i=1}^n |\langle x, x_i\rangle|^2}{\sum_{i=1}^n \|x_i\|^2} \cdot \Gamma(x_1, \dots, x_n). \tag{2.2}$$

Proof: If we use the Cauchy-Bunyakovsky-Schwarz type inequality

$$\left\| \sum_{i=1}^n \alpha_i y_i \right\|^2 \leqslant \sum_{i=1}^n |\alpha_i|^2 \sum_{i=1}^n \|y_i\|^2, \tag{2.3}$$

that can be easily deduced from the obvious identity

$$\sum_{i=1}^n |\alpha_i|^2 \sum_{i=1}^n \|y_i\|^2 - \left\| \sum_{i=1}^n \alpha_i y_i \right\|^2 = \frac{1}{2} \sum_{i,j=1}^n \left\| \overline{\alpha_i} x_j - \overline{\alpha_j} x_i \right\|^2, \tag{2.4}$$

we can state that

$$\left\| \sum_{i=1}^n \langle x, x_i\rangle x_i \right\|^2 \leqslant \sum_{i=1}^n \left| \langle x, x_i\rangle \right|^2 \sum_{i=1}^n \|x_i\|^2. \tag{2.5}$$

Note that the equality case holds in (2.5) if and only if, by (2.4),

$$\overline{\langle x, x_i\rangle} x_j = \overline{\langle x, x_i\rangle} x_i \tag{2.6}$$

for each $i, j \in \{1, \dots, n\}$.

Utilising the expression (1.3) of the distance $d(x, M)$, we have

$$d^2(x, M) = \|x\|^2 - \frac{\sum_{i=1}^n |\langle x, x_i\rangle|^2 \sum_{i=1}^n \|x_i\|^2}{\| \sum_{i=1}^n \langle x, x_i\rangle x_i \|^2} \cdot \frac{\sum_{i=1}^n |\langle x, x_i\rangle|^2}{\sum_{i=1}^n \|x_i\|^2}. \tag{2.7}$$

Since $\{x_1, \dots, x_n\}$ are linearly independent, hence (2.6) cannot be achieved and then we have strict inequality in (2.5).

Finally, on using (2.5) and (2.7) we get the desired result (2.1). ∎

Remark 1. It is known that (see (1.4)) if not all $\{x_1, \dots, x_n\}$ are orthogonal to each other, then the following result which is well known in the literature as Hadamard's inequality holds:

$$\Gamma(x_1, \dots, x_n) < \|x_1\|^2 \|x_2\|^2 \cdots \|x_n\|^2. \tag{2.8}$$

Utilising the inequality (2.2), we may write successively:

$$\begin{aligned}
\Gamma(x_1,x_2) &\leqslant \frac{\|x_1\|^2\|x_2\|^2-|\langle x_2,x_1\rangle|^2}{\|x_1\|^2}\|x_1\|^2 \leqslant \|x_1\|^2\|x_2\|^2,\\
\Gamma(x_1,x_2,x_3) &< \frac{\|x_3\|^2\sum_{i=1}^2\|x_i\|^2-\sum_{i=1}^2|\langle x_3,x_i\rangle|^2}{\sum_{i=1}^2\|x_i\|^2}\Gamma(x_1,x_2)\\
&\leqslant \|x_3\|^2\Gamma(x_1,x_2),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\Gamma(x_1,\dots,x_{n-1},x_n) &< \frac{\|x_n\|^2\sum_{i=1}^{n-1}\|x_i\|^2-\sum_{i=1}^{n-1}|\langle x_n,x_i\rangle|^2}{\sum_{i=1}^{n-1}\|x_i\|^2}\Gamma(x_1,\dots,x_{n-1})\\
&\leqslant \|x_n\|^2\Gamma(x_1,\dots,x_{n-1}).
\end{aligned}$$

Multiplying the above inequalities, we deduce

$$\begin{aligned}
\Gamma(x_1,\dots,x_{n-1},x_n) &< \|x_1\|^2\prod_{k=2}^{n}\left(\|x_k\|^2-\frac{1}{\sum_{i=1}^{k-1}\|x_i\|^2}\sum_{i=1}^{k-1}\left|\langle x_k,x_i\rangle\right|^2\right)\\
&\leqslant \prod_{j=1}^{n}\|x_j\|^2,
\end{aligned}\tag{2.9}$$

valid for a system of $n\geqslant 2$ linearly independent vectors which are not orthogonal on each other.

In [7], the author has obtained the following inequality.

Lemma 1. *Let $z_1,\dots,z_n\in H$ and $\alpha_1,\dots,\alpha_n\in\mathbb{K}$. Then one has the inequalities:*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leqslant \begin{cases}\max\limits_{1\leqslant i\leqslant n}|\alpha_i|^2\sum\limits_{i=1}^{n}\|z_i\|^2;\\[4pt]
\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\alpha}\right)^{1/\alpha}\left(\sum\limits_{i=1}^{n}\|z_i\|^{2\beta}\right)^{1/\beta}\\
\qquad\qquad\text{where } \alpha>1,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1;\\[4pt]
\sum\limits_{i=1}^{n}|\alpha_i|^2\max\limits_{1\leqslant i\leqslant n}\|z_i\|^2;\end{cases}$$
$$+\begin{cases}\max\limits_{1\leqslant i\neq j\leqslant n}\{|\alpha_i\alpha_j|\}\sum\limits_{1\leqslant i\neq j\leqslant n}\left|\langle z_i,z_j\rangle\right|;\\[4pt]
\left[\left(\sum\limits_{i=1}^{n}|\alpha_i|^{\gamma}\right)^2-\sum\limits_{i=1}^{n}|\alpha_i|^{2\gamma}\right]^{1/\gamma}\left(\sum\limits_{1\leqslant i\neq j\leqslant n}\left|\langle z_i,z_j\rangle\right|^{\delta}\right)^{1/\delta}\\
\qquad\qquad\text{where } \gamma>1,\ \dfrac{1}{\gamma}+\dfrac{1}{\delta}=1;\\[4pt]
\left[\left(\sum\limits_{i=1}^{n}|\alpha_i|\right)^2-\sum\limits_{i=1}^{n}|\alpha_i|^2\right]\max\limits_{1\leqslant i\neq j\leqslant n}\left|\langle z_i,z_j\rangle\right|;\end{cases}\tag{2.10}$$

where any term in the first branch can be combined with each term from the second branch giving 9 possible combinations.

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Out of these, we select the following ones that are of relevance for further consideration:

$$\text{(2.11)}\quad \Big\|\sum_{i=1}^{n}\alpha_i z_i\Big\|^2 \leqslant \max_{1\leqslant i\leqslant n}\|z_i\|^2\sum_{i=1}^{n}|\alpha_i|^2 + \max_{1\leqslant i<j\leqslant n}|\langle z_i, z_j\rangle|\Big[\Big(\sum_{i=1}^{n}|\alpha_i|\Big)^2 - \sum_{i=1}^{n}|\alpha_i|^2\Big]$$
$$\leqslant \sum_{i=1}^{n}|\alpha_i|^2\Big(\max_{1\leqslant i\leqslant n}\|z_i\|^2 + (n-1)\max_{1\leqslant i<j\leqslant n}|\langle z_i, z_j\rangle|\Big)$$

and

$$\text{(2.12)}\quad \Big\|\sum_{i=1}^{n}\alpha_i z_i\Big\|^2 \leqslant \max_{1\leqslant i\leqslant n}\|z_i\|^2\sum_{i=1}^{n}|\alpha_i|^2 + \Big[\Big(\sum_{i=1}^{n}|\alpha_i|^2\Big)^2 - \sum_{i=1}^{n}|\alpha_i|^4\Big]^{1/2} \times\Big(\sum_{1\leqslant i\neq j\leqslant n}|\langle z_i, z_j\rangle|^2\Big)^{1/2}$$
$$\leqslant \sum_{i=1}^{n}|\alpha_i|^2\Big[\max_{1\leqslant i\leqslant n}\|z_i\|^2 + \Big(\sum_{1\leqslant i\neq j\leqslant n}|\langle z_i, z_j\rangle|^2\Big)^{1/2}\Big].$$

Note that the last inequality in (2.11) follows by the fact that

$$\Big(\sum_{i=1}^{n}|\alpha_i|\Big)^2 \leqslant n\sum_{i=1}^{n}|\alpha_i|^2,$$

while the last inequality in (2.12) is obvious.

Utilising the above inequalities (2.11) and (2.12) which provide alternatives to the Cauchy-Bunyakovsky-Schwarz inequality (2.3), we can state the following results.

Theorem 2. *Let $\{x_1, \dots, x_n\}$, M and x be as in Theorem 1. Then*

$$\text{(2.13)}\quad d^2(x, M) \leqslant \frac{\|x\|^2[\max_{1\leqslant i\leqslant n}\|x_i\|^2 + (\sum_{1\leqslant i\neq j\leqslant n}|\langle x_i, x_j\rangle|^2)^{1/2}] - \sum_{i=1}^{n}|\langle x, x_i\rangle|^2}{\max_{1\leqslant i\leqslant n}\|x_i\|^2 + (\sum_{1\leqslant i\neq j\leqslant n}|\langle x_i, x_j\rangle|^2)^{1/2}}$$

or, equivalently,

$$\text{(2.14)}\quad \Gamma(x_1, \dots, x_n, x) \leqslant \frac{\|x\|^2[\max_{1\leqslant i\leqslant n}\|x_i\|^2 + (\sum_{1\leqslant i\neq j\leqslant n}|\langle x_i, x_j\rangle|^2)^{1/2}] - \sum_{i=1}^{n}|\langle x, x_i\rangle|^2}{\max_{1\leqslant i\leqslant n}\|x_i\|^2 + (\sum_{1\leqslant i\neq j\leqslant n}|\langle x_i, x_j\rangle|^2)^{1/2}} \times\Gamma(x_1, \dots, x_n).$$

Proof: Utilising the inequality (2.12) for $\alpha_i = \langle x, x_i\rangle$ and $z_i = x_i$, $i \in \{1, \dots, n\}$, we can write:

$$\text{(2.15)}\quad \Big\|\sum_{i=1}^{n}\langle x, x_i\rangle x_i\Big\|^2 \leqslant \sum_{i=1}^{n}|\langle x, x_i\rangle|^2\Big[\max_{1\leqslant i\leqslant n}\|x_i\|^2 + \Big(\sum_{1\leqslant i\neq j\leqslant n}|\langle x_i, x_j\rangle|^2\Big)^{1/2}\Big]$$

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for any $x \in H$.

Now, since, by the representation formula (1.3)

$$d^2(x, M) = \|x\|^2 - \frac{\sum_{i=1}^n |\langle x, x_i \rangle|^2}{\| \sum_{i=1}^n \langle x, x_i \rangle x_i \|^2} \cdot \sum_{i=1}^n \left| \langle x, x_i \rangle \right|^2, \tag{2.16}$$

for $x \notin M^{\perp}$, hence, by (2.15) and (2.16) we deduce the desired result (2.13). $\square$

REMARK 2. In 1941, Boas [2] and in 1944, Bellman [1], independent of each other, proved the following generalisation of Bessel's inequality:

$$\sum_{i=1}^n \left| \langle y, y_i \rangle \right|^2 \leqslant \|y\|^2 \left[\max_{1 \leqslant i \leqslant n} \|y_i\|^2 + \left(\sum_{1 \leqslant i \neq j \leqslant n} \left| \langle y_i, y_j \rangle \right|^2 \right)^{1/2} \right], \tag{2.17}$$

provided y and y_i $\left(i \in \{1, \dots, n\} \right)$ are arbitrary vectors in the inner product space $\left(H; \langle \cdot, \cdot \rangle \right)$. If $\{y_i\}_{i \in \{1, \dots, n\}}$ are orthonormal, then (2.17) reduces to Bessel's inequality.

In this respect, one may see (2.13) as a refinement of the Boas-Bellman result (2.17).

REMARK 3. On making use of a similar argument to that utilised in Remark 1, one can obtain the following refinement of the Hadamard inequality:

$$\begin{aligned} \Gamma(x_1, \dots, x_n) &\leqslant \|x_1\|^2 \prod_{k=2}^n \left(\|x_k\|^2 - \frac{\sum_{i=1}^{k-1} |\langle x_k, x_i \rangle|^2}{\max_{1 \leqslant i \leqslant k-1} \|x_i\|^2 + \left(\sum_{1 \leqslant i \neq j \leqslant k-1} |\langle x_i, x_j \rangle|^2 \right)^{1/2}} \right) \\ &\leqslant \prod_{j=1}^n \|x_j\|^2. \end{aligned} \tag{2.18}$$

Further on, if we choose $\alpha_i = \langle x, x_i \rangle$, $z_i = x_i$, $i \in \{1, \dots, n\}$ in (2.11), then we may state the inequality

$$\left\| \sum_{i=1}^n \langle x, x_i \rangle x_i \right\|^2 \leqslant \sum_{i=1}^n \left| \langle x, x_i \rangle \right|^2 \left(\max_{1 \leqslant i \leqslant n} \|x_i\|^2 + (n-1) \max_{1 \leqslant i \neq j \leqslant n} \left| \langle x_i, x_j \rangle \right| \right). \tag{2.19}$$

Utilising (2.19) and (2.16) we may also state the following result.

THEOREM 3. *Let $\{x_1, \dots, x_n\}$, M and x be as in Theorem 1. Then*

$$d^2(x, M) \leqslant \frac{\|x\|^2 [\max_{1 \leqslant i \leqslant n} \|x_i\|^2 + (n-1) \max_{1 \leqslant i \neq j \leqslant n} |\langle x_i, x_j \rangle|] - \sum_{i=1}^n |\langle x, x_i \rangle|^2}{\max_{1 \leqslant i \leqslant n} \|x_i\|^2 + (n-1) \max_{1 \leqslant i \neq j \leqslant n} \left| \langle x_i, x_j \rangle \right|} \tag{2.20}$$

or, equivalently,

$$\begin{aligned} &\Gamma(x_1, \dots, x_n, x) \\ &\leqslant \frac{\|x\|^2 [\max_{1 \leqslant i \leqslant n} \|x_i\|^2 + (n-1) \max_{1 \leqslant i \neq j \leqslant n} |\langle x_i, x_j \rangle|] - \sum_{i=1}^n |\langle x, x_i \rangle|^2}{\max_{1 \leqslant i \leqslant n} \|x_i\|^2 + (n-1) \max_{1 \leqslant i \neq j \leqslant n} \left| \langle x_i, x_j \rangle \right|} \\ &\qquad \times \Gamma(x_1, \dots, x_n). \end{aligned} \tag{2.21}$$

REMARK 4. The above result (2.20) provides a refinement for the following generalisation of Bessel's inequality:

$$\sum_{i=1}^{n}\left|\langle x, x_i\rangle\right|^2 \leqslant \|x\|^2\left[\max_{1\leqslant i\leqslant n}\|x_i\|^2 + (n-1)\max_{1\leqslant i\neq j\leqslant n}\left|\langle x_i, x_j\rangle\right|\right], \tag{2.22}$$

obtained by the author in [7].

One can also provide the corresponding refinement of Hadamard's inequality (1.4) on using (2.21), that is,

$$\begin{aligned}\Gamma(x_1,\dots,x_n) &\leqslant \|x_1\|^2\prod_{k=2}^{n}\left(\|x_k\|^2 - \frac{\sum_{i=1}^{k-1}|\langle x_k, x_i\rangle|^2}{\max_{1\leqslant i\leqslant k-1}\|x_i\|^2 + (k-2)\max_{1\leqslant i\neq j\leqslant k-1}|\langle x_i, x_j\rangle|}\right)\\ &\leqslant \prod_{j=1}^{n}\|x_j\|^2.\end{aligned} \tag{2.23}$$

3. OTHER UPPER BOUNDS FOR $d(x, M)$

In [8, p. 140] the author obtained the following inequality that is similar to the Cauchy–Bunyakovsky–Schwarz result.

LEMMA 2. *Let $z_1,\dots,z_n \in H$ and $\alpha_1,\dots,\alpha_n \in \mathbb{K}$. Then one has the inequalities:*

$$\begin{aligned}\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 &\leqslant \sum_{i=1}^{n}|\alpha_i|^2\sum_{j=1}^{n}\left|\langle z_i, z_j\rangle\right|\\ &\leqslant \begin{cases}\sum_{i=1}^{n}|\alpha_i|^2\max_{1\leqslant i\leqslant n}\left[\sum_{j=1}^{n}\left|\langle z_i, z_j\rangle\right|\right];\\ \left(\sum_{i=1}^{n}|\alpha_i|^{2p}\right)^{1/p}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left|\langle z_i, z_j\rangle\right|\right)^q\right)^{1/q}\\ \qquad\qquad \text{where } p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1;\\ \max_{1\leqslant i\leqslant n}|\alpha_i|^2\sum_{i,j=1}^{n}\left|\langle z_i, z_j\rangle\right|.\end{cases}\end{aligned} \tag{3.1}$$

We can state and prove now another upper bound for the distance $d(x, M)$ as follows.

THEOREM 4. *Let $\{x_1,\dots,x_n\}$, M and x be as in Theorem 1. Then*

$$d^2(x, M) \leqslant \frac{\|x\|^2\max_{1\leqslant i\leqslant n}\left[\sum_{j=1}^{n}|\langle x_i, x_j\rangle|\right] - \sum_{i=1}^{n}|\langle x, x_i\rangle|^2}{\max_{1\leqslant i\leqslant n}\left[\sum_{j=1}^{n}|\langle x_i, x_j\rangle|\right]} \tag{3.2}$$

or, equivalently,

$$\Gamma(x_1,\dots,x_n,x) \leqslant \frac{\|x\|^2\max_{1\leqslant i\leqslant n}\left[\sum_{j=1}^{n}|\langle x_i, x_j\rangle|\right] - \sum_{i=1}^{n}|\langle x, x_i\rangle|^2}{\max_{1\leqslant i\leqslant n}\left[\sum_{j=1}^{n}|\langle x_i, x_j\rangle|\right]}\cdot\Gamma(x_1,\dots,x_n). \tag{3.3}$$

PROOF: Utilising the first branch in (3.1) we may state that

$$\left\|\sum_{i=1}^{n}\langle x, x_i\rangle x_i\right\|^2 \leqslant \sum_{i=1}^{n}\left|\langle x, x_i\rangle\right|^2 \max_{1\leqslant i\leqslant n}\left[\sum_{j=1}^{n}\left|\langle x_i, x_j\rangle\right|\right] \tag{3.4}$$

for any $x \in H$.

Now, since, by the representation formula (1.3) we have

$$d^2(x, M) = \|x\|^2 - \frac{\sum_{i=1}^{n}|\langle x, x_i\rangle|^2}{\|\sum_{i=1}^{n}\langle x, x_i\rangle x_i\|^2}\cdot\sum_{i=1}^{n}\left|\langle x, x_i\rangle\right|^2, \tag{3.5}$$

for $x \notin M^{\perp}$, hence, by (3.4) and (3.5) we deduce the desired result (3.2). ∎

REMARK 5. In 1971, Bombieri [**3**] proved the following generalisation of Bessel's inequality, however not stated in the general form for inner products. The general version can be found for instance in [**9**, p. 394]. It reads as follows: if $y, y_1, \dots, y_n$ are vectors in the inner product space $(H; \langle\cdot,\cdot\rangle)$, then

$$\sum_{i=1}^{n}\left|\langle y, y_i\rangle\right|^2 \leqslant \|y\|^2 \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}\left|\langle y_i, y_j\rangle\right|\right\}. \tag{3.6}$$

Obviously, when $\{y_1, \dots, y_n\}$ are orthonormal, the inequality (3.6) produces Bessel's inequality.

In this respect, we may regard our result (3.2) as a refinement of the Bombieri inequality (3.6).

REMARK 6. On making use of a similar argument to that in Remark 1, we obtain the following refinement for the Hadamard inequality:

$$\begin{aligned}\Gamma(x_1, \dots, x_n) &\leqslant \|x_1\|^2\prod_{k=2}^{n}\left[\|x_k\|^2 - \frac{\sum_{i=1}^{k-1}|\langle x_k, x_i\rangle|^2}{\max_{1\leqslant i\leqslant k-1}[\sum_{j=1}^{k-1}|\langle x_i, x_j\rangle|]}\right] \\ &\leqslant \prod_{j=1}^{n}\|x_j\|^2.\end{aligned} \tag{3.7}$$

Another different Cauchy-Bunyakovsky-Schwarz type inequality is incorporated in the following lemma [**6**].

LEMMA 3. *Let $z_1, \dots, z_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Then*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leqslant \left(\sum_{i=1}^{n}|\alpha_i|^p\right)^{2/p}\left(\sum_{i,j=1}^{n}\left|\langle z_i, z_j\rangle\right|^q\right)^{1/q} \tag{3.8}$$

for $p > 1$, $1/p + 1/q = 1$.

If in (3.8) we choose $p = q = 2$, then we get

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leqslant \sum_{i=1}^{n}|\alpha_i|^2\left(\sum_{i,j=1}^{n}\left|\langle z_i, z_j\rangle\right|^2\right)^{1/2}. \tag{3.9}$$

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Based on (3.9), we can state the following result that provides yet another upper bound for the distance $d(x, M)$.

Theorem 5. *Let $\{x_1, \dots, x_n\}$, M and x be as in Theorem 1. Then*

$$d^2(x,M) \leqslant \frac{\|x\|^2\left(\sum_{i,j=1}^{n} |\langle x_i, x_j\rangle|^2\right)^{1/2} - \sum_{i=1}^{n} |\langle x, x_i\rangle|^2}{\left(\sum_{i,j=1}^{n} |\langle x_i, x_j\rangle|^2\right)^{1/2}} \tag{3.10}$$

or, equivalently,

$$\Gamma(x_1, \dots, x_n, x) \leqslant \frac{\|x\|^2\left(\sum_{i,j=1}^{n} |\langle x_i, x_j\rangle|^2\right)^{1/2} - \sum_{i=1}^{n} |\langle x, x_i\rangle|^2}{\left(\sum_{i,j=1}^{n} |\langle x_i, x_j\rangle|^2\right)^{1/2}} \cdot \Gamma(x_1, \dots, x_n). \tag{3.11}$$

Similar comments apply related to Hadamard's inequality. We omit the details.

4. Some Conditional Bounds

In the recent paper [5], the author has established the following reverse of the Bessel inequality.

Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $\{e_i\}_{i\in I}$ a finite family of orthonormal vectors in H, $\varphi_i, \phi_i \in \mathbb{K}$, $i \in I$ and $x \in H$. If

$$\operatorname{Re}\left\langle \sum_{i\in I} \phi_i e_i - x, x - \sum_{i\in I} \varphi_i e_i \right\rangle \geqslant 0 \tag{4.1}$$

or, equivalently,

$$\left\| x - \sum_{i\in I} \frac{\varphi_i + \phi_i}{2} e_i \right\| \leqslant \frac{1}{2}\left(\sum_{i\in I} |\phi_i - \varphi_i|^2\right)^{1/2}, \tag{4.2}$$

then

$$(0 \leqslant)\|x\|^2 - \sum_{i\in I} \left|\langle x, e_i\rangle\right|^2 \leqslant \frac{1}{4}\sum_{i\in I} |\phi_i - \varphi_i|^2. \tag{4.3}$$

The constant $1/4$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Theorem 6. *Let $\{x_1, \dots x_n\}$ be a linearly independent system of vectors in H and $M := \operatorname{span}\{x_1, \dots x_n\}$. If γ_i, $\Gamma_i \in \mathbb{K}$, $i \in \{1, \dots, n\}$ and $x \in H \backslash M^\perp$ is such that*

$$\operatorname{Re}\left\langle \sum_{i=1}^{n} \Gamma_i x_i - x, x - \sum_{i=1}^{n} \gamma_i x_i \right\rangle \geqslant 0, \tag{4.4}$$

then we have the bound

$$d^2(x,M) \leqslant \frac{1}{4}\left\| \sum_{i=1}^{n} (\Gamma_i - \gamma_i) x_i \right\|^2 \tag{4.5}$$

or, equivalently,

$$\Gamma(x_1, \dots, x_n, x) \leqslant \frac{1}{4}\left\|\sum_{i=1}^{n}(\Gamma_i - \gamma_i)x_i\right\|^2 \Gamma(x_1, \dots, x_n). \tag{4.6}$$

PROOF: It is easy to see that in an inner product space for any $x, z, Z \in H$ one has

$$\left\|x - \frac{z+Z}{2}\right\|^2 - \frac{1}{4}\|Z - z\|^2 = \operatorname{Re}\langle Z - x, x - z\rangle,$$

therefore, the condition (4.4) is actually equivalent to

$$\left\|x - \sum_{i=1}^{n} \frac{\Gamma_i + \gamma_i}{2} x_i\right\|^2 \leqslant \frac{1}{4}\left\|\sum_{i=1}^{n}(\Gamma_i - \gamma_i)x_i\right\|^2. \tag{4.7}$$

Now, obviously,

$$d^2(x, M) = \inf_{y \in M} \|x - y\|^2 \leqslant \left\|x - \sum_{i=1}^{n} \frac{\Gamma_i + \gamma_i}{2} x_i\right\|^2 \tag{4.8}$$

and thus, by (4.7) and (4.8) we deduce (4.5).

The last inequality is obvious by the representation (1.2). □

REMARK 7. Utilising various Cauchy–Bunyakovsky–Schwarz type inequalities we may obtain more convenient (although coarser) bounds for $d^2(x, M)$. For instance, if we use the inequality (2.11) we can state the inequality:

$$\left\|\sum_{i=1}^{n}(\Gamma_i - \gamma_i)x_i\right\|^2 \leqslant \sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2 \Big(\max_{1\leqslant i\leqslant n} \|x_i\|^2 + (n-1) \max_{1\leqslant i<j\leqslant n} \big|\langle x_i, x_j\rangle\big|\Big),$$

giving the bound:

$$d^2(x, M) \leqslant \frac{1}{4}\sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2 \Big[\max_{1\leqslant i\leqslant n} \|x_i\|^2 + (n-1) \max_{1\leqslant i<j\leqslant n} \big|\langle x_i, x_j\rangle\big|\Big], \tag{4.9}$$

provided (4.4) holds true.

Obviously, if $\{x_1, \dots, x_n\}$ is an orthonormal family in H, then from (4.9) we deduce the reverse of Bessel's inequality incorporated in (4.3).

If we use the inequality (2.12), then we can state the inequality

$$\left\|\sum_{i=1}^{n}(\Gamma_i - \gamma_i)x_i\right\|^2 \leqslant \sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2 \left[\max_{1\leqslant i\leqslant n} \|x_i\|^2 + \Big(\sum_{1\leqslant i\neq j\leqslant n} \big|\langle x_i, x_j\rangle\big|^2\Big)^{1/2}\right],$$

giving the bound

$$d^2(x, M) \leqslant \frac{1}{4}\sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2 \left[\max_{1\leqslant i\leqslant n} \|x_i\|^2 + \Big(\sum_{1\leqslant i\neq j\leqslant n} \big|\langle x_i, x_j\rangle\big|^2\Big)^{1/2}\right], \tag{4.10}$$

provided (4.4) holds true.

In this case, when one assumes that $\{x_1, \dots, x_n\}$ is an orthonormal family of vectors, then (4.10) reduces to (4.3) as well.

Finally, on utilising the first branch of the inequality (3.1), we can state that

$$d^2(x, M) \leqslant \frac{1}{4} \sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2 \max_{1 \leqslant i \leqslant n} \left[\sum_{j=1}^{n} |\langle x_i, x_j \rangle| \right], \tag{4.11}$$

provided (4.4) holds true.

This inequality is also a generalisation of (4.3).

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