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SPACE OF SOLUTIONS OF HOMOGENEOUS ELLIPTIC EQUATIONS

BY

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This is the continuation of our paper [1] and includes the results promised there. As in [1], we consider a homogeneous elliptic equation in two variables. In [1] we showed that all solutions of such equations can be written in a specific form, viz. in the form of an infinite series in certain specific polynomials. Here we first establish that a common solution of any two positive powers of any two linearly independent, linear elliptic polynomials can be expressed as a polynomial (Lemma 2). This leads, by induction, to the main result (Corollary 2) of §1, viz. the linear space of all vectors $(u_1, \ldots, u_r)$, $Q_i^{k_i}u_i=0$, $i=1,\ldots,r$ such that $u_1+\cdots+u_r=0$ is finite-dimensional, where $Q=Q_1^{k_1}\ldots Q_r^{k_r}$ is an elliptic homogeneous polynomial and the Q_i's are its factors ([1], Lemma 3).

Finally, using some facts about the locally convex space of entire functions, we show that the linear space of solutions of an elliptic homogeneous polynomial in two variables is isomorphic to the space of entire functions and therefore has a Schauder basis, as stated in [1].

All notation and terminology is identical to that used in [1]. We take $z=x+iy$ and $\bar{z}$ its complex congugate.

1. Intersections of solution spaces. In this section we let $P(s_1, s_2)=s_1+is_2$, $P_\nu(s_1, s_2)=\alpha_\nu s_1+\beta_\nu s_2$, $\nu=1,\ldots,r$ and we assume that the pairs $(1, i)$, $(a_1, \beta_1), \ldots, (\alpha_r, \beta_r)$ are pairwise linearly independent in $\mathbf{C}^2$. Let $k, k_1, \ldots, k_r$ be positive integers, and assume $k-1\le k_j, j=1,\ldots,r$.

LEMMA 1. *Let*

$$v(x_1, x_2) = \sum_{j=0}^{k-1}(x_1-ix_2)^j g_j(x_1, x_2),$$

$$g_j(x_1, x_2) = \sum_{n=0}^{N} a_{nj}(x_1+ix_2)^n.$$

If $P_1^{k_1}u=v$, *then* u *can be written as the sum*: $u=u_0+u_1$, *where*

$$P_1^{k_1}u_1 = 0, \qquad P_1^{k_1}u_0 = v,$$

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$$u_0(x_1, x_2) = \sum_{j=0}^{k-1} (x_1 - ix_2)^j h_j(x_1, x_2),$$

$$h_j(x_1, x_2) = \sum_{n=0}^{N+(k-j)k_1} b_{nj}(x_1 + ix_2)^n.$$

Proof. We need only obtain u_0, since then one gets the required $u_1 = u - u_0$, for $P_1^{k_1} u_1 = P_1^{k_1} u - P_1^{k_1} u_0 = v - v = 0$.

To obtain u_0 we compute $h_0, \ldots, h_{k-1}$ from the following equation:

$$\begin{aligned} P_1^{k_1} u_0 &= \sum_{j=0}^{k-1} \sum_{\mu=0}^{j} \binom{k_1}{\mu} \frac{j!}{(j-\mu)!} (i\alpha_1 + \beta_1)^\mu (x_1 - ix_2)^{j-\mu} P_1^{k_1-\mu} h_j \\ &= \sum_{j=0}^{k-1} \left(\sum_{\mu=0}^{k-1-j} \binom{k_1}{\mu} \frac{(j+\mu)!}{j!} (i\alpha_1 + \beta_1)^\mu P_1^{k_1-\mu} h_{j+\mu} \right). \end{aligned}$$

Equating coefficients of $(x_1 - ix_2)^j$ on both sides, we get:

$$\sum_{\mu=0}^{k-l-j} C_{\mu,j} P_1^{k_1-\mu} h_{j+\mu} = g_j, \quad j = 0, \ldots, k-1,$$

where

$$C_{\mu,j} = \binom{k_1}{\mu} \frac{(j+\mu)!}{j!} (i\alpha_1 + \beta_1)^\mu.$$

Note $C_{0,j} = 1$, so we can write the equations as,

$$P_1^{k_1} h_{k-1} = g_{k-1}$$

$$P_1^{k_1} h_{k-1-j} = g_{k-1-j} + f_j, \quad j = 1, 2, \ldots, k-1,$$

where f_j is a linear combination of $P_1^{k_1-j} h_{k-1}, P_1^{k_1-j+1} h_{k-2}, \ldots, P_1^{k_1-1} h_{k-j}$.

We can then verify directly that the first equation has a solution of the form

$$h_{k-1}(x_1, x_2) = \sum_{n=0}^{N+k_1} b_{n,k-1}(x+iy)^n.$$

Thus successively we obtain $h_{k-1}, \ldots, h_{k-j}$ of the form:

$$h_{k-\nu}(x_1, x_2) = \sum_{n=0}^{N+\nu k_1} b_{n,k-\nu}(x+iy)^n, \quad \nu = 1, \ldots, j$$

whence we have the required h_j's of the desired form.

Lemma 2. *Let $P^k u = P_1^{k_1} u = 0$. Then u can be written as*

$$u(x_1, x_2) = \sum_{j=0}^{k-1} (x_1 - ix_2)^j f_j,$$

$$f_j(x_1, x_2) = \sum_{n=0}^{k+k_1-1} a_{nj}(x_1 + ix_2)^n, \quad j = 0, \ldots, k-1.$$

Proof. *Case* 1. $P_1(s_1, s_2) = s_1 - is_2$. Then by [1, Theorem 1] we can write,

$$u(x_1, x_2) = \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} a_{nj}(x_1 - ix_2)^j (x_1 + ix_2)^n$$

and

$$u(x_1, x_2) = \sum_{j=0}^{k_1-1} \sum_{n=0}^{\infty} b_{nj}(x_1 + ix_2)^j (x_1 - ix_2)^n.$$

If we apply the operator $P_1^j P^n$ to each equation and equate the result, evaluating at $x_1 = x_2 = 0$, we obtain $a_{nj} = b_{jn}$. But $b_{nj} = 0$ for $j \geq k_1$, so $a_{nj} = 0$ for $n \geq k_1$ and we have

$$u(x_1, x_2) = \sum_{j=0}^{k-1} \sum_{n=0}^{k_1-1} a_{nj}(x_1 - ix_2)^j (x_1 + ix_2)^n,$$

and since $k_1 - 1 \leq k + k_1 - 1$ we have the desired result.

Case 2. $P_1(s_1, s_2)$ is not a scalar multiple of $s_1 - is_2$. In this case we can write $P_1(s_1, s_2) = \alpha_1 s_1 + \beta_1 s_2$ and $i\alpha_1 + \beta_1 \neq 0$. Since $P^k u = 0$, again by [1, Theorem 1], we have

$$u(x_1, x_2) = \sum_{j=0}^{k-1} (x_1 - ix_2)^j f_j, \qquad f_j(x_1, x_2) = \sum_{n=0}^{\infty} a_{nj}(x_1 + ix_2)^n.$$

As in the proof of Lemma 1, we have

$$0 = P_1^{k_1} u = \sum_{j=0}^{k-1} \sum_{\mu=0}^{k-1-j} \binom{k_1}{\mu} \frac{(j+\mu)!}{j!} (i\alpha_1 + \beta_1)^\mu P_1^{k_1-\mu} f_{j+\mu}(x_1 - ix_2)^j.$$

It is well known and elementary that the powers of the function $x_1 - ix_2$ are linearly independent with coefficients from the ring of entire functions. In fact this is shown inductively by applying appropriate powers of the operator P to a polynomial in $(x_1 - ix_2)$ with entire functions as coefficients. With this fact we then conclude that

$$\sum_{\mu=0}^{k-1-j} \binom{k_1}{\mu} \frac{(j+\mu)!}{\mu!} (i\alpha_1 + \beta_1)^\mu P_1^{k_1-\mu} f_{j+\mu} = 0, \quad j = 0, \ldots, k-1.$$

By successively writing out the above for $j = k-1, k-2, \ldots, 0$, we obtain

$$P_1^{k_1} f_{k-1} = P_1^{k_1+1} f_{k-2} = \cdots = P_1^{k_1+k-1} f_0 = 0.$$

But

$$P_1^\nu f_j(x_1, x_2) = \sum_{n=\nu}^{\infty} \frac{n!}{(n-\nu)!} (i\alpha_1 + \beta_1)^\nu a_{nj}(x_1 + ix_2)^{n-\nu} = 0$$

so $a_{nj} = 0$ for $n \geq \nu$ and we have

$$f_j(x_1, x_2) = \sum_{n=0}^{k+k_1-1} a_{nj}(x_1 + ix_2)^n.$$

THEOREM 1. *If $P^k\, u = 0$ and $\prod_{i=1}^r P_i^{k_i} u = 0$, then u can be written as*

$$u(x_1, x_2) = \sum_{j=0}^{k-1} (x_1 - ix_2)^j f_j(x_1, x_2),$$

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$$f_j(x_1, x_2) = \sum_{n=0}^{k(k_1+\cdots+k_r)} a_{nj}(x_1+ix_2)^n, \quad j = 0,\ldots,k-1.$$

Proof. We use induction on r. If $r=1$ the result follows from Lemma 2 and the fact that $k+k_1-1 \le kk_1$. Now suppose the result holds for $r-1$ and we have $P^k u = P_1^{k_1}\ldots P_r^{k_r}u=0$. Let $v=P_1^{k_1}u$. Then $P^k v=0$ and $P_2^{k_2}\ldots P_r^{k_r}v=0$ so by the induction hypothesis,

$$v(x_1, x_2) = \sum_{j=0}^{k-1} (x_1-ix_2)^j g_j,$$

$$g_j(x_1, x_2) = \sum_{n=0}^{k(k_2+\cdots+k_r)} b_{nj}(x_1+ix_2)^n.$$

Then by Lemma 1, we can write $u=u_0+u_1$ such that $P_1^{k_1}u=0$ and $P_1^{k_1}u_0=v$, where

$$u_0(x_1, x_2) = \sum_{j=0}^{k-1} (x_1-ix_2)^j h_j^0(x_1, x_2),$$

$$h_j^0(x_1, x_2) = \sum_{n=0}^{k(k_1+\cdots+k_r)} C_{nj}^0(x_1+ix_2)^n.$$

But $P^k u_1 = P^k u - P^k u_0 = 0$, because $P^k u=0$ and by [1, Theorem 1], $P^k u_0=0$. Hence we can apply Lemma 2 to u_1 and conclude that

$$u_1(x_1, x_2) = \sum_{j=0}^{k-1} (x_1-ix_2)^j h_j^1(x_1, x_2),$$

$$h_j^1(x_1, x_2) = \sum_{n=0}^{k+k_1-1} C_{n,j}^1(x_1+ix_2)^n.$$

Hence we have

$$u(x_1, x_2) = \sum_{j=0}^{k-1} (x_1-ix_2)^j(h_j^0(x_1, x_2)+h_j^1(x_1, x_2))$$

and since $k+k_1-1 \le k(k_1+\cdots+k_r)$ we may take $f_j=h_j^0+h_j^1$ and we have the desired expression for u.

Corollary 1. *The vector space of functions u with the property that $P^k u=0$ and $P_1^{k_1}\ldots P_r^{k_r}u=0$ is finite dimensional with dimension at most $k(k_1+\cdots+k_r+1)$.*

Proof. The vector space generated by the linearly independent functions, $(x_1-ix_2)^j$, $(x_1+ix_2)^n$, $0 \le j \le k-1$, $0 \le n \le k(k_1+\cdots+k_r)$ includes the function u.

Although the above detailed results may be of some interest, the only result to be used in the sequel is the following:

Corollary 2. *Let Q be an elliptic homogeneous polynomial with factorization, $Q=Q_1^{k_1}\ldots Q_r^{k_r}$, $k_1 \le k_2 \le \cdots \le k_r$ into independent linear factors, with $r \ge 2$. Let E_j be the vector space of solutions of $Q_j^{k_j}u=0$. Then*

$$F_r = \{(u_1,\ldots,u_r)\colon u_1+\cdots+u_r = 0, \quad Q_j^{k_j}u_j = 0, \quad j = 1,\ldots,r\}$$

is finite-dimensional.

Proof. If $u_1+u_2+\cdots+u_r=0$, then $u_1=-u_2-\cdots-u_r$. We have $Q_1^{k_1}u_1=0$, and also

$$Q_2^{k_2}\ldots Q_r^{k_r}u_1 = -Q_2^{k_2}\ldots Q_r^{k_r}(u_2+\cdots+u_r) = 0.$$

So by Corollary 1, the set of u_1 lies in a $k_1(k_2+\cdots+k_r+1)$-dimensional space. For $r=2$, $(u_1, u_2)\in F_2$ implies $u_1+u_2=0$, i.e. $u_2=-u_1$. Since the set of all u_1 lies in a finite-dimensional space, F_2 is finite-dimensional. Assume by induction that F_{r-1} is finite-dimensional. Consider the projection $P_1\colon F_r\to F_1$. Then F_r is isomorphic to $P_1^{-1}(0)\times P_1(F_r)$. But $P_1^{-1}(0)=\{(0, u_2,\ldots, u_r)\colon u_2+\cdots+u_r=0\}$ which is isomorphic to F_{r-1}. Hence F_r is finite-dimensional because $P_1^{-1}(0)$ is finite-dimensional and $P_1(F_r)=F_1$ is finite-dimensional.

2. The locally convex space of entire functions. Let H be the space of entire functions in one complex variable equipped with the compact open topology. It is well known that H is a Frechét space and furthermore, if f represents a function in H, the map

$$f\to\frac{1}{n!}\frac{d^nf}{dz^n}(0) = a_n(f)$$

is a continuous linear functional. (This can be seen, for example, by considering the Cauchy formula.) We write the power series expansion of f as $f(x)=\sum_{n=0}^{\infty} a_nz^n$, where $a_n(f)=a_n$, and observe that the map $u_n\colon H\to \mathbf{C}$, $u_n(f)=a_n$, is an element of H' (the space of all continuous linear functionals on H).

THEOREM 2. (i) *H is isomorphic to H^2,*
(ii) *H is isomorphic to H^n($n\geq 2$ finite integer), and*
(iii) *If F is a finite-dimensional subspace of H, then H/F is isomorphic to H.*

Proof. (i) We map $H\times H\to H$ by

$$\left(\sum_{n=0}^{\infty} a_nz^n, \sum_{n=0}^{\infty} b_nz^n\right)\xrightarrow{T}\sum_{n=0}^{\infty} a_nz^{2n}+\sum_{n=0}^{\infty} b_nz^{2n+1},$$

where

$$f(z) = \sum_{n=0}^{\infty} a_nz^n \quad\text{and}\quad g(z) = \sum_{n=0}^{\infty} b_nz^n.$$

Clearly, T is an algebraic isomorphism of the Frechét space $H\times H$ onto the Frechét space H. To show that it is continuous we need only show that the graph of T is closed. For $\nu=1, 2,\ldots$, let

$$f^\nu(z) = \sum_{n=0}^{\infty} a_n^\nu z^n,\qquad f(z) = \sum_{n=0}^{\infty} a_nz^n,$$

$$g^\nu(z) = \sum_{n=0}^{\infty} b_n^\nu z^n,\qquad g(z) = \sum_{n=0}^{\infty} b_nz^n,$$

$$h^\nu(z) = \sum_{n=0}^{\infty} c_n^\nu z^n,\qquad h(z) = \sum_{n=0}^{\infty} c_nz^n.$$

We assume that $h^\nu = T(f^\nu, g^\nu)$, $\lim_\nu f^\nu = f$, $\lim_\nu g^\nu = g$, $\lim_\nu h^\nu = h$ and we must show that $h = T(f, g)$. But from the definition of T, we have $c_n^\nu = a_{n/2}^\nu$ (for n even) $= b_{(n-1)/2}^\nu$ (for n odd), and from the remarks preceding this theorem we know that

$$\lim_\nu a_n^\nu = a_n, \quad \lim_\nu b_n^\nu = b_n, \quad \lim_\nu c_n^\nu = c_n^\nu, \quad n = 0, 1, 2, \ldots.$$

Hence it follows that $c_n = a_{n/2}$ (for n even) $= b_{(n-1)/2}$ (for n odd), so $T(f, g) = h$. Therefore by the closed graph theorem, T is continuous and hence it is open by the open mapping theorem.

(ii) In view of (i), this follows by induction.

(iii) For $f \in H$, consider the mapping, $f \to zf$ of H into itself. This is easily seen to be 1:1. The range of this mapping is a complementary subspace G of the space F_0 of constant functions in H. Clearly F_0 is a one-dimensional subspace of H, G a closed hyperplane of H and $G = H/F_0$ is isomorphic to H. Since it is not difficult to show that any two closed hyperplanes in a topological vector space are isomorphic, it follows that H is isomorphic to H/F, where F is any one-dimensional subspace of H. By induction then H is isomorphic to H/F, where F is any finite-dimensional vector subspace.

THEOREM 4. *H is a nuclear space, has a Schauder basis and is isomorphic to a power series space* (*see* [2], *p.* 88).

Proof. These are well-known facts about H found, for example, in [2].

3. The locally convex space $E(P)$**.** Let P be a hypoelliptic partial differential operator in two variables x_1 and x_2 so that all solutions are C^∞. Let $E(P)$ be the vector space of solutions of $Pu = 0$ equipped with the compact open topology.

THEOREM 5. *The compact-open topology on $E(P)$ is equivalent to the topology of uniform convergence of each derivative on compact sets* (*that is, the topology induced by the space C^∞*).

Proof. If K runs through compact sets then a fundamental sequence of seminorms for the compact open topology is given by $(p_K)_K$, where

$$p_K(u) = \sup\{|u(x)| : x \in K\}, \qquad u \in E(P).$$

If ν runs through the positive integers, then a fundamental sequence of seminorms for the "C^∞-topology" is given by $(p_{\nu,K})_{\nu,K}$ where

$$p_{\nu,K}(u) = \sup\left\{\left|\frac{\partial^{m_1+m_2} u}{\partial x_1^{m_1}\partial x_2^{m_2}}(x)\right| : x \in K \quad \text{and} \quad m_1 + m_2 \leq \nu\right\}.$$

Clearly $p_K(u) \leq p_{\nu,K}(u)$ for all ν, K, u. Conversely, given K, ν one has K' and $M > 0$ with $K \subset (K')^0$ and

$$p_{\nu,K}(u) \leq M \sup\{|u(x)| : x \in K'\} = Mp_{K'}(u).$$

For a proof of this fact, see [3, p. 146]. Thus the two topologies compare in both directions so they are equivalent.

COROLLARY 4. *$E(P)$ is a Frechét space.*

Proof. Clearly $P: C^\infty \to C^\infty$ is continuous and $E(P)$ is its kernel. The result then follows from the well-known fact that C^∞ is a Frechét space, and a closed subspace of a Frechét space is again a Frechét space.

THEOREM 6. *Let $P(s_1, s_2)=(\alpha s_1+\beta s_2)^k$ be elliptic. Then $E(P)$ is isomorphic to H.*

Proof. As in the proof of [1, Theorem 1], we may assume that $\alpha=1$, $\beta=i$ because the resulting solution space can be transformed into $E(P)$ by a map of the form $u \to u \circ T$ where T is a fixed, invertible linear transformation on $\mathbf{R}^2$, which clearly is an isomorphism.

With this assumption we apply [1, Theorem 1] and in complex notation, we obtain,

$$E(P) = \left\{u: u(z) = \sum_{j=0}^{k-1} (\bar{z})^j \sum_{n=0}^{\infty} a_n^j z^n;\ \lim_{n\to\infty} |a_n^j|^{1/n} = 0\right\}.$$

Let

$$E_j = \{u: u(z) = \bar{z}^j f(z), f \in H\}, \quad j = 0, \ldots, k-1.$$

The space $E(P)$ is algebraically isomorphic to $E_0 \times \cdots \times E_{k-1}$ under the map:

$$f_0+\bar{z}f_1+\cdots+(\bar{z})^{k-1}f_{k-1} \to (f_0, \bar{z}f_1, \ldots, (\bar{z})^{k-1}f_{r-1}).$$

Here we again use the fact that the powers of $\bar{z}$ are linearly independent over the ring of entire functions. One sees immediately that, say, the inverse of this map is continuous if each E_j is given the compact open topology and $E_0 \times \cdots \times E_{k-1}$ is given the product topology. Thus by the open mapping theorem, the Frechét spaces $E(P)$, $E_0 \times \cdots \times E_{k-1}$ are isomorphic.

Since H is isomorphic to each E_j ($j=0, \ldots, k-1$) under $f \to (\bar{z})^j f$, we have shown that $E(P)$ is isomorphic to the product of k copies of H and hence by Corollary 3, $E(P)$ is isomorphic to H.

THEOREM 7. *Let P be a homogeneous elliptic polynomial in two variables. Then $E(P)$ is isomorphic to H.*

Proof. We can write $P=P_1 \ldots P_r$ where $P_\nu(s_1, s_2)=(\alpha_\nu s_1+\beta_\nu s_2)^{k_\nu}$, $\nu=1, \ldots, r$ and pairs (α_ν, β_ν), $(\alpha_{\nu^1}, \beta_{\nu^1})$ $\nu \neq \nu^1$ are linearly independent in $\mathbf{C}^2$ ([1, Proposition 2]). Consider the map $T: E(P_1) \times \cdots \times E(P_r) \to E(P)$, where

$$T(u_1, \ldots, u_r) = u_1+\cdots+u_r.$$

Obviously this map is linear and continuous and by [1, Lemma 3] it is onto. Hence by the open mapping theorem, $E(P)$ is isomorphic to $[E(P_1) \times \cdots \times E(P_r)]/T^{-1}(0)$. By Theorem 6 and Corollary 3, $E(P)$ is isomorphic to $H/T^{-1}(0)$. By Corollary 2, $T^{-1}(0)$ is finite-dimensional so by Theorem 3, $E(P)$ is isomorphic to H.

COROLLARY 4. *If P is a homogeneous elliptic polynomial in two variables then $E(P)$ is a nuclear Frechét space with a Schauder basis and it is isomorphic to a power series space.*

Proof. Immediate from Theorem 7 and Theorem 4.

It would be interesting to extend these results to elliptic polynomials in more than two variables, and also to the case in which the polynomial is not homogeneous.

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