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Schmidt games and Cantor winning sets

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Abstract. Schmidt games and the Cantor winning property give alternative notions of largeness, similar to the more standard notions of measure and category. Being intuitive, flexible, and applicable to recent research made them an active object of study. We survey the definitions of the most common variants and connections between them. A new game called the Cantor game is invented and helps with presenting a unifying framework. We prove surprising new results such as the coincidence of absolute winning and 1 Cantor winning in metric spaces, and the fact that $1/2$ winning implies absolute winning for subsets of $\mathbb{R}$. We also suggest a prototypical example of a Cantor winning set to show the ubiquity of such sets in metric number theory and ergodic theory.

Key words: Schmidt games, Cantor-winning, Diophantine approximation
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1. *Introduction and results*

When attacking various problems in Diophantine approximation, the application of certain games has proven extremely fruitful. Many authors have appealed to Schmidt's celebrated game [**37**] (e.g. [**10**, **15**, **18**, **20**, **23**, **31**, **34**, **39**]), the *absolute game* [**33**] (e.g. [**11**, **14**, **26**, **29**, **32**]), and the *potential game* [**26**, Appendix C] (e.g. [**3**, **4**, **28**, **35**, **40**, **41**]). It is the amenable properties of the winning sets associated with games of this type that make the games such attractive tools. These properties most commonly include the following (here we use winning in a loose sense to mean winning for either the α game, the c potential game, or the absolute game).

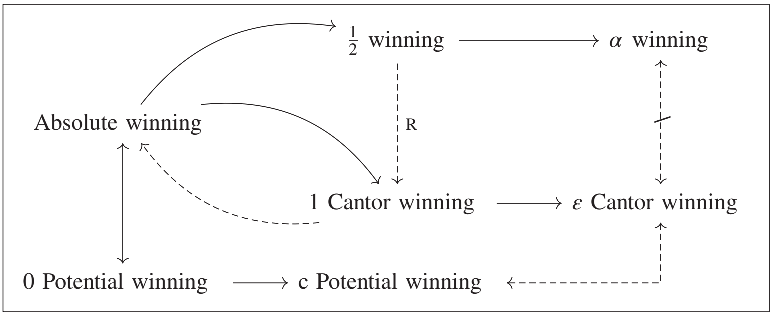

FIGURE 1. Main results.

(W1) A winning set is dense within the ambient space. In $\mathbb{R}^N$, winning sets are thick; that is, their intersection with any open set has Hausdorff dimension N.
(W2) The intersection of a countable collection of winning sets is winning.
(W3) The image of a winning set under a certain type of mapping (depending on the game in question) is winning.

At the same time, many related problems in Diophantine approximation have been solved via the method of constructing certain tree-like (or 'Cantor-type') structures inside a Diophantine set of interest [1, 2, 6–8, 16, 36, 42]. One of the key ingredients in the proof of a famous Schmidt conjecture in [6] was the construction of a certain generalized Cantor set in $\mathbb{R}$. The main ideas were formalized in a subsequent paper [7] and then in [5], the theory of *generalized Cantor sets* and *Cantor winning sets* was finalized in the general setting of complete metric spaces and a host of further applications to problems in Diophantine approximation was given. The family of ε Cantor winning sets satisfies properties (W1)–(W3).

It is therefore very natural to investigate how these two different approaches are related, and that is precisely the intention of this paper.

1.1. *Known connections.* In §2, we provide a complete set of definitions for the unfamiliar reader, along with further commentary relating to the results in this and the succeeding subsection.

The main results of this paper can be summarized in Figure 1. In Figure 1, the solid arrows represent known implications, the dashed arrows represent our new results, and a crossed arrow stands for where an implication does not hold in general.

The following connections are already known.

PROPOSITION 1.1. [33] *If $E \subseteq \mathbb{R}^N$ is absolute winning, then E is α winning for every $\alpha \in (0, 1/2)$.*

One can actually slightly improve on this via the following folklore argument (see [28, Proposition 2.1] for a related observation).

PROPOSITION 1.2. *Let X be a complete metric space. If $E \subseteq X$ is winning and $\alpha_0 := \sup\{\alpha : E$ is α winning$\}$, then E is α_0 winning.*

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For the sake of completeness, we will prove this proposition in §2. Potential winning is related to absolute winning by the following proposition.

PROPOSITION 1.3. [**26**, Theorem C.8] *Let X be a complete doubling metric space. (See Definition 2.21 for the definition of doubling metric spaces.) If $E \subseteq X$ is c potential winning for every $c > 0$ (that is, 0 potential winning), then E is absolute winning.*

Meanwhile, absolute winning is related to Cantor winning in the following way.

PROPOSITION 1.4. [**5**, Theorem 12] *Absolute winning subsets of $\mathbb{R}^N$ are 1 Cantor winning.*

1.2. *New results.* In some sense, the above statements suggest that the construction of Cantor winning sets mimics the playing of a topological game. In §3.1, we introduce a new game called the 'Cantor game'. We prove that its associated winning sets are exactly the Cantor winning sets (cf. Theorem 3.2); to do this, we use ideas related to the potential game. This serves as a justification for the term 'Cantor winning'.

Next, we prove a connection between Cantor winning and potential winning, and, in particular, provide a converse to Proposition 1.4.

THEOREM 1.5. *A set $E \subseteq \mathbb{R}^N$ is ε Cantor winning if and only if it is $N(1-\varepsilon)$ potential winning. In particular, E is Cantor winning if and only if E is potential winning, and if $E \subseteq \mathbb{R}^N$ is 1 Cantor winning, then it is absolute winning.*

Remark 1.6. It is proved in [**5**, Theorem 5.2] that hyperplane absolute winning sets (cf. §2.2 for the definition) in $\mathbb{R}^N$ are $1/N$ Cantor winning; however, the converse is not true. It is possible to develop a theory of generalized Cantor sets in which the removed sets come from neighborhoods of hyperplanes and then use the hyperplane potential game (cf. [**26**, Appendix C] or [**35**] for the definition) to state a complete analogue of Theorem 1.5. This will not be pursued in this note.

The connection between winning sets for Schmidt's game and Cantor winning sets is more delicate, as demonstrated by the existence of the following counterexamples (which are given explicitly in §5).

THEOREM 1.7. *There is a Cantor winning set in $\mathbb{R}$ that is not winning.*

THEOREM 1.8. *There is a winning set in $\mathbb{R}$ that is not Cantor winning.*

However, a partial result connecting Cantor winning sets and winning sets when some restrictions are placed on the parameters is possible.

THEOREM 1.9. *Let $E \subseteq \mathbb{R}^N$ be an ε Cantor winning set. Then for every $c > N(1-\varepsilon)$, there exists $\gamma > 0$ such that for all $\alpha, \beta > 0$ with $\alpha < \gamma(\alpha\beta)^{c/N}$, the set E is (α, β) winning.*

Even though Cantor winning does not imply winning, the intersection of the corresponding sets has full Hausdorff dimension.

THEOREM 1.10. *The intersection in $\mathbb{R}^N$ of a winning set for Schmidt's game and a Cantor winning set has Hausdorff dimension N.*

That being said, we prove a surprising result regarding subsets of the real line.

THEOREM 1.11. *If a set $E \subseteq \mathbb{R}$ is $1/2$ winning, then it is 1 Cantor winning.*

In view of Theorem 1.11, Propositions 1.1 and 1.2, and Theorem 1.5, we deduce the following equivalence.

COROLLARY 1.12. *The following statements about a set $E \subseteq \mathbb{R}$ are equivalent:*
(1) *E is $1/2$ winning;*
(2) *E is 1 Cantor winning;*
(3) *E is absolute winning.*

In [8], the notion of *Cantor rich* subsets of $\mathbb{R}$ was introduced. Cantor rich sets were shown to satisfy some of the desirable properties associated with winning sets, and have been applied since elsewhere [42]. As an application of the results in this paper, we can prove that Cantor rich and Cantor winning coincide.

THEOREM 1.13. *A set $E \subseteq \mathbb{R}$ is Cantor rich in an interval B_0 if and only if E is Cantor winning in B_0.*

Another way to measure how large a set is, is to consider its intersections with classes of tamed sets. In fact, proving non-empty intersection with certain classes of sets motivated the invention of the absolute game. For example, it is well known that absolute winning sets are large in the following dual sense [11].

(W4) The intersection of an absolute winning set with any closed non-empty uniformly perfect set is non-empty.

We demonstrate that a similar property holds for Cantor winning sets. In fact, Cantor winning Borel sets are characterized by an analogue of property (W4).

THEOREM 1.14. *Let $E \subseteq \mathbb{R}^N$. If E is ε_0 Cantor winning, then any $0 \leq \varepsilon < \varepsilon_0$ and $K \in \mathbb{R}^N$ which is $N(1-\varepsilon)$ Ahlfors regular satisfy $E \cap K \neq \varnothing$. If E is Borel, then the converse is also true.*

Moreover, the intersection of Cantor winning sets with Ahlfors regular sets is often large in terms of Hausdorff dimension (cf. Theorem 4.7). The characterization in Theorem 1.14 makes the notion of Cantor winning sets natural from a geometric point of view.

Remark 1.15. Continuing Remark 1.6, it is also possible to characterize hyperplane absolute winning sets as sets which have a non-trivial intersection with any closed *hyperplane diffuse* fractal (cf. [10, Definition 4.2]). This characterization is not discussed in this note.

As a typical example of a Cantor winning set, we suggest the following.

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THEOREM 1.16. *Let $X = \{0, 1\}^{\mathbb{N}}$ with the metric $d(x, y) = 2^{-v(x,y)}$, where $v(x, y)$ denotes the length of the longest common initial segment of x and y. Let $T : X \to X$ be the left shift map. If $K \subseteq X$ satisfies $\dim_H K < 1$, then the set*

$$E = \{x \in X : \overline{\{T^i x : i \geq 0\}} \cap K = \varnothing\} \tag{1}$$

is $1 - \dim_H K$ Cantor winning or, equivalently, $\dim_H K$ potential winning.

Here, $\dim_H$ stands for the Hausdorff dimension. The set E in equation (1) was first considered by Dolgopyat [**17**], who proved that $\dim_H E = 1$. Unlike the property of having full Hausdorff dimension, the potential winning property passes automatically to subspaces (see Lemma 4.4). Hence, in particular, using the notation of Theorem 1.16, if $Y \subseteq X$ is a subshift of finite type and $K \subseteq X$ satisfies $\dim_H(K \cap Y) < \dim_H Y$, then Theorems 1.16 and 4.7 imply that $\dim_H(E \cap Y) = \dim_H Y$, which is exactly the content of [**17**, Theorem 1].

The set of all points whose orbit closure misses a given subset is considered in various contexts, such as piecewise expanding maps of the interval and Anosov diffeomorphisms on compact surfaces [**17**], certain partially hyperbolic maps [**13**, **41**], rational maps on the Riemann sphere [**12**], and β shifts [**21**, **22**]. We suggest that Theorem 1.16 might be generalizable to these contexts; however, we choose to present the proof in the above prototypical example. See also [**9**] for a related problem where the rate at which the orbit is allowed to approach the subset K depends on the geometrical properties of K.

2. *Background, notation, and definitions*

We formally define the concepts introduced in §1. Each of the games we describe below is played between two players, Alice and Bob, in a complete metric space $(X, \mathbf{d})$. A ball in $B \subseteq X$ is specified by its center and radius, denoted by $\mathrm{cent}(B)$ and $\mathrm{rad}(B)$, respectively.

2.1. *Schmidt's game.*

Given two real parameters $\alpha, \beta \in (0, 1)$, *Schmidt's* (α, β) *game* begins with Bob choosing an arbitrary closed ball $B_0 \subseteq X$. Alice and Bob then take turns to choose a sequence of nested closed balls (Alice chooses balls A_i and Bob chooses balls B_i) satisfying both

$$B_0 \supseteq A_1 \supseteq B_1 \supseteq A_2 \supseteq \cdots \supseteq B_i \supseteq A_{i+1} \supseteq B_{i+1} \supseteq \cdots$$

and

$$\mathrm{rad}(A_{i+1}) = \alpha \cdot \mathrm{rad}(B_i) \quad \text{and} \quad \mathrm{rad}(B_{i+1}) = \beta \cdot \mathrm{rad}(A_{i+1}) \quad \text{for all } i \geq 0. \tag{2}$$

Since the radii of the balls tend to zero, at the end of the game, the intersection of Bob's (or Alice's) balls must consist of a single point. See Figure 2 for a pictorial representation of the game, in which the shaded area represents where Alice has specified Bob must play his next ball. A set $E \subseteq X$ is called (α, β) *winning* if Alice has a strategy for placing her balls that ensures

$$\bigcap_{i=0}^{\infty} B_i \subseteq E. \tag{3}$$

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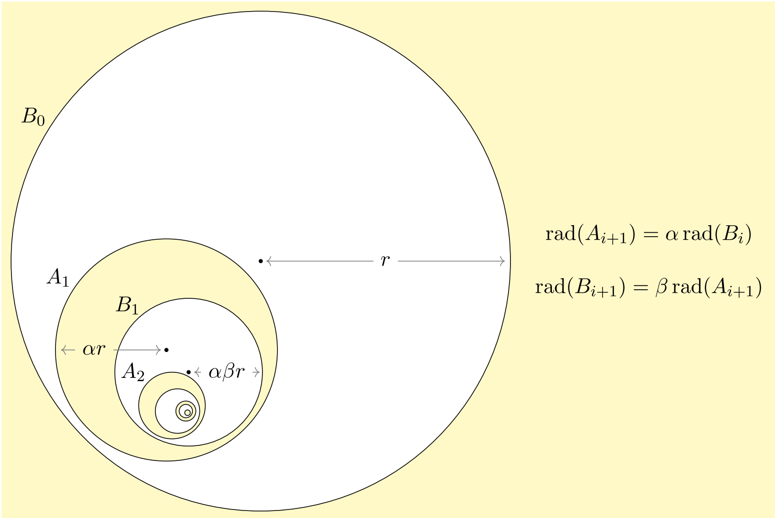

FIGURE 2. Schmidt's game.

In this case, we say that Alice wins. The α *game* is defined by allowing Bob to choose the parameter β and then continuing as in the (α, β) game. We say that $E \subseteq X$ isα *winning* if Alice has a winning strategy for the α game. Note that $E \subseteq X$ is α winning if and only if E is (α, β) winning for all $0 < \beta < 1$. Finally, *Schmidt's game* is defined by allowing Alice to choose $0 < \alpha < 1$ and then continuing as in the α game. We say $E \subseteq X$ is *winning* if Alice has a winning strategy for Schmidt's game. Note that E is winning if and only if it is α winning for some $\alpha \in (0, 1)$.

It is easily observed that if a set E is α winning for some $\alpha \in (0, 1)$, then E is α' winning for every $\alpha' \in (0, \alpha)$, and that E is α winning for some $\alpha \in (1/2, 1)$ if and only if $E = X$ (cf. [**37**], Lemmas 5 and 8, respectively); that is, the property of being $1/2$ winning is the strongest possible for a non-trivial subset.

Remark 2.1. Strictly speaking, since a ball in an abstract metric space X may not necessarily possess a unique center and radius, Alice and Bob should pick successive pairs of centers and radii for their balls, satisfying certain distance inequalities that formally guarantee that the appropriate subset relations hold between the corresponding balls. However, we will assume that this nuance is accounted for in each player's strategy.

We end the introduction of Schmidt games with the proof of the folklore Proposition 1.2.

Proof of Proposition 1.2. Fix $0 < \beta_0 < 1$, let α_0 be as in the statement. The case $\alpha_0 = 1$ is clear. For $\alpha_0 < 1$, set

$$\alpha = \alpha_0 \frac{1 - (\alpha_0\beta_0)^2 - \beta_0(1 - \alpha_0)}{1 - (\alpha_0\beta_0)^2 - \alpha_0\beta_0(1 - \alpha_0)} < \alpha_0, \quad \alpha\beta = (\alpha_0\beta_0)^2. \tag{4}$$

We describe the winning strategy for Alice for the (α_0, β_0) game (Game I) based on her strategy for the (α, β) game (Game II). It can be given as follows.

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(1) Whenever Bob plays a move $B_{2n} = B(b_{2n}, \rho_{2n})$ in Game I, he correspondingly plays the move $B'_n = B(b'_n, \rho'_n)$ in Game II, where

$$(1-\alpha)\rho'_n = (1-\alpha_0)\rho_{2n}, \quad b'_n = b_{2n}.$$

(2) When Alice responds to this by playing a move $A'_{n+1} = B(a'_{n+1}, \alpha\rho'_n)$ in Game II (according to her winning strategy), she correspondingly plays the move $A_{2n+1} = B(a_{2n+1}, \alpha_0\rho_{2n})$ in Game I, where

$$a_{2n+1} = a'_{n+1}.$$

(3) When Bob responds to this by playing a move $B_{2n+1} = B(b_{2n+1}, \rho_{2n+1})$ in Game I, Alice responds by playing the move $A_{2n+2} = B(a_{2n+2}, \alpha_0\rho_{2n+1})$ in Game I, where

$$a_{2n+2} = b_{2n+1}.$$

This sets the stage for the next iteration.

To finish the proof, it suffices to check that all of these moves are legal. When $n = 0$, move 1 is legal because any ball can be Bob's first move. Move 2 is legal because

$$\mathbf{d}(b_{2n}, a_{2n+1}) = \mathbf{d}(b'_n, a'_{n+1}) \le (1-\alpha)\rho'_n = (1-\alpha_0)\rho_{2n},$$

and move 3 is legal because $\mathbf{d}(b_{2n+1}, a_{2n+2}) = 0$. When $n \ge 1$, move 1 is legal because

$$\begin{aligned}\mathbf{d}(a'_n, b'_n) &= \mathbf{d}(a_{2n-1}, b_{2n}) \le \mathbf{d}(a_{2n-1}, b_{2n-1}) + \mathbf{d}(a_{2n}, b_{2n}) \\ &\le (1-\beta_0)\alpha_0\rho_{2n-2} + (1-\beta_0)\alpha_0(\alpha_0\beta_0)\rho_{2n-2} = (1-\beta)\alpha\rho'_{n-1}. \qquad \square\end{aligned}$$

For details and other properties of winning sets, see Schmidt's book [38].

2.2. *The absolute game.* The *absolute game* was introduced by McMullen in [33]. It is played similarly to Schmidt's game but instead of choosing a region where Bob must play, Alice now chooses a region where Bob must not play. To be precise, given $0 < \beta < 1$, the β absolute game is played as follows: Bob first picks some initial ball $B_0 \subseteq X$. Alice and Bob then take turns to place successive closed balls in such a way that

$$B_0 \supseteq B_0 \setminus A_1 \supseteq B_1 \supseteq \cdots \supseteq B_i \supseteq B_i \setminus A_{i+1} \supseteq B_{i+1} \supseteq \cdots, \tag{5}$$

subject to the conditions

$$\text{rad}(A_{i+1}) \le \beta \cdot \text{rad}(B_i) \quad \text{and} \quad \text{rad}(B_{i+1}) \ge \beta \cdot \text{rad}(B_i) \quad \text{for all } i \ge 0. \tag{6}$$

See Figure 3 for a pictorial view of the game in $\mathbb{R}^2$.

If $r_i := \text{rad}(B_i) \not\to 0$, then we say that Alice wins by default. We say a set $E \subseteq X$ is β *absolute winning* if Alice has a strategy which guarantees that either she wins by default or equation (3) is satisfied. In this case, we say that Alice wins. The *absolute game* is defined by allowing Bob to choose $0 < \beta < 1$ on his first turn and then continuing as in the β absolute game. A set is called *absolute winning* if Alice has a winning strategy for the absolute game. Note that a set is absolute winning if and only if it is β absolute winning for every $0 < \beta < 1$.

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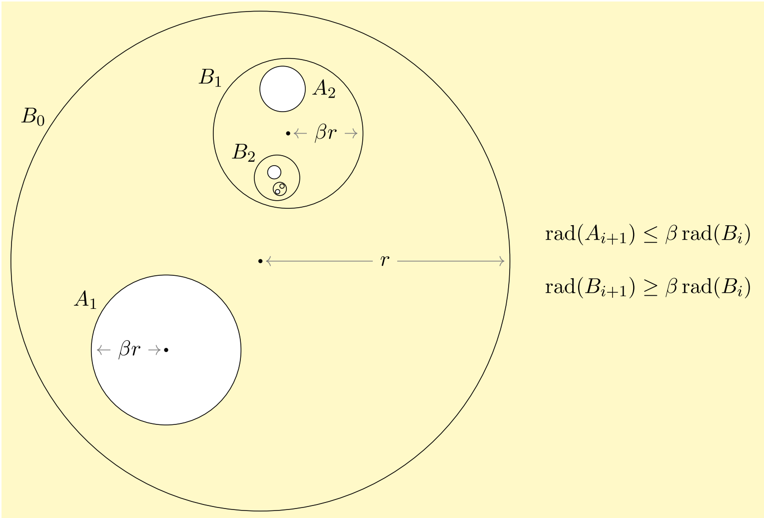

FIGURE 3. The absolute game.

A useful generalization of the absolute game was introduced in [**26**, Appendix C]. Let $\mathcal{H}$ be any collection of non-empty closed subsets of X. The $\mathcal{H}$ *absolute game* is defined in a similar way to the absolute game, but Alice is now allowed to choose a neighborhood of an element $H \in \mathcal{H}$, namely,

$$A_{i+1} = B(H, \beta \cdot \mathrm{rad}(B_i)).$$

McMullen's absolute game is the same as the $\mathcal{H}$ absolute game when $\mathcal{H}$ is the collection of all singletons. The first generalization of the absolute game was the *k-dimensional absolute game*, which is the $\mathcal{H}$ absolute game where $X = \mathbb{R}^N$ and $\mathcal{H}$ is the collection of k-dimensional hyperplanes in $\mathbb{R}^N$. When $k = N - 1$, the associated winning sets are often referred to by the acronym HAW, short for 'hyperplane absolute winning' [**10**]. Some of the results in this note regarding the absolute game may be generalized to the context of $\mathcal{H}$ absolute game but we will not pursue this line further in the current note.

Remark 2.2. These definitions deserve some justification. We do not require $0 < \beta < \frac{1}{3}$ as in the original definitions for $X = \mathbb{R}^N$ [**10**, **33**]. To cover those cases where Bob chooses $\frac{1}{3} \leq \beta < 1$, in which Bob might not have a legal move that satisfies equation (5) on one of his turns, we declare that Bob is losing if this happens. This gives the wanted effect when $X = \mathbb{R}^N$, but puts us in a situation that a subset of an abstract metric space might be winning because the space X is not large enough to allow Bob not to lose by default. In this situation, even the empty set might be absolute winning (cf. Remark 4.9). This situation, however, cannot occur when X is uniformly perfect (cf. §2.7).

Also note that it is shown in [**26**, Proposition 4.5] that replacing the inequalities in equation (6) with equalities does not change the class of absolute winning sets. The reason for introducing these inequalities is to allow an easy proof of the invariance of absolute winning sets under 'quasisymmetric maps' (cf. [**33**, Theorem 2.2]).

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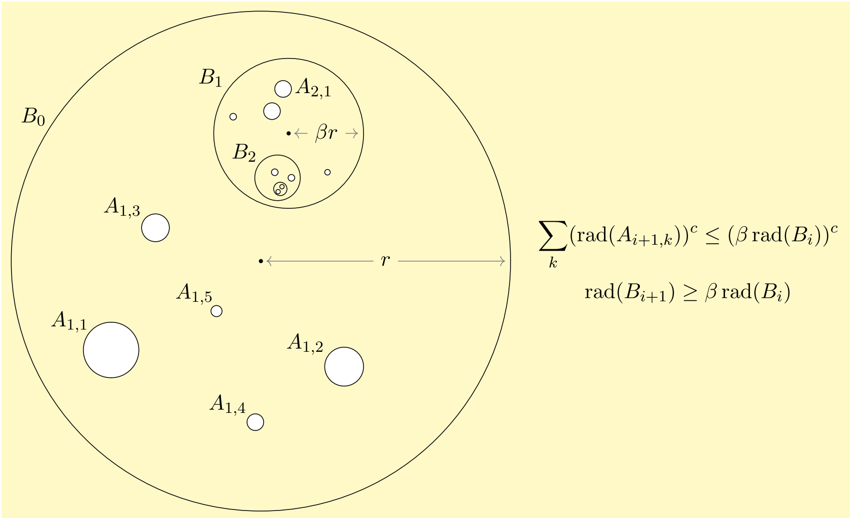

FIGURE 4. The potential game.

2.3. *The potential game.* Given two parameters $c, \beta > 0$, the (c, β) *potential game* begins with Bob picking some initial ball $B_0 \subseteq X$. On her $(i+1)$st turn, Alice chooses a countable collection of closed balls $A_{i+1,k}$, satisfying

$$\sum_k \mathrm{rad}(A_{i+1,k})^c \leq (\beta \cdot \mathrm{rad}(B_i))^c. \tag{7}$$

By convention, we assume that a ball always has a positive radius; that is, Alice cannot choose $A_{i,k}$ to be singletons. Then, Bob on his $(i+1)$st turn chooses a ball $B_{i+1} \subseteq B_i$ satisfying $\mathrm{rad}(B_{i+1}) \geq \beta \cdot \mathrm{rad}(B_i)$. See Figure 4 for a demonstration of the playing of such a game in $\mathbb{R}^2$ (in which, for simplicity, Bob's balls do not intersect Alice's preceding collections—however, it should be understood that this is not a necessary condition for Bob).

If

$$\bigcap_i B_i \subseteq \bigcup_i \bigcup_k A_{i,k} \tag{8}$$

or if $\mathrm{rad}(B_i) \not\to 0$, we say that Alice wins by default.

A set $E \subseteq X$ is said to be (c, β) *potential winning* if Alice has a strategy guaranteeing that either she wins by default or equation (3) holds. The c_0 *potential game* is defined by allowing Bob to choose in his first turn the parameters $c > c_0$ and $\beta > 0$, and then continue as in the (c, β) potential game. A set E is c_0 *potential winning* if Alice has a winning strategy for E in the c_0 potential game. Note that E is c_0 potential winning if and only if E is (c, β) potential winning for every $c > c_0$ and $\beta > 0$. If X is Ahlfors δ regular for some $\delta > 0$, the *potential game* is defined by allowing Alice to choose $c_0 < \delta$ and then continuing as in the c_0 potential game, and a set E is called *potential winning* if Alice has a winning strategy in the potential game. Note that E is potential winning if and only if it is c_0 potential winning for some $c_0 < \delta$.

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Remark 2.3. The c_0 potential game with $c_0 = 0$ was introduced in [**26**, Appendix C] as the potential game. In light of the connections between Cantor, absolute, and potential winning in §3.2 and the geometrical interpretation in §4, it is natural to adjust the definition that appears there to the one that appears here.

For the sake of completeness, we will verify properties (W1) and (W2) for potential winning sets. Property (W1) is verified for a class of Ahlfors regular spaces X in Theorem 4.7.

PROPOSITION 2.4. *Let $E_1, E_2 \subseteq X$ be c_1 and c_2 potential winning, respectively. Then $E_1 \cap E_2$ is* $\max\{c_1, c_2\}$ *potential winning.*

Proof. Fix arbitrary $c > \max\{c_1, c_2\}$ and $\beta > 0$. By assumption, E_1 and E_2 are both (c, β^2) potential winning. Now Alice uses the following strategy to win the (c, β) game on the set $E_1 \cap E_2$. After Bob's initial move B_0, Alice chooses the collection $A_{1,k}$ from her (c, β^2) winning strategy for the set E_1. After Bob's first move B_1, Alice chooses the collection $A^*_{1,k}$ from her (c, β^2) winning strategy for the set E_2, assuming that Bob's initial move was B_1.

In general, after B_{2m}, Alice chooses the collection $A_{m,k}$ following her (c, β^2) winning strategy for the set E_1; after B_{2m+1}, she chooses the collection $A^*_{m,k}$ following her (c, β^2) winning strategy for the set E_2.

It is easy to check that Alice's moves are legal, because

$$\sum_k \operatorname{rad}(A_{i+1,k})^c \le (\beta^2 \cdot \operatorname{rad}(B_i))^c \le (\beta \cdot \operatorname{rad}(B_i))^c.$$

The same inequality is true for the collections $A^*_{i+1,k}$. Also, by construction, either Alice wins by default or $\bigcap_i B_i \subseteq E_1 \cap E_2$. □

2.4. *The strong and the weak Schmidt games.* One can define further variants of Schmidt's original game. The *strong game* (introduced by McMullen in [**33**]) coincides with Schmidt's game except that the equalities in equation (2) are replaced by the inequalities

$$\operatorname{rad}(A_{i+1}) \ge \alpha \cdot \operatorname{rad}(B_i) \quad \text{and} \quad \operatorname{rad}(B_{i+1}) \ge \beta \cdot \operatorname{rad}(A_{i+1}) \quad \text{for all } i \ge 0. \tag{9}$$

Similarly, we define the *weak game* so that Alice can choose her radii with an inequality but Bob must use equality. Obviously, every weak winning set is winning but the converse is not necessarily true (however, the weak game does not have the intersection property—see later).

One could also define a *very strong game*, where Alice must use equality but Bob may use inequality.

Remark 2.5. It can be shown that a set is strong winning if and only if it is very strong winning, but this fact will not be needed and hence we skip its proof.

Later in the paper, we will need the following proposition which relates the very strong winning property of a set E and the weak winning property of its complement $X \setminus E$.

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PROPOSITION 2.6. *Let $E \subseteq X$ satisfy the following property: for any $\beta \in (0, 1)$, there exists $\alpha > 0$ such that E is (α, β) very strong winning. Then $X \setminus E$ is not weak winning.*

Proof. We need to show that for any $\beta > 0$, the set $X \setminus E$ is not β weak winning. To do that, we will provide a winning strategy for Bob that ensures that $\bigcap_{i=0}^{\infty} B_i \subseteq E$. On his first turn, Bob chooses $\alpha > 0$ such that E is (α, β) very strong winning, so that he is now playing the (β, α) weak game and chooses an arbitrary ball B_0. In his subsequent moves, Bob follows Alice's winning strategy for the (α, β) very strong game. □

2.5. *Cantor winning.* For the sake of clarity, we introduce a specialization of the framework presented in [5], tailored to suit our needs. We first define the notion of a 'splitting structure' on X. This is in some sense the minimal amount of structure the metric space needs to allow the construction of generalized Cantor sets (cf. [5]). Denote by $\mathcal{B}(X)$ the set of all closed balls. We assume that by definition, to specify a closed ball, it is necessary to specify its center and radius. Note that this means that in some cases, two distinct balls may be set-theoretically equal in the sense of containing the same points (see also Remark 2.1) in X and by $\mathcal{P}(\mathcal{B}(X))$ the set of all subsets of $\mathcal{B}(X)$. A *splitting structure* is a quadruple $(X, \mathcal{S}, U, f)$, where:

- $U \subseteq \mathbb{N}$ is an infinite multiplicatively closed set such that if $u, v \in U$ and $u \mid v$, then $v/u \in U$;
- $f : U \to \mathbb{N}$ is a totally multiplicative function;
- $\mathcal{S} : \mathcal{B}(X) \times U \to \mathcal{P}(\mathcal{B}(X))$ is a map defined in such a way that for every ball $B \in \mathcal{B}(X)$ and $u \in U$, the set $\mathcal{S}(B, u)$ consists solely of balls $S \subseteq B$ of radius $\mathrm{rad}(B)/u$.

Additionally, we require all these objects to be linked by the following properties:

(S1) $\#\mathcal{S}(B, u) = f(u)$ (in fact, one can make the definition of a splitting structure slightly more general by removing the function f out of it and replacing condition (S1) with

(S1') $\#\mathcal{S}(B, u)$ is finite.

In other words, one can let $\#\mathcal{S}(B, u)$ vary with respect to B. In this case, instead of the function $f(u)$, we define $f(B, u) := \#\mathcal{S}(B, u)$. An enthusiastic reader can verify that in this more general setting, all the results of this paper are still satisfied after minor straightforward adjustments have been made. However, for convenience, we stick with the notation from [5]);

(S2) if $S_1, S_2 \in \mathcal{S}(B, u)$ and $S_1 \neq S_2$, then S_1 and S_2 may only intersect on the boundary, that is, the distance between their centers must be at least $\mathrm{rad}(S_1) + \mathrm{rad}(S_2) = 2\mathrm{rad}(B)/u$;

(S3) for all $u, v \in U$,

$$\mathcal{S}(B, uv) = \bigcup_{S \in \mathcal{S}(B,u)} \mathcal{S}(S, v).$$

Broadly speaking, the set U determines the set of possible ratios of radii of balls in the successive levels of the upcoming Cantor set construction. For any fixed $B \in \mathcal{B}(X)$, the function f tells us how many balls inside B of a fixed radius we may choose as candidates for the next level in the Cantor set construction, and the sets $\mathcal{S}(B, u)$ describe the position of these candidate balls.

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If $f \equiv 1$, we say the splitting structure $(X, \mathcal{S}, U, f)$ is trivial. Otherwise, fix a sequence $(u_i)_{i\in\mathbb{N}}$ with $u_i \in U$ such that $u_i \mid u_{i+1}$ and $u_i \underset{i\to\infty}{\longrightarrow} \infty$. Then, one can show [5, Theorem 2.1] that for any $B_0 \in \mathcal{B}(X)$, the limit set

$$A_\infty(B_0) := \bigcap_{i=1}^{\infty} \bigcup_{B \in \mathcal{S}(B,u_i)} B$$

is compact and independent of the choice of sequence $(u_i)_{i\in\mathbb{N}}$.

Example 2.7. The standard splitting structure on $X = \mathbb{R}^N$ is with the metric that comes from the sup norm: $\mathbf{d}(x, y) := \|x - y\|_\infty$. The collection of splittings $\mathcal{S}(B, u)$ of any closed ball B according to the integer $u \in U = \mathbb{N}$, into $f(u) = u^N$ parts, is defined as follows: cut B into u^N equal square boxes whose edges have length u times less than the edges length of B. One can check that $(\mathbb{R}^N, \mathcal{S}, \mathbb{N}, f)$ satisfies properties (S1)–(S3) and is the unique splitting structure for $(\mathbb{R}^N, \mathbf{d})$ with $U = \mathbb{N}$ such that $A_\infty(B) = B$ for every $B \in \mathcal{B}(X)$.

In a similar manner, the standard splitting structure on $\{0, 1\}^{\mathbb{N}}$ is with the standard metric $\mathbf{d}(x, y) := 2^{-\min\{i\geq 0: x_i \neq y_i\}}$, $U = \{2^i : i \geq 0\}$, $f(u) = u$, and

$$\mathcal{S}([x_1, \ldots, x_n], 2^i) = \{[x_1, \ldots, x_n, y_1, \ldots, y_i] : (y_1, \ldots, y_i) \in \{0, 1\}^i\},$$

where for any $x \in X$ and $n \geq 0$, we define $[x_1, \ldots, x_n] := \{y : y_k = x_k \text{ for all } 1 \leq k \leq n\} = B(x, 2^{-n})$ to be the cylinder with prequel $(x_1, \ldots, x_n)$, thought of as a ball of radius 2^{-n} (the choice of center is arbitrary).

For any collection $\mathcal{B} \subseteq \mathcal{P}(\mathcal{B}(X))$ of balls and for any $R \in U$, we will write

$$\frac{1}{R}\mathcal{A} := \bigcup_{A\in\mathcal{A}} \mathcal{S}(A, R).$$

Of course, for higher powers of R, we can write $1/R^n\mathcal{A} = 1/R(\cdots(1/R\mathcal{A})\cdots)$.

We can now recall the definition of generalized Cantor sets. Fix some closed ball $B_0 \subseteq X$ and some $R \in U$, and let

$$\mathbf{r} := (r_{m,n}),\ \ m, n \in \mathbb{Z}_{\geqslant 0},\ \ m \leqslant n$$

be a two-parameter sequence of non-negative real numbers. Define $\mathcal{B}_0 := \{B_0\}$. We start by considering the set $1/R\mathcal{B}_0$. The first step in the construction of a generalized Cantor set involves the removal of a collection $\mathcal{A}_{0,0}$ of at most $r_{0,0}$ balls from $1/R\mathcal{B}_0$. We label the surviving collection as $\mathcal{B}_1$. Note that we do not specify here the removed balls, we just give an upper bound for their number. In general, for a fixed $n \geqslant 0$, given the collection $\mathcal{B}_n$, we construct a nested collection $\mathcal{B}_{n+1}$ as follows: let

$$\mathcal{B}_{n+1} := \left(\frac{1}{R}\mathcal{B}_n\right) \setminus \bigcup_{m=0}^{n} \mathcal{A}_{m,n} \quad \text{where } \mathcal{A}_{m,n} \subseteq \frac{1}{R^{n-m+1}}\mathcal{B}_m \quad (0 \leq m \leq n) \tag{10}$$

are collections of removed balls satisfying for every $B \in \mathcal{B}_m$, the inequality

$$\#\mathcal{A}_{m,n}(B) \leq r_{m,n} \quad \text{where } \mathcal{A}_{m,n}(B) := \{A \in \mathcal{A}_{m,n} : A \subseteq B\}.$$

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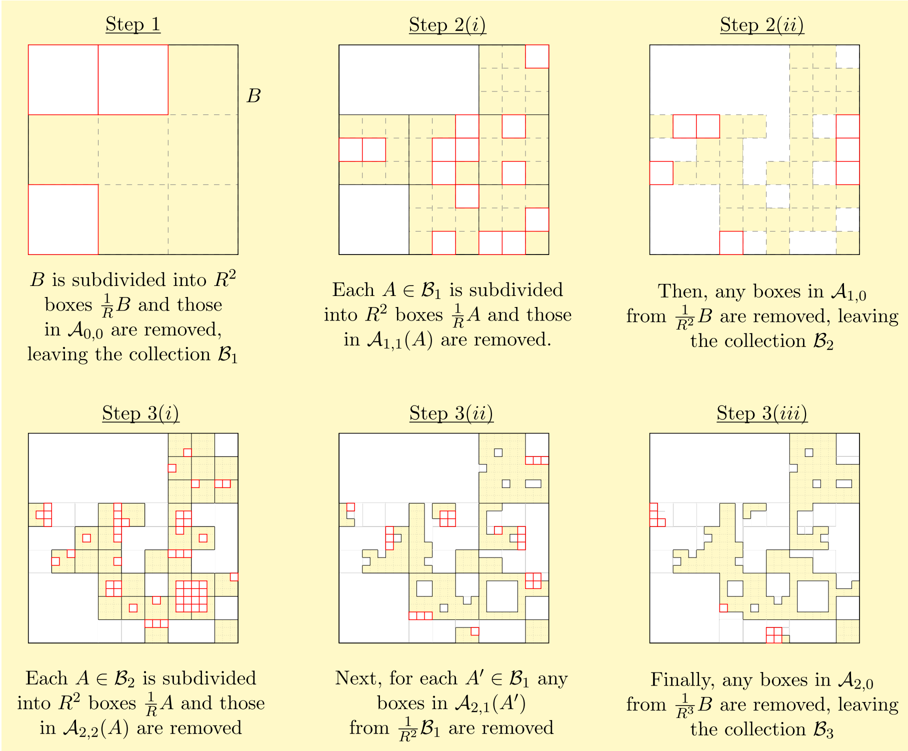

FIGURE 5. Construction of a Cantor winning set.

For a pictorial demonstration in $\mathbb{R}^2$ of such a construction, see Figure 5, in which boxes removed in the current step are colored red.

Define the *limit set* of such a construction to be

$$\bigcap_{n=0} \bigcup_{B \in \mathcal{B}_n} B \tag{11}$$

and denote it by $\mathcal{K}(B_0, R, \mathbf{r})$. Note that the triple $(B_0, R, \mathbf{r})$ does not uniquely determine $\mathcal{K}(B_0, R, \mathbf{r})$ as there is a large degree of freedom in the sets of balls $\mathcal{A}_{m,n}$ removed in the construction procedure. We say a set $\mathcal{K}$ is a *generalized Cantor set* if it can be constructed by the procedure described above for some triple $(B_0, R, \mathbf{r})$. In this case, we may refer to $\mathcal{K}$ as being $(B_0, R, \mathbf{r})$ *Cantor* if we wish to make such a triple explicit and write $\mathcal{K} = \mathcal{K}(B_0, R, \mathbf{r})$.

Example 2.8. In the case that $r_{m,n} = 0$ for every $m < n$, we refer to a $(B_0, R, \mathbf{r})$ Cantor set as a *local Cantor set* and denote it simply by $\mathcal{K}(B_0, R, \{r_n\})$, where $r_n := r_{n,n}$. The simplest class of local Cantor sets occurs when the sequence $\{r_n\}$ is taken to be constant; that is, $r_n := r$. Such (B_0, R, r) Cantor sets are well studied and their measure theoretic properties are well known.

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Example 2.9. The middle-third Cantor set $\mathcal{K}$ is an $(I, 3, 1)$ Cantor set in $\mathbb{R}$, and $\mathcal{K}^2$ is a $(I^2, 3, 5)$ Cantor set in $\mathbb{R}^2$.

Definition 2.10. Fix a ball $B_0 \subseteq X$. Given a parameter $\varepsilon_0 \in (0, 1]$, we say a set $E \subseteq X$ is ε_0 *Cantor winning on* B_0 *(with respect to the splitting structure* $(X, \mathcal{S}, U, f)$*)* if for every $0 < \varepsilon < \varepsilon_0$, there exists $R_\varepsilon \in U$ such that for every $R_\varepsilon \leq R \in U$, the set E contains a $(B_0, R, \mathbf{r})$ Cantor set satisfying

$$r_{m,n} \leq f(R)^{(n-m+1)(1-\varepsilon)} \quad \text{for every } m, n \in \mathbb{N},\ m \leqslant n.$$

If a set $E \subseteq X$ is ε_0 Cantor winning on B_0 for every ball $B_0 \subseteq X$, then we say E is ε_0 *Cantor winning*, and simply that E is *Cantor winning* if it is ε_0 Cantor winning for some $\varepsilon_0 \in (0, 1]$.

It is obvious that if a set is ε_0 Cantor winning, then it is also ε Cantor winning for any $\varepsilon \in (0, \varepsilon_0)$. In particular, the property of being 1 Cantor winning is the strongest possible Cantor winning property.

Remark 2.11. As for the generalized Cantor sets, whenever we mention Cantor winning sets not in the context of a splitting structure, we will refer to a standard splitting structure. In particular, this applies to all theorems in §1.2 where the standard splitting structure in $\mathbb{R}^N$ is considered. Some of those results remain true in a more general context of complete metric spaces.

2.6. *Cantor rich.* The class of *Cantor rich* subsets of $\mathbb{R}$ was introduced in [8]. As was noted in [5], this notion can be generalized to the context of metric spaces endowed with a splitting structure.

Definition 2.12. Let $M \geq 4$ be a real number and B_0 be a ball in X. A set $E \subseteq X$ is (B_0, M) *Cantor rich* if for every R such that

$$f(R) > M \tag{12}$$

and $0 < y < 1$, E contains a $(B_0, R, \mathbf{r})$ Cantor set, where $\mathbf{r} = (r_{m,n})_{0 \leq m \leq n}$ is a two-parameter sequence satisfying

$$\sum_{m=0}^{n} \left(\frac{4}{f(R)}\right)^{n-m+1} r_{m,n} \leq y \quad \text{for every } n \geq 0.$$

Here, E is *Cantor rich in* B_0 if it is (B_0, M) Cantor rich in B_0 for some $M \geq 4$, and E is *Cantor rich* if it is Cantor rich in some B_0.

Note that, following [8], Cantor rich sets are Cantor rich in some ball, while Cantor winning sets are Cantor winning in every ball. Also note that unlike in [8], the inequality in equation (12) is strict. This enables a clearer correspondence with the parameter ε_0 in Definition 2.10 (cf. Corollary 3.11).

2.7. *Diffuse sets and doubling spaces.* Schmidt's game was originally played on a complete metric space. However, to conclude that winning sets have properties such as large Hausdorff dimension, certain assumptions on the underlying space are required. The first time that large intersection with certain subspaces of $\mathbb{R}^n$ was proved using Schmidt's game was in [**24**]. Later, it was realized in [**11**] that, more generally, some spaces form a natural playground for the absolute game. The following notion generalizes the definition of a diffuse set in $\mathbb{R}^n$ as appeared in [**11**].

Definition 2.13. For $0 < \beta < 1$, a closed set $K \subseteq X$ is β *diffuse* if there exists $r_0 > 0$ such that for every $x \in K$, $y \in X$ and $0 < r \leq r_0$, we have

$$(B(x, (1-\beta)r) \setminus B(y, 2\beta r)) \cap K \neq \varnothing. \tag{13}$$

If K is β diffuse for some β, then we say it is *diffuse*.

The condition in equation (13) is satisfied if and only if there exists $z \in K$ such that

$$B(z, \beta r) \subseteq B(x, r) \setminus B(y, \beta r).$$

The definition of diffuse sets is designed so that the β absolute game on a β diffuse metric space cannot end after finitely many steps (as long as the first ball of Bob is small enough). So, if Alice plays the game in such way that she is able to force the outcome to lie in the target set, then the reason for her victory is her strategy and not the limitations of the space in which the game is played.

Remark 2.14. In the definition that appears in [**11**], sets as in Definition 2.13 are called 0-dimensionally diffuse. Moreover, equation (13) is replaced by

$$(B(x, r) \setminus B(y, \beta r)) \cap K \neq \varnothing. \tag{14}$$

Note that

$$B(x, r) \setminus B(y, \beta r) = B(x, (1-\beta')\rho) \setminus B(y, 2\beta'\rho)$$

for $\beta' = \beta/(2+\beta)$ and $\rho = (1+\beta/2)r$, so both equations (13) and (14) give rise to the same class of diffuse sets. We will use equations (13) and (14) interchangeably without further notice.

A more standard definition that is equivalent to diffuseness (see [**26**, Lemma 4.3]) is the following.

Definition 2.15. For $0 < c < 1$, a closed set $K \subseteq X$ is c *uniformly perfect* if there exists $r_0 > 0$ such that for every $x \in K$ and $0 < r \leq r_0$, we have

$$(B(x, r) \setminus B(x, cr)) \cap K \neq \varnothing. \tag{15}$$

If K is c uniformly perfect for some c, then we say it is *uniformly perfect*.

Being diffuse is equivalent to having no singletons as microsets [**11**, Lemma 4.4]. Instead of expanding the discussion on microsets, we refer the interested reader to [**27**], and mention that Ahlfors regular sets are diffuse. In the other direction, diffuse sets support

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Ahlfors regular measures (see Proposition 2.18). We will now give the relevant definitions and provide the results which support the statements above.

Definition 2.16. A Borel measure μ on X is called *Ahlfors regular* if there exist $C, \delta, r_0 > 0$ such that every x in the support of μ and $0 < r \leq r_0$ satisfy

$$\frac{1}{C} r^\delta \leq \mu(B(x, r)) \leq C r^\delta.$$

In this case, we call δ the *dimension* of μ and say that μ is *Ahlfors regular of dimension* δ.

We say that a closed set $K \subseteq X$ is *Ahlfors regular* if it is the support of an Ahlfors regular measure. More exactly, $K \subseteq X$ is δ *Ahlfors regular* if it is the support of an Ahlfors regular measure of dimension δ. This terminology gives rise to a dimensional property of K.

Definition 2.17. The *Ahlfors regularity dimension* of $K \subseteq X$ is

$$\dim_R K = \sup\{\delta : K \text{contains the support of an Ahlfors regular measure of dimension } \delta\}.$$

Note that it is a direct consequence of the mass distribution principle (see [**19**]) that $\dim_R K \leq \dim_H K$, where $\dim_H K$ stands for the *Hausdorff dimension* of K.

PROPOSITION 2.18. *A diffuse set* $K \subseteq X$ *satisfies* $\dim_R K > 0$. *Conversely, if* $\dim_R K > 0$, *then K contains a diffuse set.*

Proof. Suppose K is β diffuse; then every ball in K of radius ρ contains two disjoint balls of radius $\beta\rho$. Construct a Cantor set by recursively replacing every ball of radius ρ with two disjoint subballs of radius $\beta\rho$. Standard calculations show that this Cantor set is Ahlfors regular of dimension $(\log 2)/(-\log \beta) > 0$, so $\dim_R K > 0$.

Suppose $\dim_R K > 0$; then K contains an Ahlfors regular subset of positive dimension, which is diffuse (cf. e.g. [**11**, Lemma 5.1]). □

The following property of Ahlfors regular sets will be used later in the paper.

PROPOSITION 2.19. *If* (X, μ) *is* δ *Ahlfors regular, then there exists* $c > 0$ *such that for any ball* $B \subseteq X$ *and any* $\alpha < 1$, *there exist at least* $c\alpha^{-\delta}$ *balls* $B_i = B(x_i, \alpha \cdot \mathrm{rad}(B)) \subseteq B$ $(i \in \{1, \ldots, I\}, I \geqslant c\alpha^{-\delta})$ *such that* $\mathbf{d}(x_i, x_j) \geqslant 3\alpha \cdot \mathrm{rad}(B)$ *for any* $1 \leqslant i \neq j \leqslant I$.

Proof. It is enough to prove the proposition for all $\alpha \leq 1/2$. Let B' be a ball with the same center as B, but with radius $(1 - \alpha)\mathrm{rad}(B)$, and let $D \subseteq X$ be an arbitrary ball of radius $\mathrm{rad}(D) = 3\alpha \cdot \mathrm{rad}(B)$. Then

$$\mu(D) \leqslant C \cdot (3\alpha)^\delta (\mathrm{rad}(B))^\delta \leqslant C^2 \cdot \left(\frac{3\alpha}{1-\alpha}\right)^\delta \mu(B') \leq C_1 \alpha^\delta \mu(B').$$

Therefore, for any k balls D_i of radius $3\alpha \cdot \mathrm{rad}(B)$ with $k < C_1^{-1}\alpha^{-\delta}$, there exists another ball D whose center lies inside $B' \setminus \bigcup_{i=1}^k D_i$. Hence, there are at least $I = \lceil C_1^{-1}\alpha^{-\delta} \rceil$ balls D_i such that their centers lie inside B' and are distanced by at least $3\alpha \cdot \mathrm{rad}(B)$ from each other. For each $1 \leq i \leq I$, let B_i be a ball with the same center as of D_i but with the radius

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$\alpha \cdot \text{rad}(B)$. Clearly, the balls B_i satisfy all the conditions of the proposition, where $c > 0$ is chosen to ensure the inequality

$$c\alpha^{-\delta} \leqslant \lceil C_1^{-1}\alpha^{-\delta} \rceil$$

for all $\alpha \in [0, 1]$. □

Although winning sets are usually dense, the large dimension property (W1) for some types of winning sets depends on the space X being large enough. For example, absolute winning sets may be empty if the Ahlfors regularity dimension is zero (cf. Remark 4.9). Various conditions are introduced in this context; for example, see [**37**, §11] and [**24**, Theorem 3.1]. More recently, the following condition was introduced in [**5**].

(S4) There exists an absolute constant $C(X)$ such that any ball $B \in \mathcal{B}(X)$ cannot intersect more than $C(X)$ disjoint open balls of the same radius as B.

PROPOSITION 2.20. [**5**, Corollary to Theorem 4.1] *Let X be a complete metric space and let $(X, \mathcal{S}, U, f)$ be a splitting structure. If X satisfies (S4), then for any $B \in \mathcal{B}(X)$ and any Cantor winning set $E \subseteq X$, we have*

$$\dim_H(E \cap A_\infty(B)) = \dim_H A_\infty(B).$$

Condition (S4) is similar to the following definition.

Definition 2.21. A metric space X is *doubling* if there exists a constant $N \in \mathbb{N}$ such that for every $r > 0$, every ball of radius $2r$ can be covered by a collection of at most N balls of radius r.

Doubling spaces satisfy condition (S4), but the converse does not hold. For example, in an ultrametric space, any two non-disjoint balls of the same radius are set-theoretically equal. So an ultrametric space satisfies condition (S4) with $C = 1$, but is not necessarily doubling. As shown in [**5**], the space X needs to satisfy condition (S4) for Cantor winning sets in X to satisfy the large dimension condition (W1) and large intersection condition (W2), and if X is doubling, then Cantor winning sets satisfy condition (W3) with respect to bi-Lipschitz homeomorphisms of X. This makes doubling spaces a natural setting for generalized Cantor sets, and it is under this condition that many of the results of the next section hold.

3. *Connections*

3.1. *The Cantor game.* To facilitate a more natural exposition, we introduce a new game. The 'Cantor game' described below is similar to the games described earlier. The main difference is that this game is designed to be played on a metric space endowed with a splitting structure. Our nomenclature is so chosen because (as we shall demonstrate) under natural conditions, the winning sets for this game are precisely the Cantor winning sets defined in Definition 2.10.

Let X be a complete metric space and let $(X, \mathcal{S}, U, f)$ be a splitting structure. Given $\varepsilon \in (0, 1]$, the ε *Cantor game* is defined as follows: Bob starts by choosing an integer $2 \leq R \in U$ and a closed ball $B_0 \subseteq X$. For any $i \geq 0$, given Bob's ith choice B_i, Alice

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chooses a collection $\mathcal{A}_{i+1} \subseteq 1/R\{B_i\}$ satisfying

$$\#\mathcal{A}_{i+1} \leq f(R)^{1-\varepsilon}.$$

Given $\mathcal{A}_{i+1}$, Bob chooses a ball $B_{i+1} \in 1/R\{B_i\} \setminus \mathcal{A}_{i+1}$, and the game continues. We say that $E \subseteq X$ is *winning for the* ε *Cantor game* if Alice has a strategy which guarantees that equation (3) is satisfied. The *Cantor game* is defined by allowing Alice first to choose $\varepsilon \in (0, 1]$ and then continuing as in the ε Cantor game. We say that E is *winning for the Cantor game* if Alice has a winning strategy for the Cantor game. Note that E is winning for the Cantor game if and only if it is winning for the ε Cantor game for some $\varepsilon \in (0, 1]$. We say that E is *winning for the* ε *Cantor game on* B_0 if Alice has a winning strategy in the ε Cantor game given that Bob's first ball is B_0.

Remark 3.1. A version of the Cantor game may be played in metric spaces with no prescribed splitting structure. Given $c \geq 0$ and $0 < \beta < 1$, Bob will choose a sequence of decreasing balls with radii that shrink at rate β, while Alice will remove a collection of at most β^{-c} closed balls of radius βr where r is the radius of Bob's previous choice, instead of just one such ball as in the absolute game. When $c = 0$, this game coincides with the absolute game.

It is clear that winning sets for the Cantor game contain Cantor winning sets. Indeed, by the construction, for any $R \in U$, a winning set for the ε Cantor game on B_0 contains a $(B_0, R, f(R)^{1-\varepsilon})$ Cantor set which comprises all Bob's possible moves while playing against a fixed winning strategy of Alice. Therefore, it is ε Cantor winning on B_0. We show that the converse is also true.

THEOREM 3.2. *Let $E \subseteq X$. Let $B_0 \subseteq X$ be a closed ball and $0 < \varepsilon_0 \leq 1$. If E is ε_0 Cantor winning on B_0, then E is winning for the ε_0 Cantor game on B_0.*

Proof. We will define a winning strategy for Alice. Fix $R \geq 2$ and let $\delta \geq 0$ be chosen so that $f(R) = R^{\delta}$. Choose $0 < \eta < \varepsilon_0$ so small that

$$\frac{R^{\delta(1-\varepsilon_0+\eta)}}{\lfloor R^{\delta(1-\varepsilon_0)} \rfloor + 1} < 1.$$

Set

$$\varepsilon_1 = \varepsilon_0 - \frac{\eta}{2}, \tag{16}$$

$$\varepsilon_2 = \varepsilon_0 - \eta, \tag{17}$$

and let R_{ε_1} be as in Definition 2.10. Then, for any ℓ such that $R^{\ell} \geq R_{\varepsilon_1}$, since $f(R^{\ell}) = f(R)^{\ell} = R^{\delta\ell}$, there exists a $(B_0, R^{\ell}, (R^{(n-m+1)\ell\delta(1-\varepsilon_1)})_{0\leq m\leq n})$ Cantor set in E. Let ℓ be large enough so that $R^{\ell} \geq R_{\varepsilon_1}$,

$$\left(\frac{R^{\delta(1-\varepsilon_2)}}{\lfloor R^{\delta(1-\varepsilon_0)} \rfloor + 1}\right)^{\ell} < \frac{1}{2}, \tag{18}$$

and

$$R^{-\ell\delta(\eta/2)} < \frac{1}{3}. \tag{19}$$

Such a value exists because of the choice of η.

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Choose a $(B_0, R^\ell, (R^{(n-m+1)\ell\delta(1-\varepsilon_1)})_{0\leq m\leq n})$ Cantor set in E. Let $\mathcal{A} = \{\mathcal{A}_{m,n}\}_{0\leq m\leq n}$ be the removed balls, so that $\mathcal{B}_{n+1} = (1/R^\ell)\mathcal{B}_n \setminus \bigcup_{m=0}^n \mathcal{A}_{m,n}$ for all $n \geq 0$, where $\mathcal{B}_0 = \{B_0\}$.

Define a winning strategy for Alice in the ε_0 Cantor game as follows. For each $i \geq 0$, define a potential function on balls:

$$\varphi_i(B) = \sum_{m=0}^{\lfloor i/\ell\rfloor} \sum_{n=m}^{\infty} \#\{A \in \mathcal{A}_{m,n} : A \cap B \text{ has non-empty interior}\} R^{-(n+1)\ell\delta(1-\varepsilon_2)}. \tag{20}$$

Given Bob's ith move B_i, Alice's strategy is to choose the subcollection $\mathcal{A}_{i+1} \subseteq 1/R\{B_i\}$ consisting of the $\lfloor R^{\delta(1-\varepsilon_0)}\rfloor$ balls with the largest corresponding values of φ_{i+1}. We will show that this is a winning strategy for Alice. Assume Bob chose the sequence of balls $\{B_i\}_{i=0}^\infty$ on his turns, while Alice responded according to the above strategy. It is enough to prove that for every $i \geq 0$,

$$\varphi_{i\ell}(B_{i\ell}) < R^{-i\ell\delta(1-\varepsilon_2)}. \tag{21}$$

Indeed, this inequality would imply that $B_{i\ell}$ does not intersect any of the balls from $\mathcal{A}_{m,n}$ with $n < i$ except at the boundary, which implies that $B_{i\ell} \in \mathcal{B}_i$.

Fix $i \geq 0$ such that equation (21) holds. For every $0 \leq j \leq \ell - 1$, note that $\lfloor (i\ell + j)/(\ell)\rfloor = i$, so by the induction hypothesis in equation (21), we get

$$\varphi_{i\ell+j}(B_{i\ell+j}) \leqslant \varphi_{i\ell}(B_{i\ell}) < R^{-i\ell\delta(1-\varepsilon_2)}.$$

Therefore, for any $n \geq m \geq 0$, if $A \in \mathcal{A}_{m,n}$ is such that $A \cap B_{i\ell+j+1}$ has non-empty interior, then necessarily $n + 1 > i$, and thus by condition (S2), we have $A \subseteq B_{i\ell+j+1}$. So

$$\sum_{A\in \frac{1}{R}\{B_{i\ell+j}\}} \varphi_{i\ell+j}(A) = \varphi_{i\ell+j}(B_{i\ell+j}), \tag{22}$$

and the definition of Alice's strategy yields

$$\varphi_{i\ell+j}(B_{i\ell+j+1}) \leq \frac{1}{\lfloor R^{\delta(1-\varepsilon_0)}\rfloor + 1}\varphi_{i\ell+j}(B_{i\ell+j}) \tag{23}$$

for every $0 \leq j \leq l - 1$. Indeed, there are at least $\lfloor R^{\delta(1-\varepsilon_0)}\rfloor$ balls A, chosen by Alice in her turn, with $\varphi_{i\ell+j}(A) \geqslant \varphi_{i\ell+j}(B_{i\ell+j+1})$. Therefore, if equation (23) fails for some ball $B_{i\ell+j+1}$, we have

$$\varphi_{i\ell+j}(B_{i\ell+j+1}) + \sum_{A \text{ chosen by Alice}} \varphi_{i\ell+j}(A) > \varphi_{i\ell+j}(B_{i\ell+j})$$

which contradicts equation (22).

To estimate $\varphi_{(i+1)\ell}(B_{(i+1)\ell})$, we need to additionally consider the potential coming from the sets $(\mathcal{A}_{i+1,n})_{n\geqslant i+1}$. Clearly, for any $A \in \mathcal{A}_{i+1,n}$, the condition that $A \cap B_{(i+1)\ell}$ has non-empty interior is equivalent to the inclusion $A \subseteq B_{(i+1)\ell}$. Hence, we have

$$\begin{aligned}\varphi_{(i+1)\ell}(B_{(i+1)\ell}) \leq& \left(\frac{1}{\lfloor R^{\delta(1-\varepsilon_0)}\rfloor + 1}\right)^{\ell} \varphi_{i\ell}(B_{i\ell}) \\ &+ \sum_{n=i+1}^{\infty} \#\{A \in \mathcal{A}_{i+1,n} : A \subseteq B_{(i+1)\ell}\} R^{-(n+1)\ell\delta(1-\varepsilon_2)}.\end{aligned}$$

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Since $\#\{A \in \mathcal{A}_{m,n} : A \subseteq B_{m\ell}\} \leq R^{(n-m+1)\ell\delta(1-\varepsilon_1)}$ for any $n \geq m \geq 0$, by equations (16), (17), and (21), we get

$$\begin{aligned}\varphi_{(i+1)\ell}(B_{(i+1)\ell}) &\leq \bigg(\frac{1}{\lfloor R^{\delta(1-\varepsilon_0)}\rfloor + 1}\bigg)^{\ell} R^{-i\ell\delta(1-\varepsilon_2)} + \sum_{n=i+1}^{\infty} R^{(n-i)\ell\delta(1-\varepsilon_1)} R^{-(n+1)\ell\delta(1-\varepsilon_2)} \\ &= \bigg(\frac{R^{\delta(1-\varepsilon_2)}}{\lfloor R^{\delta(1-\varepsilon_0)}\rfloor + 1}\bigg)^{\ell} R^{-(i+1)\ell\delta(1-\varepsilon_2)} + \sum_{n=1}^{\infty} R^{-n\ell\delta(\eta/2)} R^{-(i+1)\ell\delta(1-\varepsilon_2)} \\ &= \bigg(\bigg(\frac{R^{\delta(1-\varepsilon_2)}}{\lfloor R^{\delta(1-\varepsilon_0)}\rfloor + 1}\bigg)^{\ell} + \frac{R^{-\ell\delta(\eta/2)}}{1 - R^{-\ell\delta(\eta/2)}}\bigg) R^{-(i+1)\ell\delta(1-\varepsilon_2)}.\end{aligned}$$

By equations (18) and (19), we get

$$\varphi_{(i+1)\ell}(B_{(i+1)\ell}) < R^{-(i+1)\ell\delta(1-\varepsilon_2)},$$

which is what we need to complete the induction. □

Theorem 3.2 can be rephrased in terms of local Cantor sets (cf. Example 2.8).

COROLLARY 3.3. *Let $E \subseteq X$. Let $B_0 \subseteq X$ be a closed ball and $0 < \varepsilon \leqslant 1$. If E is ε Cantor winning on B_0, then for any $R \geqslant 2$, it contains a $(B_0, R, f(R)^{1-\varepsilon})$ Cantor set.*

Corollary 3.3 means that in Definition 2.10, it is enough to use local Cantor sets. This will be useful in the proof of Theorem 1.13.

For the sake of completeness, let us define a version of the Cantor game called the 'Cantor potential game'. Its name is justified via the analogy with the potential game and the key role that potential functions, like the one in equation (20), play in the proof above that relates it to the Cantor game.

Given $\varepsilon_0 \in (0, 1]$, the ε_0 *Cantor potential game* is defined as follows: Bob starts by choosing $0 < \varepsilon < \varepsilon_0$ and Alice replies by choosing $R_\varepsilon \geq 2$. Then Bob chooses an integer $R \geq R_\varepsilon$ and a closed ball $B_0 \subseteq X$. For any $i \geq 0$, given Bob's ith choice B_i, Alice chooses collections $\{\mathcal{A}_{i+1,k}\}_{k=0}^{\infty}$ such that $\mathcal{A}_{i+1,k} \subseteq 1/R^k\{B_i\}$ and

$$\#\mathcal{A}_{i+1,k} \leq f(R)^{(k+1)(1-\varepsilon)}$$

for all $k \geq 0$. Given $\{\mathcal{A}_{i+1,k}\}_{k=0}^{\infty}$, Bob chooses a ball $B_{i+1} \in 1/R\{B_i\} \setminus \bigcup_{k=0}^{i} \mathcal{A}_{i+1-k,k}$ (if there is no such ball, we say that Alice wins by default). We say that $E \subseteq X$ is ε Cantor potential winning if Alice has a strategy which guarantees that either she wins by default or equation (3) is satisfied. The *Cantor potential game* is defined by allowing Alice first to choose $\varepsilon_0 \in (0, 1]$ and then continuing as in the ε_0 Cantor potential game.

By definition, ε_0 Cantor winning is the same as ε_0 Cantor potential winning. Moreover, there is a natural way to pass between the collections that define a Cantor winning set to the winning strategy in the Cantor potential game. Indeed, if $E \subseteq X$ is ε_0 Cantor winning in B_0 given by the collections $\mathcal{B}_n$ and $\mathcal{A}_{m,n}$ for any $n \geq 0$ and any $0 \leq m \leq n$, then a winning strategy for Alice for the ε_0 Cantor potential game may be given by $\mathcal{A}'_{i+1,k} = \mathcal{A}_{i+1,k+i+1}(B_i)$ where B_i is Bob's ith move. In the other direction, let $\mathcal{A}'$ denote a winning strategy for Alice for the ε_0 Cantor potential game on B_0 for E, that is, if B is the ith move

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of Bob, denote by $\mathcal{A}'_{i+1,k}(B_i)$ Alice's $i+1$st move according to this winning strategy. Define the collections $\mathcal{B}_n$ for $n \geq 0$ and $\{\mathcal{A}_{m,n}\}_{0\leq m\leq n}$ for $n \geq 1$ by recursion as follows: $\mathcal{B}_0 = \{B_0\}$,

$$\mathcal{A}_{m,n} = \bigcup_{B\in\mathcal{B}_m} \mathcal{A}'_{m+1,n-m}(B),$$

for every $n \geq 1$ and $0 \leq m \leq n$, and define $\mathcal{B}_n$ as usual via equation (10). Since $\mathcal{A}'$ is a winning strategy, it implies that the limit set as in equation (11) is contained in E. Theorem 3.2 may then be reinterpreted as saying that the winning sets for the ε Cantor potential game are the same as the winning sets for the ε Cantor game.

3.2. *Cantor, absolute, and potential winning.* We prove Theorem 1.5 as a special case of a more general result connecting potential winning sets with Cantor winning sets, corresponding to the standard splitting structure on $\mathbb{R}^N$.

THEOREM 3.4. *Let $(X, \mathcal{S}, U, f)$ be a splitting structure on a doubling metric space X, fix $B_0 \in \mathcal{B}(X)$, and denote*

$$\delta = \dim_H(A_\infty(B_0)).$$

Fix $\varepsilon_0 \in (0, 1]$ and let $c_0 = \delta(1-\varepsilon_0)$. Then a set $E \subseteq A_\infty(B_0)$ is ε_0 Cantor winning on B_0 with respect to $(X, \mathcal{S}, U, f)$ if and only if E is c_0 potential winning on $A_\infty(B_0)$.

In particular, E is Cantor winning on B_0 if and only if E is potential winning on $A_\infty(B_0)$.

Since potential winning is defined on metric spaces without using a splitting structure, we get the following observation.

COROLLARY 3.5. *Cantor winning in a doubling metric space X is independent of the choice of splitting structure, provided that the limit set $A_\infty(B)$ is fixed.*

A combination of Proposition 1.3 and Theorem 3.4 yields the following corollary.

COROLLARY 3.6. *Assume a complete metric space X is doubling and is endowed with a non-trivial splitting structure $(X, \mathcal{S}, U, f)$, and let $B_0 \in \mathcal{B}(X)$. Then, a set $E \subseteq X$ is 1 Cantor winning on B_0 if and only if E is absolute winning on $A_\infty(B_0)$. In particular, for any splitting structure such that $B = A_\infty(B)$ for every ball B, a set $E \subseteq X$ is 1 Cantor winning if and only if E is absolute winning.*

Remark 3.7. The statements in [**5**, §7] immediately provide us with applications of Corollary 3.6. In particular, various sets from [**5**, Theorems 14 and 15] are in fact absolutely winning.

We now prove Theorem 3.4.

Proof of forward direction. Since X is doubling, we may assume that $f(u) = u^\delta$ for every $u \in U$ (cf. [**5**, Corollary 1]). Suppose that E is ε_0 Cantor winning, and fix $\beta > 0$, $c > c_0$, and $\rho_0 > 0$. Fix $0 < \varepsilon < \varepsilon_0$ such that $\delta(1-\varepsilon) \in (c_0, c)$. Fix large $R \in U$. Its precise value

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will be determined later. Potentially, R may depend on ε, β, c, and ρ_0, and, in particular, it satisfies $R \geq \max(R_\varepsilon, 1/\beta)$ and $R^{\delta(1-\varepsilon)-c} < \frac{1}{2}$. Then by the definition of Cantor winning, E contains some $(B_0, R, \mathbf{r})$ Cantor set $\mathcal{K}$, where

$$r_{m,n} = f(R)^{(n-m+1)(1-\varepsilon)} = R^{\delta(n-m+1)(1-\varepsilon)} \quad \text{for all} m, n \text{with} m \leq n.$$

We now describe a strategy for Alice to win the (c, β) potential game on $A_\infty(B_0)$, assuming that Bob's starting move has radius ρ_0. Recall that for any $B \in \mathcal{B}_m$,

$$\mathcal{A}_{m,n}(B) := \{A \in \mathcal{A}_{m,n} : \ A \subseteq B\}.$$

Let ρ denote the radius of B_0, let D_k denote Bob's kth move (we are denoting Bob's kth move by D_k instead of B_m so as to reserve the letters B and m for the splitting structure framework), and for each $k \in \mathbb{N}$, let $m = m_k \in \mathbb{N}$ denote the unique integer such that $\beta \cdot \mathrm{rad}(D_k) < R^{-m}\rho \leq \mathrm{rad}(D_k)$, assuming such an integer exists. If such a number does not exist, then we say that m_k is undefined. Then Alice's strategy is as follows. On turn k, if m_k is defined, then remove all elements of the set

$$\bigcup_{n \geq m} \mathcal{A}_{m,n} = \bigcup_{B \in \mathcal{B}_m} \bigcup_{n \geq m} \mathcal{A}_{m,n}(B)$$

that intersect Bob's current choice. If m_k is undefined, then delete nothing. Obviously, this strategy, if executable, will make Alice win since the intersection of Bob's balls will satisfy $\bigcap_{k \in \mathbb{N}} D_k \subseteq \mathcal{K} \subseteq E$. To show that it is legal, we need to show that

$$\sum_{\substack{B \in \mathcal{B}_m \\ B \cap D_k \neq \varnothing}} \sum_{n \geq m} r_{m,n} (R^{-(n+1)}\rho)^c \leq (\beta \cdot \mathrm{rad}(D_k))^c. \tag{24}$$

This is because elements of $\mathcal{A}_{m,n}(B)$ all have radius $R^{-(n+1)}\rho$.

To estimate the sum in question, we use the fact that X is doubling. Since the elements of $\mathcal{B}_m$ have disjoint interiors and have radius $R^{-m}\rho \asymp_\beta \mathrm{rad}(D_k)$, the number of them that intersect D_k is bounded by a constant depending only on β. Call this constant C_1. Then the left-hand side of equation (24) is less than

$$\begin{aligned} &C_1 \sum_{n \geq m} R^{\delta(n-m+1)(1-\varepsilon)} (R^{-(n+1)}\rho)^c \\ &= C_1 (R^{-m}\rho)^c \sum_{\ell=1}^{\infty} R^{\delta(1-\varepsilon)\ell} R^{-c\ell} \\ &\leqslant 2C_1 (\mathrm{rad}(D_k))^c R^{\delta(1-\varepsilon)-c} \quad \big(\text{since } R^{\delta(1-\varepsilon)-c} \leqslant \tfrac{1}{2}\big). \end{aligned}$$

By choosing R large enough so that

$$2C_1 R^{\delta(1-\varepsilon)-c} < \beta^c,$$

we guarantee that the move is legal. □

Proof of backward direction. Suppose that $E \subseteq A_\infty(B_0)$ is c_0 potential winning and fix $0 < \varepsilon < \varepsilon_0$. Let $R_\varepsilon = 2$ and fix $R \in U$ such that $R \geq R_\varepsilon$. Fix a large integer $q \in \mathbb{N}$ to be determined, and let $\beta = 1/R^q$ and $c = \delta(1 - \varepsilon)$. Then E is (c, β) potential winning.

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Now for each $k \in \mathbb{N}$ and $B \in \mathcal{B}_{qk}$, let $\mathcal{A}(B)$ denote the collection of sets that Alice deletes in response to Bob's kth move $B_k = B$, assuming that it has been preceded by the moves $B_0, B_1, \ldots, B_{k-1}$ satisfying $B_{i+1} \in \mathcal{S}(B_i, R^q)$ for all $i = 0, \ldots, k-1$. (It is necessary to specify the history to uniquely identify Alice's response because unlike the other variants of Schmidt's game, the potential game is not a *positional* game (see §3.4 and [**37**, Theorem 7]). This is because the regions that were deleted at previous stages of the game affect the overall situation and therefore may affect Alice's strategy.) Since Alice's strategy is winning, we must have

$$E \supseteq A_\infty(B_0) \setminus \bigcup_{k\in\mathbb{N}} \bigcup_{B\in\mathcal{B}_{qk}} \bigcup_{A\in\mathcal{A}(B)} A.$$

Indeed, for every point on the right-hand side, there is some strategy that Bob can use to force the outcome to equal that point (and also not lose by default). So to complete the proof, it suffices to show that the right-hand side contains a $(B_0, R, \mathbf{r})$ Cantor set, where

$$r_{m,n} = f(R)^{(n-m+1)(1-\varepsilon)} = R^{\delta(n-m+1)(1-\varepsilon)} = R^{c(n-m+1)} \quad \text{for all } m,\ n \text{ with } m \le n.$$

For each $m = qk \le n$ and $B \in \mathcal{B}_m$, let $\mathcal{C}(n)$ be the collection of elements of $\mathcal{A}(B)$ whose radius is between $R^{-n}\rho$ and $R^{-(n+1)}\rho$, where ρ is the radius of B_0. Define

$$\mathcal{A}_{m,n}(B) := \{A \in \mathcal{S}(B, R^{n-m+1}) : \text{ there exists } C \in \mathcal{C}(n) \text{ such that } A \cap C \ne \varnothing\}.$$

By equation (7) and the definition of $\mathcal{C}(n)$, we have

$$\sum_{C\in\mathcal{C}(n)} (R^{-(n+1)}\rho)^c \le (\beta R^{-m}\rho)^c;$$

that is,

$$\#\mathcal{C}(n) \le \beta^c (R^{n-m+1})^c.$$

This implies that for $m > n$, we get $\#\mathcal{C}(n) < 1$, that is, $\mathcal{C}(n) = \varnothing$ and, in turn, $\mathcal{A}_{m,n}(B) = \varnothing$. Now we construct a Cantor set $\mathcal{K} \in E$ by removing the collections $\mathcal{A}_{m,n}(B)$ $(m \le n,\ B \in \mathcal{B}_m)$. Then to complete the proof, we just need to show that

$$\#\mathcal{A}_{m,n}(B) \le r_{m,n} = R^{c(n-m+1)}$$

for every $B \in \mathcal{B}_m$. By the doubling property, each element of $\mathcal{C}(n)$ intersects a bounded number of elements of $\mathcal{S}(B, R^{n-m+1})$. Thus,

$$\#\mathcal{A}_{m,n}(B) \le C_3 \#\mathcal{C}(n) \le C_3 \beta^c (R^{n-m+1})^c$$

for some constant $C_3 > 0$ depending on R. By letting β be sufficiently small (that is, letting q be sufficiently large), we can get $C_3\beta^c \le 1$ and therefore

$$C_3\beta^c (R^{n-m+1})^c \le R^{(n-m+1)c},$$

which completes the proof. □

Finally, we briefly mention how the second statement of Theorem 1.5 follows from the first statement. If E is 1 Cantor winning then, by the first part of the theorem, it is c potential winning for any $c > 0$. The statement follows as an immediate corollary of Proposition 1.3.

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3.3. *Schmidt's winning and Cantor winning.* We now move our attention to the links between Schmidt's original game and generalized Cantor set constructions. Thanks to Theorem 3.4, in δ Ahlfors regular spaces, we can treat the notions of Cantor winning sets and potential winning sets as interchangeable. The latter notion will be more convenient in this section. We prove Theorem 1.9 as a special case of a more general result.

THEOREM 3.8. *Let X be a δ Ahlfors regular space and let $E \subseteq X$ be a c_0 potential winning set. Then for all $c > c_0$, there exists $\gamma > 0$ such that for all $\alpha, \beta > 0$ with $\alpha < \gamma(\alpha\beta)^{c/\delta}$, the set E is (α, β) very strong winning.*

To see that Theorem 1.9 follows from Theorem 3.8, notice that $\mathbb{R}^N$ is N Ahlfors regular, and that very strong (α, β) winning sets are (α, β) winning. When E is ε Cantor winning, the desired statement follows upon taking $c_0 = N(1-\varepsilon)$ in the above. The reader may wish to compare the following proof with the proof of Theorem 3.2.

Proof. Consider a strategy for Alice to win the $(c, \widetilde{\beta})$ potential game for the set E, where $\widetilde{\beta} = (\alpha\beta)^q$ for some large $q \in \mathbb{N}$. We want to show that Alice has a winning strategy in the (α, β) very strong game. Given any sequence of Bob's choices in the (α, β) game, say $B_0, \ldots, B_{qk}$, we can consider the corresponding sequence of choices in the $(c, \widetilde{\beta})$ potential game which correspond to Bob choosing the balls $B_0, B_q, \ldots, B_{qk}$. Alice's response to this sequence of moves is to delete a collection $\mathcal{A}(B_{qk})$. Now Alice's corresponding response in the (α, β) very strong game will be: if Bob has made the sequence of moves $B_0, \ldots, B_m$, then she will choose her ball $A_{m+1} \subseteq B_m$ so as to minimize the value at A_{m+1} of the potential function

$$\varphi_m(A) := \sum_{\substack{k\in\mathbb{N}\\ qk\leq m}} \sum_{\substack{C\in\mathcal{A}(B_{qk})\\ C\cap A\neq\varnothing}} \operatorname{diam}^c(C).$$

We will now prove by induction that the inequality

$$\varphi_m(A_{m+1}) \leq (\varepsilon \cdot \operatorname{rad}(B_m))^c$$

holds for all m, where $\varepsilon > 0$ will be chosen later (small but independent of $\widetilde{\beta}$). Suppose that it holds for some m and let B_{m+1} be Bob's next move. Define $r = \alpha \cdot \operatorname{rad}(B_{m+1}) \geqslant \alpha^2\beta \cdot \operatorname{rad}(B_m)$. By Proposition 2.19, there exists a set

$$\mathcal{D} := \{B(x_i, r) \subseteq B_{m+1} : i \in \{1, \ldots, I\}\},$$

which satisfies the following conditions: $I = \#\mathcal{D} \gg \alpha^{-\delta}$, and for any distinct values $1 \leqslant i \neq j \leqslant I$, one has $d(x_i, x_j) \geqslant 3r$, i.e. all balls in $\mathcal{D}$ are separated by distances of at least r.

By the induction hypothesis, for all $C \in \bigcup_{qk\leq m} \mathcal{A}(B_{qk})$ such that $C \cap A_{m+1} \neq \varnothing$, we have $\operatorname{diam}(C) \leq \varepsilon \cdot \operatorname{rad}(B_m)$. So by letting ε be small enough (that is, $\varepsilon \leq \alpha^2\beta$), we can guarantee that all balls in $\mathcal{D}$ make disjoint contributions to $\varphi_m(A_{m+1})$. In other words,

$$\sum_{D\in\mathcal{D}} \varphi_m(D) \leqslant \varphi_m(A_{m+1}).$$

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By choosing A_{m+2} to be the ball from $\mathcal{D}$ which minimizes the function φ_m, we get

$$\alpha^{-\delta}\varphi_m(A_{m+2}) \ll \varphi_m(A_{m+1}).$$

By choosing γ small enough, the inequality $\alpha < \gamma(\alpha\beta)^{c/\delta}$ implies that

$$\varphi_m(A_{m+2}) \leq (\alpha\beta)^c\varphi_m(A_{m+1})/2 = \tfrac{1}{2}(\varepsilon \cdot \mathrm{rad}(B_{m+1}))^c.$$

Note that for the base of induction, $m = -1$, the last estimate is straightforward, since $\varphi_m(A) = 0$ for all balls A, and we can set $A_0 = B_0$.

When we consider $\varphi_{m+1}(A_{m+2})$, there may be a new term coming from a new collection $\mathcal{A}(B_{m+1})$. By the definition of a $(c, \widetilde{\beta})$ potential game, that term is at most $(\widetilde{\beta} \cdot \mathrm{rad}(B_{m+1}))^c$, and thus by choosing q large enough, we can guarantee that

$$\varphi_{m+1}(A_{m+2}) \leqslant (\varepsilon(\alpha\beta)^{m+1}\rho)^c.$$

This finishes the induction.

Since Alice followed the winning strategy for $(c, \widetilde{\beta})$ potential game, we have either

$$\bigcap_{k=1}^{\infty} B_{kq} \subseteq E \quad \text{or} \quad \bigcap_{k=1}^{\infty} B_{kq} \subseteq \bigcup_{k=1}^{\infty} \bigcup_{A\in\mathcal{A}(B_{kq})} A.$$

To finish the proof of the theorem, we need to show that the second inclusion is impossible. If we assume the contrary, then there exists $A \in \mathcal{A}(B_{kq})$ for some k such that every ball B_m and, in turn, every ball A_m has non-empty intersection with A. However, in that case, we have $\varphi_m(A_{m+1}) \geq \mathrm{diam}^c(A)$, which contradicts the fact that $\varphi_m(A_{m+1}) \to 0$ as $m \to \infty$. □

As a corollary of Theorem 3.8, we get a general form of Theorem 1.10.

COROLLARY 3.9. *In an Ahlfors regular space, the intersection of a weak winning set and a potential winning set has full Hausdorff dimension.*

Proof. Let $E \subseteq X$ be a c_0 potential winning set for some $c_0 < \delta$. Fix $c_0 < c < \delta$. By Theorem 3.8, there exists $\gamma > 0$ such that for all $\alpha, \beta > 0$ with $\alpha < \gamma(\alpha\beta)^{c/\delta}$, E is (α, β) very strong winning. In particular, for all $\beta > 0$, there exists $\alpha > 0$ such that E is (α, β) very strong winning. Hence, we can apply Proposition 2.6 which says that $X \setminus E$ is not weak winning (and thus does not contain a weak winning set). So every potential winning set intersects every weak winning set non-trivially.

Now by contradiction, suppose that T is a weak winning set such that $\dim_H(E \cap T) < \delta$. This implies that for any $c \in (\dim_H(E \cap T), \delta)$ and for any $\varepsilon > 0$, one can construct a cover of $E \cap T$ by a countable collection $(A_i)_{i\in\mathbb{N}}$ of balls such that

$$\sum_{i=1}^{\infty} \mathrm{diam}^c(A_i) < \varepsilon.$$

In other words, this cover can be chosen by Alice in a single move of the c potential game, so $X \setminus (E \cap T)$ is potential winning. So by the intersection property of potential winning sets, $E \setminus T = E \cap (X \setminus (E \cap T))$ is potential winning. However, then $X \setminus T$ is a

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potential winning set whose complement is weak winning, contradicting what we have just shown. □

3.4. *Proof of Theorem 1.11.* In this section, we prove that 1/2 winning implies absolute winning for subsets of the real line. The same notation, terminology, and observations as in [37] are employed. Recall that $\mathcal{B}(\mathbb{R})$ stands for the set of all closed intervals in $\mathbb{R}$.

In Schmidt's (α, β) game, the method Alice uses to determine where to play her intervals A_i is called a *strategy*. Formally, a strategy $F := \{f_1, f_2, \ldots\}$ is a sequence of functions $f_i : \mathcal{B}(\mathbb{R})^i \to \mathcal{B}(\mathbb{R})$ satisfying $\mathrm{rad}(f_i(B_0, B_1, \ldots, B_{i-1})) = \alpha \cdot \mathrm{rad}(B_{i-1})$ and $f_i(B_0, B_1, \ldots, B_{i-1}) \subseteq B_{i-1}$. A set E is a winning set for the (α, β) game if and only if there exists a strategy determining where Alice should place her intervals $A_i := f_i(B_0, B_1, \ldots, B_{i-1})$, so that however Bob chooses his intervals $B_i \subseteq A_i$, the intersection point $\bigcap_i B_i$ lies in E. Such a strategy is referred to as a *winning strategy (for E)*. Schmidt [37, Theorem 7] observed that the existence of a winning strategy guarantees the existence of a *positional* winning strategy; that is, a winning strategy for which the placement of a given interval by Alice needs only to depend upon the position of Bob's immediately preceding interval, not on the entirety of the game so far. To be precise, a winning strategy $F := (f_1, f_2, \ldots)$ is *positional* if there is a function $f_0 : \mathcal{B}(X) \to \mathcal{B}(\mathbb{R})$ such that each function $f_i \in F$ satisfies $f_i(B_0, \ldots, B_{i-1}) = f_0(B_{i-1})$. Employing a positional strategy will not give us any real advantage in the proof that follows; however, it will allow us to simplify our notation somewhat. Without confusion, we will write $A_i = F(B_{i-1}) = f_0(B_{i-1})$ where F is a given positional winning strategy for Alice and f_0 is the function witnessing F's positionality.

We refer to a sequence $\{B_0, B_1, \ldots\}$ of intervals as an *F-chain* if it consists of the moves Bob has made during the playing out of an (α, β) game with target set E in which Alice has followed the winning strategy F. By definition, we must have $\bigcap_i B_i \subseteq E$. Furthermore, we say a finite sequence $\{B_0, \ldots B_{n-1}\}$ is an *F_n-chain* if there exist $B_n, B_{n+1}, \ldots$ for which the infinite sequence $\{B_0, \ldots B_{n-1}, B_n, B_{n+1} \ldots\}$ is an F-chain. One can readily verify (see [37, Lemma 1]) that if $\{B_0, B_1, \ldots\}$ is a sequence of intervals such that for every $n \in \mathbb{N}$, the finite sequence $\{B_0, B_1, \ldots B_{n-1}\}$ is an F_n-chain, then $\{B_0, B_1, \ldots\}$ is an F-chain.

Outline of the strategy. We begin with some interval $b_0 \subseteq \mathbb{R}$ of length $2 \cdot \mathrm{rad}(b_0)$, some $\varepsilon \in (0, 1)$, some R sufficiently large, and a 1/2 winning set $E \subseteq \mathbb{R}$. Fix $\alpha = 1/2$ and $\beta = 2/R$, so that $\alpha\beta = 1/R$. We will construct for every $R \geq 10^{1/(1-\varepsilon)}$, a local $(b_0, R, 10)$ Cantor set $\mathcal{K}$ lying inside E. This is sufficient to prove that E is 1 Cantor winning.

By assumption, the set E is $(1/2, 2/R)$ winning; thus, there exists a positional winning strategy F such that however we choose to place Bob's intervals B_i in the game, the placement of Alice's subsequent intervals $A_i = F(B_{i-1})$ guarantees that the unique element of $\bigcap_i B_i$ will fall inside E. We will, in a sense, play as Bob in many simultaneous $(1/2, 2/R)$ games, each of which Alice will win by following her prescribed strategy F.

Our procedure is as follows. The construction of the local Cantor set $\mathcal{K}$ comprises the construction of subcollections $\mathcal{B}_n$ of intervals in $1/R^n \mathcal{B}_0$ for each $n \in \mathbb{N}$, where $\mathcal{B}_0 := \{b_0\}$. Construction will be carried out iteratively in such a way that for any interval $b \in \mathcal{B}_n$, there exists an F_n-chain $\{B_0, B_1, \ldots, B_{n-1}\}$ satisfying $b \subseteq B_{n-1}$. Moreover,

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for each *ancestor* b' of b (that is, the unique interval $b' \in \mathcal{B}_k$ with $b \subseteq b'$, for some given $0 \leq k < n$), the subsequence $\{B_0, B_1, \ldots, B_{k-1}\}$ will coincide with the F_k-chain constructed for b' satisfying $b' \subseteq B_{k-1}$ from the earlier inductive steps. We write $B_n(b)$ if the dependence of the interval B_n on an interval b is not clear from context. In view of the previous discussion, upon completion of the iterative procedure, it follows that for any sequence of intervals $\{b_i\}_{i\in\mathbb{N}}$ with $b_i \in \mathcal{B}_i$, the associated sequence $\{B_0, B_1, \ldots\}$ must be an F-chain satisfying $b_i \subseteq B_{i-1}$. In this way, for every point $\mathbf{x}$ in the Cantor set $\mathcal{K}$, we will establish the existence of an F-chain $\{B_0, B_1, \ldots\}$ for which $\{\mathbf{x}\} = \bigcap_i B_i$; namely, we may simply choose the F-chain associated with a sequence of intervals $\{b_i\}_{i\in\mathbb{N}}$ with $b_i \in \mathcal{B}_i$ satisfying $\bigcap_{i\in\mathbb{N}} b_i = \{\mathbf{x}\}$. It follows that $\mathbf{x}$ must fall within E and, since $\mathbf{x} \in \mathcal{K}$ was arbitrary, that $\mathcal{K} \subseteq E$ as required.

We begin the first step of the inductive process. Given $\mathcal{B}_0 = \{b_0\}$, the idea is to construct the collection $\mathcal{B}_1 \subseteq 1/R\mathcal{B}_0$ in such a way that for every $b \in \mathcal{B}_1$, there exists a place Bob may play his first interval B_0 in a $(1/2, 2/R)$ game so that Alice's first interval $A_1 := F(B_0)$ (as specified by the winning strategy F) contains b; that is, we construct $\mathcal{B}_1$ so that for every $b \in \mathcal{B}_1$, there exists an F_1-chain $\{B_0\}$ satisfying $b \subseteq A_1 := F(B_0)$. For technical reasons, in practice, we will ensure the stronger condition that $b \subseteq (1 - 2/R)A_1$, where κB denotes the interval with the same center as B but with radius multiplied by κ.

Recall that for his opening move in a $(1/2, 2/R)$ game, Bob may choose any interval in $\mathbb{R}$. In particular, he may choose any interval of radius $2 \cdot \mathrm{rad}(b_0)$ with center lying in b_0, therefore covering b_0. Let $\mathcal{W}_1(b_0) \subseteq \mathcal{B}(\mathbb{R})$ be the set of all 'winning' intervals for Alice (as prescribed by the winning strategy F) corresponding to any opening interval of this kind that Bob might choose to begin the game with; that is, let

$$\mathcal{W}_1(b_0) := \{F(B) : B \in \mathcal{B}(\mathbb{R}),\ \mathrm{rad}(B) = 2 \cdot \mathrm{rad}(b_0),\ \mathrm{cent}(B) \in b_0\}.$$

Note that this set could either be uncountable or, since for each $A \in \mathcal{W}_1(b_0)$, we have $\mathrm{rad}(A) = \alpha \cdot (2 \cdot \mathrm{rad}(b_0)) = \mathrm{rad}(b_0)$, it could simply coincide with the singleton $\mathcal{B}_0$. Moreover, since $\alpha = 1/2$, we must have for every interval B that $\mathrm{cent}(B)$ is contained in $F(B)$, and so $\mathcal{W}_1(b_0)$ is a cover of b_0. This property is unique to the case $\alpha = 1/2$.

LEMMA 3.10. *Let $b \in \mathcal{B}(\mathbb{R})$ be any closed interval and let $\mathcal{W} \subseteq \mathcal{B}(\mathbb{R})$ be a cover of b by closed intervals of radius* $\mathrm{rad}(b)$*. Then, for any strictly positive $\varepsilon <$* $\mathrm{rad}(b)$*, there exists a subset $\mathcal{W}^*(\varepsilon) \subseteq \mathcal{W}$ of cardinality at most two that covers b except for an open interval of length at most ε.*

Proof of Lemma 3.10. Fix n such that $n^{-1}\mathrm{rad}(b) \leq \varepsilon$, and let $\mathrm{cent}(b) - \mathrm{rad}(b) = x_0, \ldots, x_n = \mathrm{cent}(b) + \mathrm{rad}(b)$ be an evenly spaced sequence of points starting and ending at the two endpoints of b. For each $i = 0, \ldots, n$, let W_i be an interval in $\mathcal{W}$ containing x_i. Now let j be the smallest integer such that $x_n \in W_j$. If $j = 0$, then $W_0 = b$ and we are done. Otherwise, since $x_n \notin W_{j-1}$, we have $x_0 \in W_{j-1}$ and thus $[x_0, x_{j-1}] \subseteq W_{j-1}$, whereas $[x_j, x_n] \subseteq W_j$. Thus, the intervals W_{j-1} and W_j cover b except for the interval (x_{j-1}, x_j), which is of length at most ε. □

We continue the proof of Theorem 1.11. Take $b = b_0$, $\varepsilon = \mathrm{rad}(b_0)/R$, and $\mathcal{W} = \mathcal{W}_1$ in Lemma 3.10. Then there exists a subcollection $\mathcal{W}_1^* := \{A_1^L, A_1^R\}$ of $\mathcal{W}_1$ of cardinality at most two covering b_0 except for an open interval $I(b_0)$ of length at most $\mathrm{rad}(b)/R$. It is possible that $I(b_0)$ is empty. If $\mathcal{W}_1^*$ contains only one interval, say $\mathcal{W}_1^* = \{b'\}$, then set $A_1^L = A_1^R = b'$. Let B_0^L and B_0^R be two choices for Bob's opening move with centers in b_0 that satisfy $A_1^L = F(B_0^L)$ and $A_1^R = F(B_0^R)$; such choices exist by the definition of $\mathcal{W}_1$. If we had $A_1^L = A_1^R$, then assume $B_0^L = B_0^R$.

We now discard the 'bad' intervals that will not appear in the generalized Cantor set $\mathcal{K}$. Let

$$\mathbf{Bad}_1 := \left\{ b' \in \frac{1}{R}\mathcal{B}_0 : b' \cap I(b_0) \neq \varnothing \text{ or } b' \not\subseteq ((1-2/R)A_1^L \cup (1-2/R)A_1^R) \right\}.$$

Since any interval of length $\mathrm{rad}(b_0)/R$ may intersect at most two intervals from $1/R\mathcal{B}_0$, and there are at most five intervals (of length at most $\mathrm{rad}(b_0)/R$) that need to be intersected by b' for it to be in $\mathbf{Bad}_1$, it is immediate that $\#\mathbf{Bad}_1 \leq 10$. It follows that $\mathcal{B}_1 := 1/R\mathcal{B}_0 \setminus \mathbf{Bad}_1$ satisfies the required conditions for our generalized Cantor set. For each $b \in \mathcal{B}_1$, we associate one of the F_1-chains $\{B_0^L\}$ or $\{B_0^R\}$, depending on whether b is a subset of A_1^L or A_1^R, respectively. If b lies in both A_1^L and A_1^R (which is possible if $I(b_0)$ was empty), then we will assign as a precedent whichever of the intervals B_0^L or B_0^R has the 'leftmost' center. This completes the first step of the inductive construction process.

Assume now that we have constructed the collection $\mathcal{B}_i$ so that for each $b \in \mathcal{B}_i$, we have an F_i-chain $\{B_0, \ldots, B_{i-1}\}$ with $\mathrm{rad}(B_{i-1}) = 2R \cdot \mathrm{rad}(b)$ and $b \subseteq (1-2/R)F(B_{i-1})$. We outline the procedure to construct the subsequent collection $\mathcal{B}_{i+1}$ and the corresponding F_{i+1}-chains.

Fix $b \in \mathcal{B}_i$ with associated F_i-chain $\{B_0, \ldots, B_{i-1}\}$. Assume Alice has just played her ith move $A_i := F(B_{i-1})$, an interval of radius $R \cdot \mathrm{rad}(b)$, in a $(1/2, 2/R)$ game corresponding to this F_i-chain. Bob may then choose any interval of radius $2 \cdot \mathrm{rad}(b)$ inside A_i. Since $b \subseteq (1-2/R)A_i$, Bob is free to choose as B_i any interval of radius $2 \cdot \mathrm{rad}(b)$ centered in b; every such choice b' will constitute a legal move since $\mathrm{rad}(b') = 2/R\mathrm{rad}(A_i)$ and $b \subseteq (1-2/R)A_i$ together imply $b' \subseteq A_i$. In particular, the collection

$$\mathcal{W}_{i+1}(b) := \{F(b') : b' \in \mathcal{B}(\mathbb{R}), \mathrm{rad}(b') = 2 \cdot \mathrm{rad}(b_0)/R^i, \mathrm{cent}(b') \in b\}$$

is a cover of b and consists of candidate 'winning' moves that could be played by Alice after any legal move B_i (with center in b) played by Bob. As before, we apply Lemma 3.10 to find a subset of $\mathcal{W}_{i+1}(b)$ of cardinality at most two that covers b except for an open subinterval $I(b)$ of length at most $\mathrm{rad}(b)/R$. If the subcover contains two intervals, denote them by $A_{i+1}^L(b)$ and $A_{i+1}^R(b)$; otherwise, if the subcover consists of a single ball b', then let $A_{i+1}^L(b) = A_{i+1}^R(b) = b'$. Denote by $B_i^L(b)$ and $B_i^R(b)$ some choice of intervals centered in b for which $A_{i+1}^L(b) = F(B_i^L(b))$ and $A_{i+1}^R(b) = F(B_i^R(b))$, respectively. Define

$$\mathbf{Bad}_{i+1}(b) := \left\{ b' \in \frac{1}{R}\{b\} : b' \cap I(b) \neq \varnothing \text{ or } b' \right.$$
$$\left. \not\subseteq ((1-2/R)A_{i+1}^L(b) \cup (1-2/R)A_{i+1}^R(b)) \right\}$$

and let $\mathcal{B}_{i+1}(b) := 1/R\{b\} \setminus \mathbf{Bad}_{i+1}(b)$. By the same arguments as before, it is clear that $\#\mathbf{Bad}_{i+1}(b) \leq 10$ for each $b \in \mathcal{B}_i$.

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Finally, for each $b' \in \mathcal{B}_{i+1}(b)$, we assign the F_{i+1}-chain $\{B_0, \ldots, B_{i-1}, B_i^L(b)\}$ or $\{B_0, \ldots, B_{i-1}, B_i^R(b_i)\}$ depending on whether b' lies in $A_{i+1}^L(b_i)$ or $A_{i+1}^R(b_i)$, respectively. Again, if $I(b)$ was empty and b was contained in both $A_{i+1}^L(b)$ and $A_{i+1}^R(b)$, then we choose whichever of $B_i^L(b)$ and $B_i^R(b)$ has the leftmost center as a convention.

Upon setting

$$\mathcal{B}_{i+1} := \bigcup_{b \in \mathcal{B}_i} \mathcal{B}_{i+1}(b),$$

we see that the conditions of our generalized Cantor set $\mathcal{K}(b_0, R, 10)$ are satisfied. This completes the inductive procedure.

3.5. *Cantor rich and Cantor winning.* In this section, we prove the following statement which implies Theorem 1.13.

THEOREM 3.11. *Let $(X, \mathcal{S}, U, f)$ be a splitting structure, let $B_0 \subseteq X$ be any ball, and let $\varepsilon > 0$ be a real number. If $E \subseteq X$ is ε Cantor winning in B_0, then it is (B_0, M) Cantor rich with*

$$M = 4^{1/\varepsilon}.$$

Conversely, assume that X is doubling and let $M \geqslant 4$ be a real number. If E is (B_0, M) Cantor rich, then it is ε Cantor winning in B_0, for any ε for which there exists $R \in U$ such that

$$M < f(R) \leq 4^{1/\delta\varepsilon}. \tag{25}$$

In particular, if $X = \mathbb{R}$ with the standard splitting structure, then E is M Cantor rich if and only if it is ε Cantor winning with

$$\varepsilon = \log_4 \left(\frac{1}{\lfloor M \rfloor + 1} \right).$$

Proof. Assume E is ε Cantor winning in B_0. Then, by Corollary 3.3, for every $2 \leq R \in U$, there exists a $(B_0, R, f(R)^{1-\varepsilon})$ Cantor set contained in E. Fix an integer R such that $f(R) > M$ and a real number $y > 0$. Since $4/f(R)^\varepsilon < 1$, we can choose $\ell > 0$ such that $4/f(R) \cdot (4/f(R)^\varepsilon)^\ell < y$. Consider a $(B_0, R^\ell, f(R)^{\ell(1-\varepsilon)})$ Cantor set contained in E and denote its Cantor sequence by $\{\mathcal{B}_n\}_{n=0}^\infty$. Define another Cantor sequence $\{\mathcal{B}'_n\}_{n=0}^\infty$ by setting $\mathcal{B}'_{\ell k} = \mathcal{B}_k$ for all $k \geq 0$ and $\mathcal{B}'_n = 1/R\mathcal{B}_{n-1}$ whenever $n \neq \ell k$. Then the limit set

$$\bigcap_{n=1}^{\infty} \bigcup_{B \in \mathcal{B}'_n} B$$

is a $(B_0, R, (r_{m,n})_{0 \leq m \leq n})$ generalized Cantor set, with $r_{\ell(k-1),\ell k} = f(R)^{\ell(1-\varepsilon)}$ for all $k \geq 0$, and $r_{m,n} = 0$ for all other pairs. Then, for any $k > 0$ and $n = \ell k$, we get

$$\sum_{m=0}^{\ell k} \left(\frac{4}{f(R)} \right)^{\ell k - m + 1} r_{m,\ell k} = \left(\frac{4}{f(R)} \right)^{\ell+1} f(R)^{\ell(1-\varepsilon)} = \frac{4}{f(R)} \cdot \left(\frac{4}{f(R)^\varepsilon} \right)^\ell < y.$$

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For every $n \neq \ell k$, we have

$$\sum_{m=0}^{n} \left(\frac{4}{f(R)}\right)^{n-m+1} r_{m,n} = 0 < y.$$

For the other direction, suppose that E is (B_0, M) Cantor rich. Let $c_0 = \delta(1-\varepsilon)$. By Theorem 3.4, it is enough to show that E is c_0 potential winning in B_0. We argue as in the proof of Theorem 3.4. Choose R such that $M < f(R) \leq 4^{1/\varepsilon}$ and fix any $0 < \beta \leq 1/R$, $c > c_0$. Fix a small enough $y > 0$ whose precise value will be determined later. By the definition of Cantor rich, E contains some $(B_0, R, \mathbf{r})$ Cantor set $\mathcal{K}$, where $\mathbf{r}$ satisfies

$$\sum_{m=0}^{n} \left(\frac{4}{f(R)}\right)^{n-m+1} r_{m,n} < y$$

for every $n \in \mathbb{N}$. In particular, for every $m, n \in \mathbb{N}$, we have

$$r_{m,n} < y\left(\frac{f(R)}{4}\right)^{n-m+1} \leq yR^{(n-m+1)c_0}.$$

We now describe a strategy for Alice to win the (c, β) potential game on $A_\infty(B_0)$. As in the proof of Theorem 3.4, let ρ denote the radius of B_0, let D_k denote Bob's kth move, and for each $k \in \mathbb{N}$, let $m = m_k \in \mathbb{N}$ denote an integer such that $\beta \cdot \mathrm{rad}(D_k) < R^{-m}\rho \leq \mathrm{rad}(D_k)$. The inequality $\beta \leqslant 1/R$ ensures that such m exists. Then Alice's strategy is as follows: on turn k, remove all elements of the set

$$\bigcup_{B \in \mathcal{B}_m} \bigcup_{n \geq m} \mathcal{A}_{m,n}(B)$$

that intersect Bob's current choice. This strategy, if executable, will make Alice win since the intersection of Bob's balls will satisfy $\bigcap_{k\in\mathbb{N}} D_k \subseteq \mathcal{K} \subseteq E$. To show that it is legal, we need to show that

$$\sum_{\substack{B \in \mathcal{B}_m \\ B \cap D_k \neq \varnothing}} \sum_{n \geq m} r_{m,n}(R^{-(n+1)}\rho)^c \leq (\beta \cdot \mathrm{rad}(D_k))^c. \tag{26}$$

This is enough because elements of $\mathcal{A}_{m,n}(B)$ all have radius $R^{-(n+1)}\rho$. Since the elements of $\mathcal{B}_m$ are disjoint and have radius $R^{-m}\rho \asymp_\beta \mathrm{rad}(D_k)$, the number of them that intersect D_k is bounded by a constant depending only on β. Call this constant C_1. Then the left-hand side of equation (26) is less than

$$\begin{aligned} C_1 \sum_{n \geq m} yR^{(n-m+1)c_0}\left(R^{-(n+1)}\rho\right)^c &= C_1(R^{-m}\rho)^c y \sum_{\ell=1}^{\infty} R^{(c_0-c)\ell} \\ &\leqslant C_1(\mathrm{rad}(D_k))^c \frac{R^{(c_0-c)}}{1-R^{(c_0-c)}} y. \end{aligned}$$

By choosing y so that

$$y < \frac{\beta^c(1-R^{(c_0-c)})}{C_1 R^{(c_0-c)}},$$

we guarantee that the move is legal. □

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4. *Intersection with fractals*

The concept of winning sets gives a notion of largeness which is orthogonal to both category and measure. It was first noticed in [11] that absolute winning subsets of $\mathbb{R}^N$ are also large in a dual sense: they intersect every non-empty diffuse set. The Cantor and the potential games, together with the definition of the Ahlfors regularity dimension, allow us to quantify this observation. Based on the Borel determinacy property for Schmidt games [25], we also prove a partial converse.

THEOREM 4.1. *If $E \subseteq X$ is c_0 potential winning, then $E \cap K \neq \varnothing$ for every closed set $K \subseteq X$ with $\dim_R K > c_0$. If E is Borel, then the converse holds.*

Combining this with Propositions 2.18 and 1.3 yields the following corollary.

COROLLARY 4.2. *If $E \subseteq X$ is Borel and $E \cap K \neq \varnothing$ for any diffuse set K, then E is absolute winning.*

Remark 4.3. One can check that Theorem 1.14 follows from Theorem 4.1 together with Theorem 3.4. Indeed, by Theorem 3.4, the set $E \in \mathbb{R}^N$ is ε_0 Cantor winning if and only if it is c_0 potential winning for $c_0 = N(1 - \varepsilon_0)$. Then, Theorem 1.14 is a straightforward corollary of Theorem 4.1.

We will need the following observation regarding the potential game.

LEMMA 4.4. *Let X be a complete metric space, $K \subseteq X$ a closed subset, and let $c > 0$. If $E \subseteq X$ is c potential winning in X, then $E \cap K$ is c potential winning in K.*

Proof. Assume $c' > c$ and $\beta > 0$, and fix a winning strategy for Alice on E for the (c', β) potential game on X. Define a winning strategy for Alice on $E \cap K$ for the $(c', 3\beta)$ potential game on K as follows. Assume $i \geq 0$ and that $B_i = B(x_i, r_i) \subseteq K$ with $x_i \in K$ is chosen by Bob in his ith move. Consider the ball with the same center and radius in X, and apply the winning strategy of Alice on X to get a collection of balls $\mathcal{A}_{i+1}$. For every $A \in \mathcal{A}_i$ for which $A \cap K \neq \varnothing$, fix a point $z(A) \in A \cap K$. The triangle inequality implies that $A \cap K \subseteq B(z(A), 3\text{rad}(A))$ for every $A \in \mathcal{A}_{i+1}$. For the game in K, Alice will choose the collection of balls in K defined by

$$\mathcal{A}'_{i+1} = \{B(z(A), 3\text{rad}(A)) : A \in \mathcal{A}_{i+1} \text{ and } A \cap K \neq \varnothing\}.$$

On one hand,

$$\sum_{A \in \mathcal{A}'_{i+1}} \text{rad}(A)^{c'} \leq 3^{c'} \sum_{A \in \mathcal{A}_{i+1}} \text{rad}(A)^{c'} < 3^{c'} (\beta \cdot \text{rad}(B_i))^{c'} = (3\beta \cdot \text{rad}(B_i))^{c'}$$

and, therefore, this move is legal. On the other hand,

$$\bigcup_{A \in \mathcal{A}_{i+1}} A \cap K \subseteq \bigcup_{A \in \mathcal{A}'_{i+1}} A.$$

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Since K is closed, applying this strategy for every i guarantees that, if $\mathrm{rad}(B_i) \to 0$, then either

$$\bigcap_{i=0}^{\infty} B_i \subseteq E \cap K$$

or

$$\bigcap_{i=0}^{\infty} B_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{A \in \mathcal{A}'_i} A.$$

This shows that $E \cap K$ is $(c', 3\beta)$ potential winning for every $c' > c$ and $\beta > 0$, and hence, it is c potential winning. □

Proof of Theorem 4.1. Assume that $E \subseteq X$ is c_0 potential winning and let $K \subseteq X$ be a c Ahlfors regular set with $c > c_0$. Lemma 4.4 implies that $E \cap K$ is winning for the c_0 potential game restricted to K. Therefore, by Theorem 3.8, $E \cap K$ contains an (α, β) winning set for certain pairs (α, β) and hence it is non-empty.

For the converse, assume E is Borel, and that E is not c_0 potential winning. By the Borel determinacy theorem [25, Theorem 3.1], Bob has a winning strategy to make sure the c_0 potential game ends in $X \setminus E$. (In fact, [25, Theorem 3.1] does not deal with potential games. However, its proof can be easily modified to cover them. In the notation of that paper, it is clear from the definition of who wins the potential game that there exists a Borel set $A \subseteq Z \times X$ such that if $\omega \in E_\Gamma$ is a play of the game, then ω is a win for Alice if and only if either $\omega \notin Z$ or $(\omega, \iota(\omega)) \in B$. Combining with [25, Lemma 3.3] shows that the set of plays of the game that result in a win for Alice is Borel.) Fix this strategy and let $\beta > 0$ and $c > c_0$ be Bob's choices according to this strategy. We will show that $X \setminus E$ contains Ahlfors regular sets of dimension arbitrarily close to c.

Fix a small number $\gamma > 0$. Denote by $\mathcal{A}_{i+1}$ the collection of balls chosen by Alice in response to Bob's ith move B_i. We will consider strategies for Alice where she only chooses non-empty collections on some turns, namely $\mathcal{A}_{i+1} \neq \varnothing$ only if

$$\mathrm{rad}(B_i) \leq \gamma^n \mathrm{rad}(B_0) < \mathrm{rad}(B_{i-1}) \quad \text{for some } n \in \mathbb{N}.$$

Such turns $i = i(n)$ will be called *good turns*. If $B_0, \ldots, B_{i(n)}$ versus $\mathcal{A}_1, \ldots, \mathcal{A}_{i(n)+1}$ is a history of the game, then we let $f(\mathcal{A}_{i(n)+1})$ denote the next ball that Bob plays on a good turn if the game continues with Bob playing his winning strategy and Alice playing dummy moves (that is, choosing the empty collection), that is, $f(\mathcal{A}_{i(n)+1}) := B_{i(n+1)}$. Next, we let

$$N = \left\lfloor \left(\frac{\beta^2}{3\gamma} \right)^c \right\rfloor.$$

If $B_0, \ldots, B_{i(n)}$ versus $\mathcal{A}_1, \ldots, \mathcal{A}_{i(n)}$ is a history of the game, then we define a sequence of balls $(g_k(B_{i(n)}))_{k=1}^N$ by recursion: we let $g_1(B_{i(n)}) := f(\varnothing)$, and if $g_1, \ldots, g_{k-1}$ are defined for some $k \geq 1$, then we let

$$g_k(B_{i(n)}) = f(\{\mathcal{N}(g_1(B_{i(n)}), \gamma^{n+1}\mathrm{diam}(B_0)), \ldots, \mathcal{N}(g_{k-1}(B_{i(n)}), \gamma^{n+1}\mathrm{diam}(B_0))\}).$$

Here, $\mathcal{N}(A, d)$ denotes the d-neighborhood of a set A, that is, $\mathcal{N}(A, d) := \{x \in X : \mathbf{d}(A, x) \leqslant d\}$. We will show later (using the definition of N) that it is legal for Alice

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to play the collection appearing in the right-hand side, and hence the right-hand side is well defined. Note that since $B_{i(n)}$ is the nth good turn in the history $B_0, \ldots, B_{i(n)}$ versus $\mathcal{A}_1, \ldots, \mathcal{A}_{i(n)}$, for each $k = 1, \ldots, N$, $g_k(B_{i(n)})$ is the $(n+1)$st good turn in its corresponding history. We call these balls the *children* of $B_{i(n)}$. The children of $B_{i(n)}$ are disjoint, because if $g_j(B_{i(n)}) \cap g_k(B_{i(n)}) \neq \varnothing$ for some j, k with $j < k$, then $g_k(B_{i(n)}) \subseteq \mathcal{N}(g_j(B_{i(n)}), \gamma^{n+1}\mathrm{diam}(B_0))$, and thus Alice wins by default in the game in which Bob played his winning strategy ending in move $g_k(B_{i(n)})$, which is a contradiction.

Now construct a Cantor set $\mathcal{K}$ as follows: let $\mathcal{B}_0 = \{B_0\}$, and if $\mathcal{B}_n$ is defined, then let $\mathcal{B}_{n+1}$ be the collection of children of elements of $\mathcal{B}_n$ according to the above construction. Note that $\mathcal{B}_n$ consists of moves for Bob that are the nth good turns with respect to their corresponding histories and, in particular, $\beta\gamma^n\mathrm{rad}(B_0) < \mathrm{rad}(B) \leq \gamma^n\mathrm{rad}(B_0)$ for all $B \in \mathcal{B}_n$. Finally, let $\mathcal{K} = \bigcap_{n\in\mathbb{N}} \bigcup_{B\in\mathcal{B}_n} B$. Any element of $\mathcal{K}$ is the outcome of some game in which Bob played his winning strategy, so $\mathcal{K} \subseteq X \setminus E$. Moreover, from the construction of $\mathcal{K}$, it is clear that $\mathcal{K}$ is Ahlfors regular of dimension $(\log N)/(-\log\gamma)$. From the definition of N, we see that this dimension tends to c as $\gamma \to 0$.

To complete the proof, we need to show that it is legal for Alice to play the collection

$$\{\mathcal{N}(g_1(B_{i(n)}), \gamma^{n+1}\mathrm{diam}(B_0)), \ldots, \mathcal{N}(g_{k-1}(B_{i(n)}), \gamma^{n+1}\mathrm{diam}(B_0))\}$$

for any $k \leq N$. Indeed, the cost of this collection is

$$\sum_{j=1}^{k-1}(\mathrm{rad}(g_j(B_{i(n)})) + \gamma^{n+1}\mathrm{diam}(B_j))^c \leq N(3\gamma^{n+1}\mathrm{rad}(B_0))^c$$
$$\leq (\beta^2\gamma^n\mathrm{rad}(B_0))^c \leq (\beta \cdot \mathrm{rad}(B_{i(n)}))^c. \qquad \square$$

Remark 4.5. A similar characterization is possible in the generality of $\mathcal{H}$ potential games.

In fact, potential winning sets not only have non-empty intersection with Ahlfors regular sets, but the intersection has full Hausdorff dimension. To prove this, let us first prove the following.

PROPOSITION 4.6. *If* $\dim_H(E) \leq c_0$*, then* $X \setminus E$ *is* c_0 *potential winning.*

Proof. Fix $\beta > 0$, $c > c_0$, and $r_0 > 0$. Since the c-dimensional Hausdorff measure of E is zero, there exists a cover $\mathcal{C}$ of E such that

$$\sum_{C\in\mathcal{C}} \mathrm{diam}^c(C) \leq (\beta r_0)^c.$$

Thus, if Bob's first ball has radius r_0, then Alice can legally remove the collection $\mathcal{C}$, thus deleting the entire set E on her first turn. $\square$

THEOREM 4.7. *If* $c_0 < \delta = \dim_R X$*, then every* c_0 *potential winning set in* X *has Hausdorff dimension* δ*.*

Proof. By contradiction, suppose that $E \subseteq X$ is c_0 potential winning but $c_1 := \dim_H(E) < \delta$. Then by Proposition 4.6, $X \setminus E$ is c_1 potential winning, so by the intersection property (W2) (see Proposition 2.4), the empty set $\varnothing$ is c_2 potential winning,

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where $c_2 = \max(c_0, c_1) < \delta$. However, then by Theorem 3.8, the empty set $\varnothing$ is (α, β) winning for some $\alpha, \beta > 0$, which is a contradiction. □

Theorem 4.7 may be seen as a quantitative version of the full dimension intersection property of absolute winning sets [11, Corollary 5.4].

Remark 4.8. In view of Theorem 3.4, if a complete doubling metric space admits a splitting structure $(X, \mathcal{S}, U, f)$ such that $B = A_\infty(B)$ for every ball B, then the notions of Cantor winning and potential winning sets are equivalent. However, not every metric space X admits such a splitting structure. Indeed, spaces with splitting structures satisfying condition (S4) must have dimensions of the form $(\log n)/(\log m)$ $(m, n \in \mathbb{N})$ (cf. [5, Theorem 3]), whereas one can construct a complete doubling metric space of arbitrary Hausdorff dimension. In that sense, the notion of potential winning sets is strictly more general than the notion of Cantor winning sets.

Remark 4.9. In the above proof, we needed to be careful due to the fact that c_0 potential winning sets are not automatically non-empty (as the winning sets for Schmidt's game are); indeed, by Proposition 4.6, the empty set is $\dim_H(X)$ potential winning.

Lastly, the same observation that is used to prove Proposition 4.6 may be used to prove Theorem 1.16.

Proof of Theorem 1.16. Recall that every closed ball in X of radius ≤ 1 is of the form

$$B(x, r) = \{(y_k)_{k=1}^\infty : y_k = x_k \text{ for any } 1 \leq k \leq n\}$$

whenever $x \in X$ and $2^{-n} \leq r < 2^{-n+1}$. Note that X is 1 Ahlfors regular so it is enough to prove that the exceptional set defined in equation (1), that is,

$$E = \{x \in X : \overline{\{T^i x : i \geq 0\}} \cap K = \varnothing\},$$

is $\dim_H K$ potential winning. We now define a winning strategy for Alice. Assume Bob chooses $0 < \beta < 1$ and $c > \dim_H K$. Without loss of generality, assume that $B_0 = X$ and $r_0 = 1$. Then there exists a unique positive integer ℓ such that

$$2^{-\ell} \leq \beta < 2^{-\ell+1}.$$

Since $\dim_H K < c$, we may choose a collection of open balls $\mathcal{C}$ such that $K \subseteq \bigcup_{C \in \mathcal{C}} C$ and

$$\sum_{C \in \mathcal{C}} \mathrm{rad}(C)^c < \ell^{-1}(2^{-2\ell}\beta)^c. \tag{27}$$

Let $\{B_i\}_{i=0}^\infty$ denote any sequence of balls that are chosen by Bob which satisfy $2^{-(i+2)\ell} \leq \mathrm{rad}(B_i) < 2^{-(i+1)\ell}$. If this sequence is not infinite, then Alice wins by default. By abuse of notation, we will think of B_i as Bob's ith choice, and it will be sufficient to define Alice's reaction to these moves. On her $(i+1)$st move, Alice will remove the collection

$$\mathcal{A}_{i+1} = \bigcup_{j=0}^{\ell-1} \{C : C \in T^{-(i\ell+j)}\mathcal{C} \text{ and } C \cap B_i \neq \varnothing\}. \tag{28}$$

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Here, $T^{-j}\mathcal{C}$ denotes the set of all images of elements of $\mathcal{C}$ under the inverse branches of T^j.

Note that if Alice still does not win by default, that is, if equation (8) is not satisfied, then necessarily

$$\bigcap_{i=0}^{\infty} B_i \subseteq \left\{x \in X : \{T^i x : i \geq 0\} \subseteq X \setminus \bigcup_{C \in \mathcal{C}} C\right\} \subseteq E.$$

Therefore, we are done once we show that the collection in equation (28) is a legal move for Alice, that is, that it satisfies equation (7). First, for any $C \in \mathcal{C}$ and $j = 0, \ldots, \ell - 1$, we have that $T^{-(i\ell+j)}(\{C\})$ is a collection of balls of radius $2^{-(i\ell+j)}\mathrm{rad}(C)$, pairwise separated by distances of at least $2^{-(i\ell+j)} \geq 2^{-i\ell-\ell}$. Since $\mathrm{rad}(B_i) < 2^{-i\ell-\ell}$, at most one of them intersects B_i. Therefore,

$$\sum_{C \in \mathcal{A}_{i+1}} \mathrm{rad}(C)^c \leq \sum_{j=0}^{\ell-1} \sum_{C \in \mathcal{C}} (2^{-(i\ell+j)} \cdot \mathrm{rad}(C))^c \leq \ell 2^{2c\ell} \sum_{C \in \mathcal{C}} \mathrm{rad}(C)^c \cdot \mathrm{rad}(B_i)^c,$$

which is smaller than $(\beta \cdot \mathrm{rad}(B_i))^c$ by equation (27). □

5. *Counterexamples*

The original papers of Schmidt [**37**, Theorem 5] and McMullen [**33**] provide examples of sets that are winning (respectively, strong winning) but not absolute winning. We provide two additional counterexamples.

5.1. *Cantor winning but not winning.*

We give an explicit example of a set which is potential winning in $\mathbb{R}$ (or equivalently, by Theorem 1.5, is Cantor winning), but is not winning. In fact, we will prove more by showing that the set in question is not weak winning (see §2.4).

Proof of Theorem 1.7. By an *iterated function system* on $\mathbb{R}$, or *IFS*, we mean a finite collection $\{f_i : \mathbb{R} \to \mathbb{R}\}$, $i = 1, 2, \ldots, N$, of contracting similarities. By the *limit set* of the IFS, we mean the unique compact set S which is equal to the union of its images $f_i(S)$ under the elements of the IFS. We call an IFS on $\mathbb{R}$ *rational* if its elements all preserve the set of rationals. Note that there are only countably many rational IFS.

Let $E \subseteq \mathbb{R}$ be the complement of the union of the limit sets of all rational IFS whose limit sets have Hausdorff dimension $\leq 1/2$. We will show that E is not winning for Schmidt's game. However, since $\dim(\mathbb{R} \setminus E) = 1/2 < 1$, it follows from Proposition 4.6 that E is $1/2$ potential winning.

Fix $\alpha > 0$ and let $\beta > 0$ be a small number, to be determined later. We will show that Alice cannot win the weak (α, β) game. Let I be a rational interval contained in Alice's first move, whose length is at least half of the length of Alice's first move. Without loss of generality, suppose that $I = [-1, 1]$. Let $\lambda < 1$ be a rational number large enough so that $\lambda + \lambda\alpha \geq 1$, and let $\gamma > 0$ be a rational number small enough so that

$$\lambda^{1/2} + 2\gamma^{1/2} \leq 1. \tag{29}$$

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Consider the IFS on the interval I consisting of the following three contractions:

$$u_0(x) = \lambda x, \quad u_1(x) = \gamma(x-1)+1, \quad u_{-1}(x) = \gamma(x+1)-1,$$

and note that the condition in equation (29) guarantees that the dimension of the limit set of this IFS is at most $1/2$ (see, for example, [**30**, §5.3, Theorem 1]). Bob's strategy is as follows: on any given turn k, if Alice just made the move A_k, then find the largest interval $J_k \subseteq A_k$ which can be written as the image of I under a composition of elements of the IFS, and select the unique ball B_k of radius $\beta \cdot \mathrm{rad}(A_k)$ whose center is the midpoint of this interval. We will show by induction that $B_k \subseteq J_k$, thus proving that this choice is legal. Indeed, suppose that this holds for k. By 'blowing up the picture' using inverse images of the elements of the IFS, we may without loss of generality suppose that $J_k = I$. Then we have $B_k = [-r, r]$, where $r = \mathrm{rad}(B_k) \leq 1$. By 'blowing up the picture' further using an iterate of the inverse image of the contraction u_0, we may assume without loss of generality that $r > \lambda$. Then Alice's next choice A_{k+1} is an interval of length at least $2\lambda\alpha$ contained in $[-1, 1]$. However, the set $\{\pm\lambda^q : q = 1, 2, \ldots, N\}$ intersects every interval of length $1-\lambda$ contained in $[-1, 1]$, where N is a large constant. Since $\lambda\alpha \geq 1-\lambda$ by assumption, there exist $q \in \{1, \ldots, N\}$ and $\varepsilon \in \{-1, 1\}$ such that $\varepsilon\lambda^q \in \frac{1}{2}A_{k+1}$. However, then $J = u_0^q u_\varepsilon(I)$ is an interval coming from a cylinder in the IFS construction which intersects $\frac{1}{2}A_{k+1}$, and whose length is $2\gamma\lambda^q \in [2\lambda^N\gamma, 2\lambda\gamma]$. By choosing $\gamma \leq \frac{1}{4}\alpha$, we can guarantee that the length of J satisfies $2\mathrm{rad}(J) \leq \frac{1}{2}\lambda\alpha \leq \frac{1}{2}\mathrm{rad}(A_{k+1})$. Combining with the fact that $J \cap \frac{1}{2}A_{k+1} \neq \varnothing$ shows that $J \subseteq A_{k+1}$. So then by the definition of J_{k+1}, we have $2\mathrm{rad}(J_{k+1}) \geq 2\mathrm{rad}(J) \geq 2\lambda^N\gamma$, and thus by letting $\beta = \lambda^N\gamma$, we get $B_{k+1} \subseteq J_{k+1}$, completing the induction step. □

5.2. *Winning but not Cantor winning.* Finally, we will show that there exists a set in $\mathbb{R}$ which is winning, but is not potential winning (or equivalently, by Theorem 1.5, Cantor winning).

Remark 5.1. The above statement remains true if 'winning' is replaced by 'weak winning', but the resulting statement is weaker and so we prove the original (stronger) statement instead. We will however need the weak (α, β) game in the proof. It appears that this is a non-trivial application of the weak game.

By Corollary 3.9, it suffices to show that there exists a winning set E whose complement is weak winning. Note that this also shows that the weak game does not have the intersection property (W2), since E and its complement are both weak winning, but their intersection is empty, and the empty set is not weak winning. We will use the result below as a substitute for the intersection property.

Definition 5.2. A set E is *finitely weak winning* if Alice can play the weak game so as to guarantee that after finitely many moves, her ball A_n is a subset of E.

LEMMA 5.3. *The intersection of countably many finitely weak α winning sets is weak α winning.*

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Proof. Let $(E_n)_{n\in\mathbb{N}}$ be a countable collection of finitely weak α winning sets. Alice can use the following strategy to ensure that $\bigcap_{m=0}^{\infty} B_m$ lies inside $\bigcap_{n=0}^{\infty} E_n$. She starts by following the strategy to ensure that $A_{m_1} \subseteq E_1$. After that, she starts playing the weak winning game for the set E_2, assuming that Bob's initial move is B_{m_1}. Therefore, she can ensure that $A_{m_1+m_2} \subseteq E_1 \cap E_2$. Then she switches her strategy to E_3 and so on.

Finally, we have that for any N,

$$\bigcap_{m=1}^{\infty} A_m \subseteq A_{\sum_{k=1}^{N} m_k} \subseteq \bigcap_{n=1}^{N} E_n.$$

By letting N tend to infinity, we prove the lemma. □

In what follows, the intersection of countably many finitely weak α-winning sets will be called σ finitely weak α-winning.

Proof of Theorem 1.8. Throughout this proof, we let $Z = \{0, 1\}$ and we let π denote the coding map for the binary expansion, so that $\pi : Z^{\mathbb{N}} \to [0, 1]$.

Definition 5.4. A *Bohr set* is a set of the form

$$A(\gamma, \delta) := \{n \in \mathbb{N} : n\gamma \in (-\delta, \delta) \bmod 1\},$$

where $\gamma, \delta > 0$.

For each $\omega = (\omega_n)_{n\in\mathbb{N}} \in Z^{\mathbb{N}}$, we let $E(\omega) = \{n \in \mathbb{N} : \omega_n = 1\}$.

We now define the set $E \subseteq \mathbb{R}$:

$$E_0 : = \mathbb{Z} + \{\pi(\omega) : \omega \in Z^{\mathbb{N}},\ \ E(\omega) \text{ contains a Bohr set}\}$$
$$E : = \bigcup_{r\in\mathbb{Q}} rE_0.$$

We claim that E is $1/3$ winning, but that its complement is weak $1/3$ winning.

Fix $1/4 < \alpha < 1/3$ and $\beta, \rho_0 > 0$. We show that Alice has a strategy to win the (α, β) Schmidt game assuming that Bob has chosen a ball of radius ρ_0 with his first move. Choose $0 < r \in \mathbb{Q}$ so that $3 < 2\rho_0/r < \alpha^{-1}$, and consider the set

$$A := \{n \in \mathbb{Z} :\ \text{there exists } m = m_n \in \mathbb{N} \text{ such that } (\alpha\beta)^m \rho_0/r \in (3 \cdot 2^{-(n+1)}, \alpha^{-1} \cdot 2^{-(n+1)})\}.$$

The bounds on α guarantee that the values m_n for all $n \in A$ are distinct. By taking logarithms and rearranging the expression for the set A, we get

$$\begin{aligned} n \in A \Leftrightarrow\ & \text{there exists } m \quad \log 3 < (n+1)\log 2 + m\log(\alpha\beta) + \log(\rho_0/r) < \log(\alpha^{-1}) \\ \Leftrightarrow\ & n\frac{\log 2}{-\log(\alpha\beta)} \in \left(\frac{\log(3/2\rho_0/r)}{-\log(\alpha\beta)}, \frac{\log(\alpha^{-1}/2\rho_0/r)}{-\log(\alpha\beta)}\right) \bmod 1. \end{aligned}$$

Since the interval on the right-hand side contains 0, this implies that A contains a Bohr set.

Now fix $n \in A$ and let $m = m_n \in \mathbb{N}$ be the corresponding value. On the mth turn, Bob's interval B_m is of length $2(\alpha\beta)^m\rho_0 \geq 3 \cdot 2^{-n}r$, so if we subdivide $\mathbb{R}$ into intervals of length $2^{-n}r$, then B_m must contain at least two of them. Since Alice's next interval is of length $2\alpha(\alpha\beta)^m\rho_0 \leq 2^{-n}r$, this means that Alice can choose a subinterval of either one of these

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intervals on her turn. So Alice can control the nth binary digit of x/r, where x is the outcome of the game, and, in particular, she can set the nth digit to 1. Since every value m corresponds to only one value $n \in A$, Alice can set all of the digits of x/r whose indices are in A to 1. This means that $x/r \in E_0$ and thus $x \in E$, so Alice can force the outcome of the game to lie in E. So by Proposition 1.2, E is 1/3 winning.

Now if we show that $\mathbb{R} \setminus E_0$ is σ finitely weak 1/3 winning, then by symmetry, $\mathbb{R} \setminus rE_0$ is σ finitely weak 1/3 winning for all $r > 0$, and since the intersection of countably many σ finitely weak 1/3 winning sets is σ finitely weak 1/3 winning, it follows that $\mathbb{R} \setminus E$ is σ finitely weak 1/3 winning. By symmetry, it suffices to consider the case $r = 1$. To prove that $\mathbb{R} \setminus E_0$ is indeed σ finitely weak 1/3 winning, we will use the following lemma.

LEMMA 5.5. *Fix $0 < \alpha < 1/3$ and $\beta > 0$. Then there exists $N \in \mathbb{N}$ such that for any $\rho_0 > 0$, there exists $a_1(\rho_0)$ so that for any sequence of integers $(a_k)_{1 \leqslant k \leqslant M}$ with*

$$a_{k+1} - a_k \geq N \quad \text{for all } k, \alpha_1 \geqslant \alpha_1(\rho_0), \tag{30}$$

Alice has a strategy in the finite weak (α, β) game to control the digits indexed by the integers $(a_k)_{1 \leqslant k \leqslant M}$.

Proof of Lemma 5.5. Suppose that Bob has just played a move of radius ρ. Then Alice can play any radius between $\alpha\rho$ and ρ, forcing Bob to play a predetermined radius between $\alpha\beta\rho$ and $\beta\rho$. After iterating n times, Alice can force Bob to play a ball of any given radius between $\alpha^n\beta^n\rho$ and $\beta^n\rho$. Since $\alpha^n < \beta$ for sufficiently large n, there exists $\gamma > 0$ such that Alice can force Bob to play a ball of any prescribed radius $\leq \gamma\rho$ (after a sufficient number of turns depending on the radius). Now Alice uses the following strategy: for each $k = 1, \ldots, M$, first force Bob to play a ball whose radius is $3 \cdot 2^{-a_k}$, and then respond to Bob's choice in a way such that the a_kth digit of the outcome is guaranteed (this is possible by the argument from the previous proof). Forcing Bob to play these radii is possible as long as $3 \cdot 2^{-a_1} \leq \gamma\rho_0$ and $3 \cdot 2^{-a_{k+1}} \leq \gamma\alpha 3 \cdot 2^{-a_k}$ for all k. By equation (30), the latter statement is true as long as N and $a_1(\rho_0)$ are sufficiently large. □

To continue the proof of Theorem 1.8, fix $0 < \alpha < 1/3$ and $\beta > 0$, and let $N \in \mathbb{N}$ be as in Lemma 5.5. Then we get the following.

- For all $k \in \mathbb{N}$, the set

  $$A_k := \mathbb{Z} + \{\pi(\omega) : T(\omega) \text{ contains an arithmetic progression of length } k \text{ and gap size } N\}$$

  is finitely weak winning, where $T(\omega) := \mathbb{N} \setminus E(\omega) = \{n \in \mathbb{N} : \omega_n = 0\}$.
- For all $q, i \in \mathbb{N}$, the set

  $$A_{q,i} := \mathbb{Z} + \{\pi(\omega) : T(\omega) \cap (q\mathbb{N} + i) \neq \varnothing\}$$

  is finitely weak winning.

So the intersection

$$A = \bigcap_k A_k \cap \bigcap_{q,i} A_{q,i}$$

is σ finitely weak winning. To finish the proof, we need to show that $\mathbb{R} \setminus E_0 \supseteq A$. Indeed, fix $n + \pi(\omega) \in A$. Then $T(\omega)$ contains arbitrarily long arithmetic progressions of gap size N, and intersects every infinite arithmetic progression non-trivially.

Now let $A(\gamma, \delta)$ be a Bohr set, and we will show that $A(\gamma, \delta)$ intersects $T(\omega)$ non-trivially. First suppose that γ is irrational. Then so is $N\gamma$, so by the minimality of irrational rotations, there exists M such that the finite sequence $0, N\gamma, \ldots, MN\gamma$ is δ-dense in $\mathbb{R}/\mathbb{Z}$. Now let $n, n+N, \ldots, n+MN$ be an arithmetic progression of length M and gap size N contained in $T(\omega)$. Then $n\gamma, n\gamma + N\gamma, \ldots, n\gamma + MN\gamma$ is also δ-dense in $\mathbb{R}/\mathbb{Z}$ and, in particular, must contain an element of $(-\delta, \delta)$. So $(n + iN)\gamma \in (-\delta, \delta)$ mod 1 for some $i = 0, \ldots, M$. However, then $n + iN \in A(\gamma, \delta) \cap T(\omega)$.

Alternatively, suppose that γ is rational, say $\gamma = p/q$. Then $A(\gamma, \delta)$ contains the infinite arithmetic progression $q\mathbb{N}$, which by assumption intersects $T(\omega)$ non-trivially.

Thus, every Bohr set intersects $T(\omega)$ non-trivially, so $E(\omega)$ does not contain any Bohr set, that is, $n + \pi(\omega) \notin E_0$. So $A \subseteq \mathbb{R} \setminus E_0$, completing the proof. □

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