Can. J. Math.Vol. 44 (5), 1992 pp. 1014–1029

POSITIVE FINITE ENERGY SOLUTIONS OF CRITICAL SEMILINEAR ELLIPTIC PROBLEMS

EZZAT S. NOUSSAIR, CHARLES A. SWANSON AND YANG JIANFU

1. **Introduction.** Existence theorems and asymptotic properties will be obtained for boundary value problems of the form

$$(1.1)\qquad \begin{cases} -\Delta u = p(x)u^\tau + f(x,u), & x \in \Omega \\ u(x) > 0, & x \in \Omega,\ u \in D_0^{1,2}(\Omega) \end{cases}$$

in an unbounded domain $\Omega \subseteq R^N (N \geq 3)$ with smooth boundary, where Δ denotes the N-dimensional Laplacian, $\tau = (N+2)/(N-2)$ is the critical Sobolev exponent, and $D_0^{1,2}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the $L^2(\Omega)$ norm of $|\nabla u|$. Detailed hypotheses on the functions $p: \overline{\Omega} \longrightarrow \overline{\mathbf{R}}_+$ and $f: (\overline{\Omega} \setminus \{0\}) \times \mathbf{R} \longrightarrow \overline{\mathbf{R}}_+$ will be listed in §2, where $\overline{\mathbf{R}}_+ = [0,\infty)$ and $\overline{\Omega} = \Omega \cup \partial\Omega$; $\partial\Omega$ is understood to be void if $\Omega = \mathbf{R}^N$. In particular, $f(x,u)$ will be assumed to be a more slowly growing nonlinearity than u^τ, *i.e.*, $\lim_{u\to\infty} u^{-\tau} f(x,u) = 0$ uniformly in Ω.

Critical semilinear elliptic equations arise from widely diverse problems in differential geometry, quantum physics, astrophysics, and other scientific areas. Many of these problems are set in unbounded domains Ω, causing mathematical difficulties from the lack of compactness of associated functionals and embeddings. Some examples are the Yamabe problem for prescribed scalar curvature [18, pp. 171–185 and references therein], the Yang-Mills equation in nonlinear field theory [23], the Eddington-Matukuma model in astrophysics [15, 20], and many variational problems related to Sobolev, isoperimetric, and trace inequalities [18].

If the perturbation term $f(x,u)$ is deleted, problem (1.1) generally has no solution; for example, Proposition 6.1 shows that no solution exists if $p(x)$ is nonconstant with $x \cdot (\nabla p)(x)$ either nonnegative or nonpositive in $\mathbf{R}^N$. If the perturbation is linear of type $\lambda q(x)u$, solutions exist only for λ in some finite positive interval; such problems in various geometric structures were treated in depth by Benci and Cerami [2], Brezis and Nirenberg [5], Egnell [8, 9], Escobar [12], Guedda and Veron [14]; accordingly we do not consider them here. Our objectives and methods also are not of the type in [4, 7, 13, 15, 20, 21, 24], mostly concerning bounded domains and/or radial coefficients.

One of our primary goals is to obtain solutions with the asymptotic behaviour $u(x) = 0(|x|^{2-N})$ as $|x| \longrightarrow \infty$. This sharp asymptotic decay law is important for various applications, *e.g.*, to obtain a solution of Matukuma's equation corresponding to finite total

The first author was supported by the Australian Research Council.
The second author was supported by NSERC (Canada) under Grant 5-83105.
Received by the editors September 20, 1990.
AMS subject classification: 35J60.

mass of a globular star structure. We note that the classical one-instanton solution of the Yang-Mills equation has this asymptotic decay at ∞, as indicated in (7.7).

In particular, our results apply to the prototype problem

$$(1.2)\qquad \begin{cases} -\Delta u = p(x)u^{\tau} + q(x)u^{\gamma}, & x \in \Omega \\ u > 0 \text{ in } \Omega, & u \in D_0^{1,2}(\Omega) \end{cases}$$

under the following conditions:

(A_1) $1 < \gamma < \tau$ if $N \geq 4$; $3 < \gamma < 5$ if $N = 3$.

(A_2) $p(x)$ is nonnegative and bounded in $\bar{\Omega}$.

(A_3) $q(x)$ is nonnegative and locally bounded in $\bar{\Omega} \setminus \{0\}, q(x) = o(|x|^{\mu})$ as $|x| \longrightarrow 0$, and $q(x) = o(|x|^{\nu})$ as $|x| \longrightarrow \infty$ for constants μ and ν satisfying $-2 < \nu \leq \mu \leq 0$, $\gamma < (N+2)/(N-2)$, and

$$(1.3)\qquad \frac{N+2\nu+2}{N-2} \leq \gamma \leq \frac{N+2\mu+2}{N-2}.$$

(A_4) There exists a bounded domain $G \subset \Omega$ and $x_0 \in G$ such that $q(x) > 0$ on $\bar{G}$ and

$$(1.4)\qquad 0 < p(x_0) = \sup_{x \in G} p(x) = \sup_{x \in \Omega} p(x) \equiv ||p||_{\infty},$$

$$(1.5)\qquad p(x) = p(x_0) + 0(|x - x_0|^2) \text{ near } x_0.$$

THEOREM 1.1. *Conditions (A_1)–(A_4) imply that problem (1.2) has a weak solution $u(x)$ in Ω such that $u(x) = 0(|x|^{2-N})$ as $|x| \longrightarrow \infty$ uniformly in Ω. If in addition* $\inf_{x \in G} q(x)$ *is sufficiently large, the same conclusion extends to all $\gamma \in (1,5)$, $N = 3$.*

Theorem 1.1 is a specialization of our main Theorem 5.1 to the prototype (1.2). The necessity of conditions (A_1)–(A_4) is indicated in §3 and §6.

§7 contains an extension of Theorem 1.1 to a critical problem (7.1) with a singularity in both the critical term and the subcritical perturbation.

The Referee has suggested the interesting problem of obtaining an analogue of Theorem 1.1 under alternatives to hypothesis (A_4) for which $\sup_{\Omega} p$ is not attained in Ω. We note that additional structure conditions on p would be necessary, as demonstrated by Ding and Ni [7, Theorem 5.13] in the radial case; in particular, no positive solution of (1.1) exists in $\mathbf{R}^N$ if p is radial and increasing for large $|x|$ and q is identically zero. For a bounded domain Ω, however, Escobar [12, Theorem 3.1, Conditions (3.2), (4.2)′] allows p to have a maximum at a boundary point x_0 provided all partial derivatives of p up to appropriate order (depending on N) vanish at x_0.

Our procedure is to first establish local solutions $u_k(x)$ in bounded subdomains Ω_k of Ω via the mountain pass theorem of Ambrosetti and Rabinowitz [1], and then show convergence of $\{u_k(x)\}$ in a suitable topology to a positive solution of (1.1) in Ω. §2 contains preliminary material including the hypotheses for 1.1, some known theorems to be applied later, and a sketch of our method. §3 contains a crucial estimate needed for the mountain pass theorem and some consequences of this estimate. §4 is a verification that the functional used in the mountain pass theorem satisfies a Palais-Smale compactness condition. The main existence theorem for (1.1) is proved in §5.

It would be desirable to carry out the proof directly in Ω, thereby removing the need to consider the sequence of problems $(2.3)_k$ (although $(2.3)_k$ has independent interest, as indicated by Remark 5.4). Our proof in §5 appeals to the Stampacchia maximum principle for weak solutions $u_k \in W_0^{1,2}(\Omega_k)$ of $-\Delta u_k \geq 0$ in order to establish the nonnegativity of local solutions u_k in Ω_k. A direct global approach would require a suitable replacement of this maximum principle for weak solutions $u \in D_0^{1,2}(\Omega)$.

We are grateful to the Referee for his interesting comments and suggestions.

2. **Preliminaries.** We use the notation $\Omega_r = \Omega \cap B_r(0)$ and $\Omega_\infty = \Omega$ for convenience, where $B_r(x)$ is the ball in $\mathbf{R}^N$ of radius r centred at x. The standard norm in $L^\rho(B)$ will be denoted by $\| \; \|_{\rho,B}, \rho \geq 1, B \subseteq \mathbf{R}^N$. The Sobolev space $E_r = D_0^{1,2}(\Omega_r)$ is defined as the completion of $C_o^\infty(\Omega_r)$ in the norm $\| |\nabla u| \|_{2,\Omega_r}$, $0 < r \leq \infty$.

The hypotheses for (1.1) are as follows:

(H_1) $p: \bar{\Omega} \longrightarrow \bar{\mathbf{R}}_+$ is bounded and (1.4), (1.5) hold for some bounded domain $G \subset \Omega$ and some $x_0 \in G$.

(H_2) $f: (\bar{\Omega} \setminus \{0\}) \times \bar{\mathbf{R}}_+ \longrightarrow \bar{\mathbf{R}}_+$ is nontrivial, $f(x,\cdot): \bar{\mathbf{R}}_+ \longrightarrow \bar{\mathbf{R}}_+$ is continuous for almost all $x \in \bar{\Omega}$, and

$$f(x,u) \leq \sum_{j=1}^{m} q_j(x) u^{\gamma(j)}, \quad x \in \Omega, \; u \geq 0$$

for nonnegative locally bounded functions q_j in $\bar{\Omega} \setminus \{0\}$ such that $q_j(x) = o(|x|^\mu)$ as $|x| \longrightarrow 0$ and $q_j(x) = o(|x|^\nu)$ as $|x| \longrightarrow \infty, j = 1, \ldots, m$, for constants $\mu \in (-2, 0], \nu$, and $\gamma(j)$ satisfying (1.3).

(H_3) $F(x,t) \leq (\gamma + 1)^{-1} t f(x,t)$ for all $x \in \Omega$, $t > 0$, where $\gamma = \min_{1 \leq j \leq m} \gamma(j)$ and $F(x,t) = \int_0^t f(x,s)\, ds$.

(H_4) There exists a nonnegative function h such that $f(x,u) \geq h(u)$ for all $u > 0$ and a.e. in G, where the primitive $H(u) = \int_0^u h(t)\, dt$ satisfies

$$\lim_{\epsilon \to 0} \epsilon^M \int_0^{\epsilon^{-1}} H\Big[\Big(\frac{\epsilon^{-1}}{1+t^2}\Big)^{\frac{N-2}{2}}\Big] t^{N-1}\, dt = +\infty, \text{ and} \\ M = \max\{N-2, 2\}, \quad N \geq 3. \tag{2.1}$$

For the prototype (1.2) it is clear that (H_4) holds since $(\gamma + 1)(N-2) > 2M$ under condition (A_1) for (1.2), and $q(x) \geq q_0 > 0$ in G by condition (A_4).

Since only positive solutions of (1.1) are under consideration, we define $f(x,u) \equiv 0$ if $u \leq 0$ and $u_+(x) = \max\{u(x), 0\}$. Let J_r be the functional on E_r defined by

$$J_r(u) = \int_{\Omega_r} \Big[\frac{1}{2}|\nabla u|^2 - \frac{1}{\tau+1} p(x) u_+^{\tau+1} - F(x,u)\Big]\, dx, \quad u \in E_r, \; 0 < r \leq \infty, \tag{2.2}$$

for which (1.1) is the associated Euler-Jacobi equation. It is known, *e.g.*, [10], that $J_r(u)$ is well defined and continuously Fréchet differentiable on $E_r, 0 < r \leq \infty$. Our method consists of an analysis of a *sequence* of problems

$$\begin{cases} -\Delta u = p(x) u^\tau + f(x,u) & x \in \Omega_k, \\ u > 0 \text{ in } \Omega_k, u \in E_k, & k = 1, 2, \ldots, \end{cases} \tag{2.3$_k$}$$

where we can assume that $G \subset \Omega_1$ (relabelling if necessary). A (weak) solution u_k of $(2.3)_k$ is defined as a positive function $u_k \in E_k$ such that $J'_k(u_k) = 0$ in the dual space E_k^*, *i.e.*,

$$\int_{\Omega_k} \nabla u_k \cdot \nabla \phi \, dx = \int_{\Omega_k} [p(x)u_k^\tau \phi + f(x, u_k)\phi] \, dx \tag{2.4}$$

for all $\phi \in E_k$, $k = 1, 2, \ldots, \infty$.

LEMMA 2.1 (BREZIS AND LIEB [6]). *If $\{u_n\}$ is a sequence in $L^\sigma(\Omega)(\sigma > 1)$ such that $u_n \longrightarrow u$ weakly in $L^\sigma(\Omega)$ and $u_n(x) \longrightarrow u(x)$ a.e. in Ω as $n \longrightarrow \infty$, then*

$$\lim_{n\longrightarrow\infty} [\|u_n\|_{\sigma,\Omega}^\sigma - \|u_n - u\|_{\sigma,\Omega}^\sigma] = \|u\|_{\sigma,\Omega}^\sigma. \tag{2.5}$$

(This generalizes Fatou's lemma).

We also require the compactness of the embedding of E_∞ into a suitable weighted Lebesgue space $L^\rho(\Omega, q)$, with standard norm

$$\|u\|_{\rho,\Omega,q} = \left[\int_\Omega |u(x)|^\rho q(x) \, dx\right]^{1/\rho}, \quad \rho \geq 1.$$

The version to be used here is essentially Egnell's Lemma 10 [10], as follows:

LEMMA 2.2 (EGNELL). *If $q(x)$ satisfies condition (A_3), then the embedding $E_\infty \hookrightarrow L^{\gamma+1}(\Omega, q)$ is compact.*

3. **An estimate for J_∞ on a path in E_∞.** In order to apply the mountain pass theorem [1] to J_∞, we first construct a function $v_\epsilon \in E_\infty$ with $J_\infty(t_0 v_\epsilon) < 0$ for sufficiently large $t_0 > 0$ and sufficiently small $\epsilon > 0$ such that a sharp upper bound can be obtained for $J_\infty(\phi)$ on a path in E_∞ joining $\mathbf{0}$ to $t_0 v_\epsilon$. To construct v_ϵ, we note that the special critical equation

$$-\Delta u = u^\tau \text{ in } \mathbf{R}^N \tag{3.1}$$

has the well known minimal decaying positive solution

$$u = u_\epsilon(x) = K\left[\frac{\epsilon}{\epsilon^2 + |x - x_0|^2}\right]^{\frac{N-2}{2}}, \ K = [N(N-2)]^{\frac{N-2}{4}}$$

for arbitrary $x_0 \in \mathbf{R}^N$ and $\epsilon > 0$. Let G and $x_0 \in G$ be as in condition (H_1) and choose $R > 0$ small enough that $B_{2R}(x_0) \subset G$. We shall abbreviate $B_r(x_0)$ to B_r since x_0 is fixed in the proof below. Define

$$w_\epsilon(x) = \phi(x)u_\epsilon(x), \quad x \in \mathbf{R}^N, \ \epsilon > 0, \tag{3.2}$$

where ϕ is a piecewise smooth radial function with support B_{2R} such that $0 \leq \phi(x) \leq 1$ on B_{2R}, $\phi(x) = 1$ on B_R, and $|\nabla\phi(x)| \leq 1/R$ on $B_{2R} \setminus B_R$. Let

$$v_\epsilon(x) = w_\epsilon(x)\left[\int_G p(x)w_\epsilon^{\tau+1}(x) \, dx\right]^{-1/(\tau+1)}. \tag{3.3}$$

The constant S in the proposition below is defined by

$$S = \inf\{\|\nabla u\|_{2,\Omega}^2 : u \in E_\infty, \|u\|_{\tau+1,\Omega} = 1\},$$

corresponding to the best constant for the Sobolev embedding $E_\infty = D_0^{1,2}(\Omega) \hookrightarrow L^{\tau+1}(\Omega)$.

PROPOSITION 3.1. *If conditions* (H_1)–(H_4) *hold, there exist positive numbers* ϵ *and* t_0 *such that* $J_\infty(t_0 v_\epsilon) < 0$ *and*

$$0 < \sup_{t \geq 0} J_\infty(tv_\epsilon) < \frac{1}{N} S^{N/2} \|p\|_\infty^{(2-N)/2}. \tag{3.4}$$

PROOF. Since $\partial u_\epsilon / \partial r \leq 0$, integration by parts of (3.1) gives

$$\int_{B_R} |\nabla w_\epsilon|^2 \, dx = \int_{B_R} |\nabla u_\epsilon|^2 \, dx \leq \int_{B_R} u_\epsilon^{\tau+1} \, dx. \tag{3.5}$$

On account of (1.4) and (1.5), it can be verified easily that

$$p(x_0) \int_{B_R} u_\epsilon^{\tau+1} \, dx \leq \int_{B_R} p(x) u_\epsilon^{\tau+1} \, dx + 0(\epsilon^2), \tag{3.6}$$

$$\int_{\mathbf{R}^N \setminus B_R} u_\epsilon^{\tau+1} \, dx = 0(\epsilon^N), \tag{3.7}$$

and

$$A_\epsilon \equiv \int_{\Omega \setminus B_R} |\nabla w_\epsilon|^2 \, dx = 0(\epsilon^{N-2}) \tag{3.8}$$

as $\epsilon \longrightarrow 0$. From the well known fact [22] that S is attained by u_ϵ and since

$$\int_{\mathbf{R}^N} |\nabla u_\epsilon|^2 \, dx = \int_{\mathbf{R}^N} u_\epsilon^{\tau+1} \, dx$$

by (3.1), it follows that

$$S = \left[\int_{\mathbf{R}^N} u_\epsilon^{\tau+1} \, dx \right]^{2/N}. \tag{3.9}$$

Then (3.5)–(3.9) yield the estimate

$$\begin{aligned} \int_\Omega |\nabla w_\epsilon|^2 \, dx &= \int_{B_R} |\nabla w_\epsilon|^2 \, dx + A_\epsilon \leq \int_{B_R} u_\epsilon^{\tau+1} \, dx + A_\epsilon \\ &= S \left[\int_{B_R} u_\epsilon^{\tau+1} \, dx \right]^{2/(\tau+1)} + A_\epsilon \\ &\leq S \|p\|_\infty^{-2/(\tau+1)} \left[\int_{B_R} p(x) w_\epsilon^{\tau+1} \, dx \right]^{2/(\tau+1)} + 0(\epsilon^2) + 0(\epsilon^{N-2}). \end{aligned} \tag{3.10}$$

Hypothesis (H_1) implies that $p(x)$ is bounded below by a positive constant if R is selected sufficiently small, and hence also $\int_G p(x) w_\epsilon^{\tau+1} \, dx$ is bounded below by a positive constant, independent of ϵ. Therefore (3.3) and (3.10) imply the inequality

$$V_\epsilon \equiv \int_\Omega |\nabla v_\epsilon|^2 \, dx \leq S \|p\|_\infty^{-2/(\tau+1)} + 0(\epsilon^{N-2}) + 0(\epsilon^2). \tag{3.11}$$

Since supp $v_\epsilon \subset G$, use of (2.2), (3.3), and (3.11) gives

$$J_\infty(tv_\epsilon) = \frac{1}{2} t^2 V_\epsilon - \frac{1}{\tau+1} t^{\tau+1} - \int_\Omega F(x, tv_\epsilon) \, dx. \tag{3.12}$$

Clearly $\lim_{t\to\infty} J_\infty(tv_\epsilon) = -\infty$ for all $\epsilon > 0$, and hence $\sup_{t\geq 0} J_\infty(tv_\epsilon)$ is attained at some number $t_\epsilon \geq 0$. We can assume that $t_\epsilon > 0$ for all $\epsilon > 0$; otherwise there would be nothing to prove. It follows from $J'_\infty(t_\epsilon v_\epsilon) = 0$ and the boundedness of V_ϵ that

(3.13) $$t_\epsilon \leq V_\epsilon^{1/(\tau-1)} \leq C_o, \quad \epsilon > 0$$

for some constant C_o, independent of ϵ. The fact that $\frac{1}{2}t^2 V_\epsilon - (\tau+1)^{-1}t^{\tau+1}$ is increasing in $t \in [0, V_\epsilon^{1/(\tau-1)}]$ implies from (3.11)–(3.13) that

(3.14) $$\begin{aligned}\sup_{t\geq 0} J_\infty(tv_\epsilon) = J_\infty(t_\epsilon v_\epsilon) &\leq \frac{1}{N}V_\epsilon^{N/2} - \int_{B_{2R}} F(x, t_\epsilon v_\epsilon)\,dx \\ &\leq \frac{1}{N}S^{N/2}\|p\|_\infty^{(2-N)/2} - \int_{B_{2R}} F(x, t_\epsilon v_\epsilon)\,dx + 0(\epsilon^L)\end{aligned}$$

where $L = \min(N-2, 2)$. Virtually the same procedure as in [5, pp. 465-466] shows via (3.3), (3.13), and (H_2) that $\lim_{\epsilon\to 0+} t_\epsilon > 0$. It is then a consequence of (3.2), (3.14), and (H_4) that a positive constant C, independent of ϵ, exists such that

(3.15) $$\sup_{t\geq 0} J_\infty(tv_\epsilon) \leq \frac{1}{N}S^{N/2}\|p\|_\infty^{(2-N/2} - \int_{B_{2R}} H(Cv_\epsilon)\,dx + 0(\epsilon^L)$$

for sufficiently small ϵ. A change of variable yields

(3.16) $$\lim_{\epsilon\to 0+} \epsilon^{-L} \int_{B_{2R}} H(Cv_\epsilon)\,dx = +\infty$$

because of (H_4), and hence (3.15) implies the conclusion (3.4) of Proposition 3.1.

REMARK 3.2. Proposition 3.1 applies to the prototype (1.2) under the stated conditions (A_1)–(A_4) following (1.2); it was already mentioned that (H_4) is implied by (A_1) and (A_4). If $q_* = \inf_{x\in G} q(x)$ is sufficiently large, we also note that (3.4) holds for the *full range* $1 < \gamma < 5, N = 3$. In fact, in (3.14)

$$\begin{aligned}\int_{B_{2R}} F(x, t_\epsilon v_\epsilon)\,dx &\geq \frac{1}{\gamma+1}\int_{B_R} q(x)u_\epsilon^{\gamma+1}\,dx \\ &\geq K_o q_* \int_0^R \left(\frac{\epsilon}{\epsilon^2 + r^2}\right)^{(\gamma+1)/2} r^2\,dr \geq K_\epsilon q_*\end{aligned}$$

for some positive constants K_o and K_ϵ. Thus, for any choice of ϵ for which $t_\epsilon > 0$, (3.14) implies (3.4) if q_* is large enough. It is worth noticing that

$$K_\epsilon = \begin{cases} 0(\epsilon^{(\gamma+1)/2}) & \text{if } 1 < \gamma < 2 \\ 0(\epsilon^{3/2}\log\frac{1}{\epsilon} & \text{if } \gamma = 2 \\ 0(\epsilon^{(5-\gamma)/2}) & \text{if } 2 < \gamma < 5. \end{cases}$$

These estimates for $1 < \gamma \leq 3$ are not sufficient for (3.16) if $N = 3, L = 1$, and hence (3.4) does not follow, unless q_* is sufficiently large.

REMARK 3.3. Reindexing, if necessary, so that $G \subset \Omega_1$, the functional J_∞ in Proposition 3.1 can be replaced by $J_k, k = 1, 2 \ldots$. It then follows that $J_k(t_o v_\epsilon) < 0$ and

(3.17) $$\sup_{k\geq 1}\sup_{t\geq 0} J_k(tv_\epsilon) < \frac{1}{N}S^{N/2}\|p\|_\infty^{(2-N)/2}$$

for a sufficiently large choice of t_o and small choice of $\epsilon > 0$.

4. **Verification of the Palais-Smale condition.** A similar analysis to that in [5] will now be given to verify that the functionals J_k in (2.2) satisfy the Palais-Smale condition $(\mathrm{PS})_a$ for $k \geq 1$ and any a such that

$$0 < a < \frac{1}{N} S^{N/2} \|p\|_\infty^{(2-N)/2}. \tag{4.1}$$

PROPOSITION 4.1. *If conditions* (H_1)–(H_4) *and (4.1) hold, then* J_k *satisfies the* $(PS)_a$-*condition for* $k = 1, 2, \ldots$.

PROOF. For fixed $k \geq 1$, let $\{u_n\}$ be a sequence in E_k satisfying $J_k(u_n) \longrightarrow a$ and $J_k'(u_n) \longrightarrow 0$ in E_k^* as $n \longrightarrow \infty$. Then

$$J_k(u_n) = \int_{\Omega_k} \Big[\frac{1}{2}|\nabla u_n|^2 - \frac{1}{\tau+1} p(x)(u_n^{\tau+1})_+ - F(x,u_n)\Big]\, dx = a + o(1) \tag{4.2}$$

and

$$\int_{\Omega_k} \big[\nabla u_n \cdot \nabla \phi - p(x)(u_n^\tau)_+ \phi - f(x,u_n)\phi\big]\, dx = o(1)\|\phi\|_{E_k} \tag{4.3}$$

as $n \longrightarrow \infty$ for arbitrary $\phi \in E_k$. With the choice $\phi = u_n$ and the definition $b_n = \|u_n\|_{E_k}$, it follows from (4.2), (4.3), and (H_3) that

$$\Big(\frac{\gamma+1}{2} - 1\Big) b_n^2 \leq (\gamma+1)a + o(1) + o(1)b_n, \tag{4.4}$$

implying the boundedness of $\{b_n\}$ since $\gamma > 1$. In view of condition (1.3) of (H_2), Lemma 2.2 and standard embedding theorems show that $\{u_n\}$ has a subsequence, still denoted by $\{u_n\}$, for which

$$\begin{cases} u_n \longrightarrow u & \text{weakly in } E_k \\ u_n \longrightarrow u & \text{in } L^{\gamma(j)+1}(\Omega_k, q_j) \text{ for } j = 1, \ldots, m \\ u_n \longrightarrow u & \text{a.e. in } \Omega_k. \end{cases} \tag{4.5}$$

Consider now the sequence $\{v_n\}$, $v_n = u_n - u$. Using (4.3) with $\phi = u_n$, the boundedness of $\{b_n\}$ and Lemma 2.1, we obtain

$$\int_{\Omega_k} \big[|\nabla u|^2 + |\nabla v_n|^2 - p(x)(u^{\tau+1})_+ - p(x)(v_n^{\tau+1})_+ - uf(x,u)\big]\, dx = o(1) \tag{4.6}$$

as $n \longrightarrow \infty$. It is easy to see from (4.3), with $\phi = u$, by passing to the limit $n \longrightarrow \infty$ that

$$\int_{\Omega_k} \big[|\nabla u|^2 - p(x)u_+^{\tau+1} - uf(x,u)\big]\, dx = 0. \tag{4.7}$$

It is a consequence of (4.6) and (4.7) that

$$\int_{\Omega_k} |\nabla v_n|^2\, dx = \int_{\Omega_k} p(x)(v_n^{\tau+1})_+\, dx + o(1). \tag{4.8}$$

Use of Lemmas 2.1 and 2.2 yields, in view of (2.2 and (4.8)

$$\begin{aligned} J_k(u) &= J_k(u_n) - \int_{\Omega_k} \Big[\frac{1}{2}|\nabla v_n|^2 - \frac{1}{\tau+1} p(x)(v_n^{\tau+1})_+\Big]\, dx \\ &\quad + \int_{\Omega_k} [F(x,u_n) - F(x,u)]\, dx \\ &= a - \Big(\frac{1}{2} - \frac{1}{\tau+1}\Big) \int_{\Omega_k} p(x)(v_n^{\tau+1})_+\, dx + o(1) \end{aligned}$$

and hence

$$a = J_k(u) + \frac{1}{N}\int_{\Omega_k} p(x)(v_n^{\tau+1})\,dx + o(1). \tag{4.9}$$

A simple consequence of (2.2), (4.7), and (H_3) is that $J_k(u) \geq 0$; in fact

$$J_k(u) \geq \int_{\Omega_k}\Big[\frac{1}{N}p(x)u_+^{\tau+1} + \Big(\frac{1}{2}-\frac{1}{\gamma}\Big)uf(x,u)\Big]\,dx > 0. \tag{4.10}$$

For a subsequence of $\{v_n\}$, denoted the same way, we define

$$\ell = \lim_{n\to\infty} \|v_n\|_{E_k}^2 = \lim_{n\to\infty} \|u_n - u\|_{E_k}^2.$$

The embedding $E_k \hookrightarrow L^{\tau+1}(\Omega_k)$ together with (4.8) gives

$$\begin{aligned}\ell &= \lim_{n\to\infty}\int_{\Omega_k} p(x)(v_n^{\tau+1})_+\,dx \\ &\leq \|p\|_\infty S^{-(\tau+1)/2}\lim_{n\to\infty}\|v_n\|_{E_k}^{\tau+1}.\end{aligned}$$

If $\ell > 0$, this implies that

$$\ell \geq S^{N/2}\|p\|_\infty^{(2-N)/2}. \tag{4.11}$$

By (4.8)–(4.10), it follows that $\ell \leq Na$, and hence (4.11) yields the contradiction

$$a \geq \frac{\ell}{N} \geq \frac{1}{N}S^{N/2}\|p\|_\infty^{(2-N)/2}.$$

Then $\ell = 0$, proving Proposition 4.1.

LEMMA 4.2. *If (H_1)–(H_4) hold, for arbitrary $\delta > 0$ there exists $\rho \in (0,\delta)$ and $\alpha > 0$, independent of k, such that $J_k(\phi) \geq \alpha$ for all $\phi \in E_k$ with $\|\phi\|_{E_k} = \rho$, $k = 1,2,\ldots$.*

PROOF. Hypothesis (H_3) and the continuity of the embedding $E_\infty \hookrightarrow L^{\gamma(j)+1}(\Omega, q_j)$, $j = 1,\ldots,m$, from Lemma 2.2, imply that

$$\int_\Omega F(x,\phi)\,dx \leq C\sum_{j=1}^m \|\phi\|_E^{\gamma(j)+1}, \quad \phi \in E$$

for some constant $C > 0$ independent of ϕ. The embedding $E \hookrightarrow L^{\tau+1}(\Omega)$ then yields

$$J_\infty(\phi) \geq \frac{1}{2}\|\phi\|_E^2 - \tilde{C}\Big[\|\phi\|_E^{2N/(N-2)} + \sum_{j=1}^m \|\phi\|_E^{\gamma(j)+1}\Big]$$

for another positive constant $\tilde{C}$. It follows that $\rho \in (0,\delta)$ can be chosen small enough that $J_\infty(\phi) \geq \frac{1}{4}\rho^2 = \alpha$ for all ϕ with $\|\phi\|_E = \rho$.

If $\psi \in E_k$ and $\|\psi\|_{E_k} = \rho$, we extend ψ to Ω by defining supp $\psi = \Omega_k$. For this extension, obviously $\|\psi\|_E = \|\psi\|_{E_k} = \rho$, and therefore $J_k(\psi) = J_\infty(\psi) \geq \alpha$. This completes the proof of Lemma 4.2.

5. **Existence of solutions.** The results of §§3 and 4 enable us to prove the following main theorem, generalizing Theorem 1.1 to the problem (1.1).

THEOREM 5.1. *Conditions (H_1)–(H_4) imply that problem (1.1) has a solution u such that $u(x) = 0(|x|^{2-N})$ as $|x| \longrightarrow \infty$, uniformly in Ω.*

PROOF. It will first be shown that problem $(2.3)_k$ has a solution u_k for every $k = 1, 2, \ldots$. The mountain pass theorem [1] will be applied with $v = t_o v_\epsilon$ selected as in Proposition 3.1 and α, ρ as in Lemma 4.2 with $\delta = \|t_o v_\epsilon\|_E$. We may assume $G \subset \Omega_k$ for every $k = 1, 2, \ldots$ without loss of generality, as already mentioned. We define

$$a_k = \inf_{g \in \Gamma} \max_{\phi \in g} J_k(\phi), \quad k = 1, 2, \ldots,$$

where Γ denotes the class of all continuous paths g in E_k joining $\mathbf{O}$ to $t_0 v_\epsilon$, and conclude from Proposition 3.1 and Remark 3.3 that

$$0 < a_k < \frac{1}{N} S^{N/2} \|p\|_\infty^{(2-N)/2}, \quad k = 1, 2, \ldots.$$

By Proposition 4.1, J_k satisfies the $(PS)_{a_k}$-condition, and hence the mountain pass theorem implies that J_k has a critical point u_k with corresponding critical value a_k, *i.e.*,

$$0 < a_k = \int_{\Omega_k} \left[\frac{1}{2} |\nabla u_k|^2 - \frac{1}{\tau + 1} p(x)(u_k^{\tau+1})_+ - F(x, u_k) \right] dx, \tag{5.1}$$

and

$$\int_{\Omega_k} \nabla u_k \cdot \nabla \phi \, dx = \int_{\Omega_k} [p(x)(u_k^\tau)_+ \phi + f(x, u_k)\phi] \, dx \tag{5.2}$$

for all $\phi \in E_k, k = 1, 2, \ldots$. In particular, u_k is a weak solution of the equation

$$-\Delta u_k = p(x)(u_k^\tau)_+ + f(x, u_k), \quad x \in \Omega_k,$$

and therefore $u_k \geq 0$ in Ω_k by the Stampacchia maximum principle, from which u_k is a solution of the equation in $(2.3)_k$. Since u_k is nonnegative and nontrivial by (5.1), the strong maximum principle for $-\Delta u_k \geq 0$ implies that $u_k > 0$ in Ω_k, and accordingly u_k solves problem $(2.3)_k, k = 1, 2, \ldots$. By extending u_k to be zero outside Ω_k, we can regard $\{u_k\}$ as a sequence in $E = D_0^{1,2}(\Omega)$.

The definition of a_k implies that $\{a_k\}$ is nonincreasing, and consequently

$$0 < a_k \leq a_1 < \frac{1}{N} S^{N/2} \|p\|_\infty^{(2-N)/2}, \quad k = 1, 2, \ldots. \tag{5.3}$$

The proof in Proposition 4.1 can therefore be repeated to conclude that $\{\|u_k\|_E\}$ is a bounded sequence, so $\{u_k\}$ has a subsequence converging weakly in E to a weak limit $u \in E$, and also [10] converging to u in $L^{\gamma(j)+1}(\Omega, q_j), j = 1, \ldots, m$.

To show that u is nontrivial, suppose to the contrary that $u \equiv 0$ in Ω so $u_k \longrightarrow 0$ in $L^{\gamma(j)+1}(\Omega, q_j)$ as $k \longrightarrow \infty$. By (H_2) and (H_3), the integrals $\int_\Omega u_k f(x, u_k) \, dx$ and $\int_\Omega F(x, u_k) \, dx$ also converge to 0 as $k \longrightarrow \infty$. We can then use (5.1) and (5.2), with $\phi = u_k$, to obtain

$$\left(\frac{\tau + 1}{2} - 1 \right) \int_\Omega |\nabla u_k|^2 \, dx = (\tau + 1)a_k + o(1)$$

as $k \longrightarrow \infty$. Since $a_k \geq \alpha > 0$ by Lemma 4.2, this implies

$$\int_\Omega |\nabla u_k|^2\, dx + o(1) = Na_k \geq N\alpha > 0. \tag{5.4}$$

Thus, if $u = \lim u_k$ is identically zero we would have

$$L \equiv \liminf_{k\to\infty} \|u_k\|_E^2 \geq N\alpha > 0, \tag{5.5}$$

where L is defined as the inferior limit in (5.5). To show that (5.5) is impossible, we note that the same procedure used for (4.11) yields, in view of (5.2) (with $\phi = u_k$),

$$L \geq S^{N/2}\|p\|_\infty^{(2-N)/2}. \tag{5.6}$$

On the other hand, (5.3) and (5.4) give

$$\|u_k\|_E^2 + o(1) = Na_k \leq Na_1 < S^{N/2}\|p\|_\infty^{(2-N)/2},$$

and therefore $L < S^{N/2}\|p\|_\infty^{(2-N)/2}$, contrary to (5.6). The contradiction (5.5) proves that u is a nontrivial solution of the equation in problem (1.1).

The asymptotic estimate in Theorem 5.1 can be proved in exactly the same way as Egnell's recent *a priori* decay estimate for finite energy solutions in Ω [11, Theorem 2]. Hence the positivity of u in Ω is a consequence of the strong maximum principle for $-\Delta u \geq 0$.

REMARK 5.2. Theorem 1.1 is a corollary of Theorem 5.1 on account of Remark 3.2.

REMARK 5.3. If $0 \in \Omega$, a result of Egnell [11, Corollary 4] shows that u is bounded in a deleted neighborhood of 0. Available elliptic regularity theorems can then be used to show that our solution u is a classical (regular) solution in $\Omega \setminus \{0\}$ under suitable regularity assumptions on p and f. If $\partial\Omega$ is bounded, the procedure in [11] sharpens the asymptotic decay law in Theorem 5.1 to $u(x) \sim C|x|^{2-N}$ as $|x| \longrightarrow \infty$ for some positive constant $C = C(u)$.

REMARK 5.4. Our procedure applies without essential change to the Dirichlet problem

$$\begin{cases} -\Delta u = p(x)u^\tau + f(x,u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega, \quad u|_{\partial\Omega} = 0 \end{cases}$$

in a bounded domain Ω with smooth boundary $\partial\Omega$. The existence of a weak solution u follows under obvious analogues of conditions (H_1)–(H_4) for a bounded domain. Some of the results in [5] are thereby extended to a more general setting.

6. **Necessary conditions.** The necessity of the conditions (H_1) and (H_2) for (1.1) to have a solution u can be seen from the modified Pohožaev-type identity (6.1) in the Proposition below.

PROPOSITION 6.1. *Let $\Omega = \mathbf{R}^N$ in (1.2) and suppose $p, q \in C^1(\mathbf{R}^N \setminus \{0\})$. If u is locally bounded in $\mathbf{R}^N \setminus \{0\}$ and solves (1.2), then u satisfies the identity*

$$\int_{\mathbf{R}^N}\Big[\Big(\frac{N}{\gamma+1}-\frac{N-2}{2}\Big)q(x)u^{\gamma+1}+\frac{N-2}{2N}x\cdot(\nabla p)(x)u^{\tau+1} + \frac{1}{\gamma+1}x\cdot(\nabla q)(x)u^{\gamma+1}\Big]\,dx = 0. \tag{6.1}$$

This identity follows, for example from [10, Corollary A2], and can be proved by the procedure of Berestycki and Lions [3, Proposition 1].

EXAMPLE 6.2. The necessity of condition (H_2) will be indicated by (1.2) in the case

$$p(x) \equiv 1, q(x) = \min\{|x|^\mu,\ |x|^\nu\}, \quad \nu < \mu. \tag{6.2}$$

If u solves (1.2), then (6.1) reduces to

$$\int_{|x|\le 1}\Big(\frac{N+\mu}{\gamma+1}-\frac{N-2}{2}\Big)|x|^\mu u^{\gamma+1}\,dx + \int_{|x|>1}\Big(\frac{N+\nu}{\gamma+1}-\frac{N-2}{2}\Big)|x|^\nu u^{\gamma+1}\,dx = 0. \tag{6.3}$$

Therefore problem (1.2) has no solution if either

$$\gamma+1 \le \frac{2(N+\nu)}{N-2} \text{ or } \gamma+1 \ge \frac{2(N+\mu)}{N-2}.$$

Suppose ν is replaced by $\tilde\nu = \nu - \epsilon$ and μ is replaced by $\tilde\mu = \mu + \epsilon$ in (6.2), $\epsilon > 0$. Then $q(x) = o(|x|^\mu)$ as $|x| \to 0$, $q(x) = o(|x|^\nu)$ as $|x| \to \infty$ and (6.3) shows that (1.2) has no solutions if (1.3) does not hold. The same argument applies if $q(x)u^\gamma$ in (1.2) is replaced by $\sum_{j=1}^m q_j(x)u^{\gamma(j)}$, where each $q_j(x) = \min\{|x|^{\tilde\mu}, |x|^{\tilde\nu}\}$ and no exponent $\gamma(j)$ is in the interval (1.3).

EXAMPLE 6.3. To show the necessity of condition (1.4) of (H_1), consider problem (1.2) with $\Omega = \mathbf{R}^N$, $q(x)$ as in (6.2), $p(x)$ bounded in $\mathbf{R}^N$, $p \in C^1(\mathbf{R}^N)$, and $x\cdot(\nabla p)(x) > 0$ in $\mathbf{R}^N$. If γ, μ, ν satisfy (1.3), then all the conditions for Theorem 1.1 hold except condition (1.4), but the left side of (6.1) is positive by a calculation as in (6.3). This contradiction shows that condition (1.4) is necessary in general for (1.2) to have a solution.

7. Equations with a singular critical term.

Theorem 1.1 will now be extended to the problem

$$\begin{cases} -\Delta u = |x|^\lambda m(x)u^\tau + q(x)u^\gamma & x \in \Omega \\ u(x) > 0 \text{ in } \Omega, & u \in D_0^{1,2}(\Omega),\ -2 < \lambda < 0, \end{cases} \tag{7.1}$$

with a singular critical term, where the critical Sobolev exponent is defined to be

$$\tau = \frac{N+2\lambda+2}{N-2}, \quad -2 < \lambda < 0. \tag{7.2}$$

The hypotheses for (7.1) are as follows:

https://doi.org/10.4153/CJM-1992-062-2 Published online by Cambridge University Press

(A_1') $1 < \gamma < \tau$ if $N \geq 4$; $3 < \gamma < \frac{5+2\lambda}{N-2}$ if $N = 3$.
(A_2') m is a nonnegative bounded function in $\overline{\Omega}$ such that

$$0 < m(0) = \sup_{x\in\Omega} m(x) \tag{7.3}$$

and

$$m(x) = m(0) + 0(|x|^2) \text{ as } |x| \longrightarrow 0. \tag{7.4}$$

(A_3') Identical to (A_3).
(A_4') $q(x) > 0$ in some deleted neighborhood $B_\delta(0) \setminus \{0\}$ of $x = 0$.

LEMMA 7.1 [10, LEMMA 9]. *If* $-2 \leq \lambda \leq 0$ *and* $N \geq 3$ *the space* $D_0^{1,2}(\mathbf{R}^N)$ *is continuously embedded into* $L^{\tau+1}(\mathbf{R}^N, |x|^\lambda)$, *where* τ *is given by (7.2).*

The constant S in §3 will be replaced by

$$S_\lambda = \inf\{\|\nabla u\|_{2,\Omega}^2 : u \in E_\infty, \|u\|_{\tau+1,\Omega,\lambda} = 1\},$$

where

$$\|u\|_{\rho,\Omega,\lambda} = \left[\int_\Omega |u(x)|^\rho |x|^\lambda \, dx\right]^{1/\rho}, \quad \rho \geq 1.$$

Then S_λ corresponds to the best constant for the embedding in Lemma 7.1.

THEOREM 7.2. *Conditions* (A_1')–(A_4') *imply that problem (7.1) has a solution* $u(x)$ *in* Ω *such that* $u(x) = 0(|x|^{2-N})$ *as* $|x| \longrightarrow \infty$. *If in addition* $\inf_{x\in B_\delta(0)} q(x)$ *is sufficiently large, the same conclusion extends to all* $\gamma \in (1,5), N = 3$.

The proof of this theorem requires the following modification of the functional (2.2):

$$J_r(u) = \int_{\Omega_r} \left[\frac{1}{2}|\nabla u|^2 - \frac{1}{\tau+1}|x|^\lambda m(x) u_+^{\tau+1} - \frac{1}{\gamma+1} q(x) u_+^{\gamma+1}\right] dx, \tag{7.5}$$
$$u \in E_r, \quad 0 < r \leq \infty.$$

It follows from Lemma 7.1 and known results (*e.g.*, [10]) that J_r is a well-defined C^1-functional on E_r, $0 < r \leq \infty$.

In analogy with (3.1), the natural "simplest" critical equation associated with (7.1) is

$$-\Delta u = |x|^\lambda u^\tau, \quad x \in \mathbf{R}^N, \ -2 \leq \lambda < 0. \tag{7.6}$$

For arbitrary $\epsilon > 0$, routine calculations show that (7.6) has the minimal decaying positive solution

$$u_\epsilon(x) = K\left[\frac{\epsilon^{(\lambda+2)/2}}{\epsilon^{\lambda+2} + |x|^{\lambda+2}}\right]^{\frac{N-2}{\lambda+2}}, \ K = [(N+\lambda)(N-2)]^{\frac{N-2}{2\lambda+4}}. \tag{7.7}$$

If $\lambda > -2$, Talenti [22] proved that S_λ is attained by $u_\epsilon(x)$ (and also by translations of $u_\epsilon(x)$ if $\lambda = 0$, as in §3).

Integration of (7.6) by parts yields

$$\int_{\mathbf{R}^N} |\nabla u_\epsilon|^2\,dx = \int_{\mathbf{R}^N} u_\epsilon^{\tau+1}|x|^\lambda\,dx,$$

implying that

$$S_\lambda = \Big[\int_{\mathbf{R}^N} u_\epsilon^{\tau+1}|x|^\lambda\,dx\Big]^{\frac{2+\lambda}{N+\lambda}}. \tag{7.8}$$

We choose $R > 0$ small enough that $B_{2R}(0) \subset \Omega, m(x) \geq m_* > 0$ in $B_{2R}(0)$, and $q(x) \geq q_* > 0$ in $B_{2R(0)} \setminus \{0\}$, possible by assumptions (A_2'), (A_4'). Let $w_\epsilon(x)$ and $v_\epsilon(x)$ be defined by analogues of (3.2) and (3.3), respectively, with G replaced by $B_R(0)$ and τ as in (7.2).

PROPOSITION 7.3. *Conditions (A_1')–(A_4') imply that there exist positive numbers ϵ and t_0 such that $J_\infty(t_0 v_\epsilon) < 0$ and*

$$0 < \sup_{t\geq 0} J_\infty(tv_\epsilon) < \frac{2+\lambda}{2(N+\lambda)} S_\lambda^{(N+\lambda)/(2+\lambda)}[m(0)]^{(2-N)/(2+\lambda)}. \tag{7.9}$$

PROOF. Integration by parts of (7.6) gives, as a replacement for (3.5),

$$\int_{B_R(0)} |\nabla w_\epsilon|^2\,dx \leq \int_{B_R(0)} u_\epsilon^{\tau+1}|x|^\lambda\,dx. \tag{7.10}$$

Computations lead to the following analogues of (3.6)–(3.8):

$$m(0)\int_{B_R(0)} u_\epsilon^{\tau+1}|x|^\lambda\,dx \leq \int_{B_R(0)} u_\epsilon^{\tau+1}m(x)|x|^\lambda\,dx + 0(\epsilon^2), \tag{7.11}$$

$$\int_{\Omega\setminus B_R(0)} u_\epsilon^{\tau+1}m(x)|x|^\lambda\,dx = 0(\epsilon^{N+\lambda}), \tag{7.12}$$

and

$$A_\epsilon \equiv \int_{\Omega\setminus B_R(0)} |\nabla w_\epsilon|^2\,dx = 0(\epsilon^{N-2}) \tag{7.13}$$

as $\epsilon \longrightarrow 0$. We can then use (7.8) and (7.10)–(7.13) to obtain

$$\begin{aligned}\int_\Omega |\nabla w_\epsilon|^2\,dx &= \int_{B_R(0)} |\nabla u_\epsilon|^2\,dx + A_\epsilon\\ &\leq S_\lambda\Big[\int_{B_R(0)} u_\epsilon^{\tau+1}|x|^\lambda\,dx\Big]^{\frac{2}{\tau+1}} + A_\epsilon\\ &\leq S_\lambda[m(0)]^{-2/(\tau+1)}\Big[\int_{B_R(0)} u_\epsilon^{\tau+1}m(x)|x|^\lambda\,dx\Big]^{\frac{2}{\tau+1}} + 0(\epsilon^L)\end{aligned} \tag{7.14}$$

as $\epsilon \longrightarrow 0$, where $L = \min(N-2, 2)$. The integral in (7.14) is the same as that in (3.3), with $p(x) = |x|^\lambda m(x)$ and G replaced by $B_R(0)$. Since it can be verified easily that this integral is bounded below by a positive constant, independent of ϵ, (3.3) and (7.14) imply the estimate

$$V_\epsilon \equiv \int_\Omega |\nabla v_\epsilon|^2\,dx \leq S_\lambda[m(0)]^{-2/(\tau+1)} + 0(\epsilon^L). \tag{7.15}$$

The analogue of $J_\infty(tv_\epsilon)$ in (3.12) attains its maximum at a number $t_\epsilon \geq 0$ (and we can assume $t_\epsilon > 0$ without loss of generality), from which

(7.16) $$0 = J'_\infty(t_\epsilon v_\epsilon) = t_\epsilon V_\epsilon - t_\epsilon^\tau - t_\epsilon^\gamma \int_\Omega q(x) v_\epsilon^{\gamma+1}\, dx.$$

This shows that (3.13) still holds, and therefore (3.12) and (7.15) yield the estimate

(7.17) $$\begin{cases} \sup_{t\geq 0} J_\infty(tv_\epsilon) = J_\infty(t_\epsilon v_\epsilon) \\ \leq \frac{\tau-1}{2(\tau+1)} V_\epsilon^{(\tau+1)/(\tau-1)} - \frac{1}{\gamma+1} t_\epsilon^{\gamma+1} \int_{B_{2R}(0)} q(x) v_\epsilon^{\gamma+1}\, dx \\ \leq \frac{2+\lambda}{2(N+\lambda)} S_\lambda^{(N+\lambda)/(2+\lambda)} [m(0)]^{\frac{2-N}{2+\lambda}} - \frac{1}{\gamma+1} t_\epsilon^{\gamma+1} \int_{B_{2R}(0)} q(x) v_\epsilon^{\gamma+1}\, dx + 0(\epsilon^L). \end{cases}$$

We use the abbreviation

(7.18) $$\beta = \frac{1}{2}(N-2)(\gamma+1) < N + \mu,$$

where the inequality is a consequence of assumption (1.3). It follows from (3.3), (7.7), and the remark preceding (7.15) that there exist positive constants C_1, C_2, and C_3, independent of ϵ, such that

(7.19) $$\begin{aligned} \int_\Omega q(x) v_\epsilon^{\gamma+1}\, dx &\leq C_1 \epsilon^\beta \int_0^{2R} \frac{r^{\mu+N-1}\, dr}{(\epsilon^{\lambda+2} + r^{\lambda+2})^{2\beta/(\lambda+2)}} \\ &= C_1 \epsilon^{N+\mu-\beta} \int_0^{2R/\epsilon} \frac{t^{\mu+N-1}\, dt}{(1+t^{\lambda+2})^{2\beta/(\lambda+2)}} \\ &\leq C_1 \epsilon^{N+\mu-\beta} \Big[\frac{1}{N+\mu} + \frac{1}{N+\mu-2\beta} \Big\{ \Big(\frac{2R}{\epsilon}\Big)^{N+\mu-2\beta} - 1 \Big\} \Big] \\ &\leq C_2 \epsilon^{N+\mu-\beta} + C_3 \epsilon^\beta. \end{aligned}$$

The definitions of v_ϵ and V_ϵ imply that $V_\epsilon \geq KS_\lambda$ for some positive constant K, independent of ϵ. Then (7.16) gives

$$t_\epsilon^{\tau-1} \geq KS_\lambda - t_\epsilon^{\gamma-1} \int_\Omega q(x) v_\epsilon^{\gamma+1}\, dx,$$

and (3.13) and (7.19) show that $\lim_{\epsilon\to 0} t_\epsilon = t_0 > 0$. As a consequence of this, it follows from (7.17) that a constant $C > 0$ exists, independent of ϵ, such that

(7.20) $$\begin{aligned} \sup_{t\geq 0} J_\infty(tv_\epsilon) \leq &\frac{2+\lambda}{2(N+\lambda)} S_\lambda^{(N+\lambda)/(2+\lambda)} [m(0)]^{(2-N)/(2+\lambda)} \\ &- C \int_{B_{2R}(0)} q(x) v_\epsilon^{\gamma+1}\, dx + 0(\epsilon^L). \end{aligned}$$

Assumption (A_4'), (3.3), and (7.7) show, similarly to (7.19), that

(7.21) $$\epsilon^{-L} \int_{B_{2R}(0)} q(x) v_\epsilon^{\gamma+1}\, dx \geq C_4 \epsilon^{N-L-\beta}$$

for another positive constant C_4, independent of ϵ. We note that

$$N - L - \beta = \begin{cases} \frac{1}{2}(N-2)(1-\gamma) & \text{if } N \geq 4 \\ \frac{1}{2}(3-\gamma) & \text{if } N = 3, \end{cases}$$

from which $N - L - \beta < 0$ by assumption (A_1'). Therefore (7.20) and (7.21) imply that (7.9) holds for sufficiently small ϵ.

PROPOSITION 7.4. *If (A_1')–(A_4') hold, then J_k satisfies the Palais-Smale condition $(PS)_a$ for $k = 1, 2, \ldots$ and any a such that*

$$0 < a < \frac{2+\lambda}{2(N+\lambda)} S_\lambda^{(N+\lambda)/(2+\lambda)} [m(0)]^{(2-N)/(2+\lambda)}.$$

The proof is virtually identical to that of Proposition 4.1, where now the best constant S_λ for the embedding in Lemma 7.1 is given by formula (7.8). The estimate (4.4) is still obtained using obvious analogues of (4.2) and (4.3), implying the boundedness of $b_n = \|u_n\|_{E_k}$.

Theorem 7.2 can then be proved via Propositions 7.3 and 7.4 almost exactly as in §5.

It is interesting that a slight modification of our proof using the "uncertainty principle" can be used to solve a *linear* singular problem (7.1) in the case $\lambda = -2, \tau = 1, q(x) \equiv 0$. In contrast, it is well-known that (1.2) has no solution if $q(x) \equiv 0$.

REFERENCES

1. A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14**(1973), 349–381.

2. V. Benci and G. Cerami, *Existence of positive solutions of the equation* $-\Delta u + a(x)u = u^{(N+2)/(N-2)}$, preprint.

3. H. Berestycki and P. L. Lions, *Nonlinear scalar field equations, I & II*, Arch. Rational Mech. Anal. **82**(1983), 313–375.

4. H. Brezis, *Some variational problems with lack of compactness*, Proc. Symp. in Pure Math. **45**(1986), 165–201.

5. H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math **36**(1983), 437–477.

6. H. Brezis and E. H. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88**(1983), 486–490.

7. W.-Y. Ding and W.-M. Ni, *On the elliptic equation* $\Delta u + Ku^{(n+2)/(n-2)} = 0$ *and related topics*, Duke Math. J. **52**(1985), 485–506.

8. H. Egnell, *Semilinear elliptic equations involving critical Sobolev exponents*, Arch. Rational Mech. Anal. **104**(1988), 27–56.

9. ———, *Elliptic boundary value problems with singular coefficients and critical nonlinearities*, Indiana Univ. Math. J. **38**(1989), 235–251.

10. ———, *Existence results for some quasilinear elliptic equations*, Proc. of the conference "Variational Problems", Paris, June, 1988, to appear.

11. ———, *Asymptotic results for finite energy solutions of semilinear elliptic equations*, preprint.

12. J. F. Escobar, *Positive solutions for some semilinear elliptic equations with critical Sobolev exponents*, Comm. Pure Appl. Math. **40**(1987), 623–657.

13. B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34**(1981), 525–598.

14. M. Guedda and L. Veron, *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal. **13**(1989), 879–902.

15. Y. Li and W.-M. Ni, *On conformal scalar curvature in* R^N, Duke. Math. J. **57**(1988), 895–924.

16. E. H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev inequality and related inequalties*, Ann. of Math. (2) **118**(1983), 349–374.

17. P. L. Lions, *The concentration-compactness principle in the Calculus of Variations. The locally compact case, Parts 1 & 2*, Ann. Inst. H. Poincaré: Anal. non linéaire **1**(1984), 109–145 & 223–283.

18. ———, *The concentration-compactness principle in the Calculus of Variations. The limit case, Parts 1 & 2*, Revista Math. Iberoamericana (1) **1**(1985), 145–201 & (2) **1**(1985), 46–120.

19. W.-M. Ni, *On the elliptic equation* $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$, *its generalizations, and applications to geometry*, Indiana Univ. Math. J. **31**(1982), 493–529.

20. W.-M. Ni and S. Yotsutani, *Semilinear elliptic equations of Matukuma-type and related topics*, Japan J. Appl. Math. **5**(1988), 1–32.

21. E. S. Noussair and C. A. Swanson, *Ground states for critical semilinear scalar field equations*, Differential and Integral Equations, **5**(1990), 875–887.

22. G. Talenti, *Best constants in Sobolev inequality*, Ann. Mat. Pura Appl. **110**(1976), 353–372.

23. C. Taubes, *The existence of a non-minimal solution to the* SU(2) *Yang-Mills-Higgs equations on* R^3, Comm. Math. Phys. **86**(1982), 257–298.

24. Dong Zhang, *Positive solutions of nonlinear scalar field equations involving critical Sobolev exponent*, Acta Math. Sinica N.S. **3**(1987), 27–37.

School of Mathematics
University of New South Wales
Kensington, N.S.W.
Australia 2033

Department of Mathematics
University of British Columbia
Vancouver, British Columbia
V6T 1Y4

Department of Mathematics
Jiangxi University
Nanchang, Jiangxi 330047
People's Republic of China