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Part 2. Lévy processes

LOSS RATE FOR A GENERAL LÉVY PROCESS WITH DOWNWARD PERIODIC BARRIER

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LOSS RATE FOR A GENERAL LÉVY PROCESS WITH DOWNWARD PERIODIC BARRIER

BY ZBIGNIEW PALMOWSKI AND PRZEMYSŁAW ŚWIĄTEK

Abstract

In this paper we consider a general Lévy process X reflected at a downward periodic barrier A_t and a constant upper barrier K, giving a process $V_t^K = X_t + L_t^A - L_t^K$. We find the expression for a loss rate defined by $l^K = \mathrm{E}\, L_1^K$ and identify its asymptotics as $K \to \infty$ when X has light-tailed jumps and $\mathrm{E}\, X_1 < 0$.

Keywords: Loss rate; Lévy process; barrier; queueing process

2010 Mathematics Subject Classification: Primary 60K05; 60K25
Secondary 60K10

1. Introduction

In this paper we consider a general Lévy process X reflected at a downward periodic barrier $A_t = \varphi(t + U)$ (for a periodic nonnegative function $\varphi(t)$ with period length s and U having uniform distribution on $[0, s]$) and at the constant upper barrier K, giving a process

$$V_t^K = X_t + L_t^A - L_t^K, \tag{1.1}$$

which is the solution of the corresponding Skorokhod problem on each period where reflection is within bounded and convex sets (see [5], [6], and [13]). In the above we have assumed that $\varphi(t) \in [0, a]$ for some $a < K$. Process X is defined on the filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{P})$, with the natural filtration satisfying the usual assumptions of right continuity and completion. From now on we will also assume that the jump measure ν of X is nonlattice.

In this paper we find the expression for a loss rate defined by

$$l^K = \mathrm{E}\, L_1^K, \tag{1.2}$$

where E denotes the expectation when the reflected process is stationary with stationary measure π_K (that is, $\mathrm{E}[\cdot] = \int_0^\infty \mathrm{E}[\cdot \mid V_0^K = x]\pi_K(\mathrm{d}x)$), and prove that $l^K \sim D\mathrm{e}^{-\gamma K}$ as $K \to \infty$, where γ solves $\kappa(\gamma) = 0$ for a Laplace exponent $\kappa(\alpha) = \log \mathrm{E} \exp\{\alpha X(1)\}$ (which is well defined in some set Θ) when X has light-tailed jumps and $\mathrm{E}\, X_1 < 0$.

The motivation for this work comes from various queueing and telecommunication models (see [1], [3], [4], [7], [8], [12], and [14]). Applications, where the reflected Lévy process considered in this paper is natural, are models where in addition to the input and output mechanisms modeled by a Lévy process there is a constant input given by a downward barrier A_t. This additional input is not available on a liquid basis, but can only be used after some maturity date has been reached. We choose this time lag to be fixed and equal to the length s of the period of φ (for an exponential time delay, see [9]). For example, in view of Internet networking applications, we consider the combined behavior of two services (e.g. streaming video and some other data). The first input behaves like a Lévy process. The other input grows deterministically and can be served only at some fixed time s. The combined workload now behaves like a Lévy process reflected at a lower barrier A_t.

Fluid models with finite buffers are useful to model systems where losses are of crucial importance, as in inventory theory and telecommunications. Indeed, in recent years real-time applications, such as video streaming and interactive games, have become increasingly popular among users. These applications are generally delay sensitive and require some preferential treatment in order to satisfy a desired level of quality of service. Traditionally, finite-capacity buffer mechanisms have been employed in the network routers, in which arriving packets are dropped when the workload reaches its maximum capacity. In this paper we analyze the intensity of packet loss given by the so-called loss rate in (1.2).

There has been a great deal of work on overflow probabilities in various fluid and queueing models, but there have been relatively few studies on the loss rate in finite-buffer systems. When jumps of the Lévy process are heavy tailed, then there is hope of finding a relationship between these two notions (see [3], [7], and [10] for more classical models). In our model we focus on the light-tailed case and, therefore, we choose Kella–Whitt's martingale approach [8]. In fact, we follow the ideas included in the seminal paper of Asmussen and Pihlsgård [2], where both barriers are constant (see also [11] for the matrix analytic method). This case corresponds to the assumption that $A(t) \equiv 0$. Denoting the loss rate by $l^{K,0}$, as in [2], we of course immediately obtain the bounds

$$l^{K,0} \le l^K \le l^{K-a,0},$$

from which, together with [2, Theorem 4.1], it follows that, e.g. $(1/K)\log l^K = -\gamma$. In this paper we focus on more precise exact asymptotics.

The paper is organized as follows. In Section 2 we give preliminary results, and in Section 3 we give the main results with proofs.

2. Preliminaries

Assume from now on that $\varphi \in \mathcal{C}^1(\operatorname{int} J_k)$ is invertible on some disjoint intervals J_k satisfying $\bigcup_{k=1}^n J_k = [0, s]$ with $\varphi'(x) \neq 0$ for $x \in \operatorname{int} J_k$.

Lemma 2.1. *The process $A_t = \varphi(t + U)$ has the invariant measure*

$$\xi(\mathrm{d}y) = \sum_{k=1}^{n} \frac{1}{s} |h_k'(y)|\, \mathbf{1}_{\varphi(\operatorname{int} J_k)}(y)\, \mathrm{d}y,$$

where h_k is an inverse of φ on int J_k.

Proof. It is sufficient to check that, for $t \in [0, s)$,

$$\mathrm{P}(A_t \le x) = \mathrm{P}(\varphi(t+U) \le x) = \frac{1}{s}\int_0^s \mathbf{1}_{\{\varphi(t+u)\le x\}}\, \mathrm{d}u = \frac{1}{s}\int_0^s \mathbf{1}_{\{\varphi(u)\le x\}}\, \mathrm{d}u,$$

where the last equality is a consequence of the periodicity of φ. The second part of the theorem follows from straightforward arguments concerning the distribution of a piecewise, strictly monotone function of random variables.

Example 2.1. The most interesting case for applications is a saw-like lower boundary modeling constant intensity input (with rate 1 for simplicity):

$$\varphi(t) = t \bmod a. \tag{2.1}$$

Here $0 < a < K$. In this case $n = 1$, $J_1 = [0, a]$, $s = a$, and $\xi(\mathrm{d}y) = (\mathrm{d}y/a)\, \mathbf{1}_{\{y\in[0,a]\}}$.

Example 2.2. A more complex situation will appear when φ is composed from a number of lines with different slopes:

$$\varphi(t)=\begin{cases} t & \text{for } t\in[0,1),\\ 1-2(t-1) & \text{for } t\in\left[1,\tfrac{3}{2}\right),\\ 3\left(t-\tfrac{3}{2}\right) & \text{for } t\in\left[\tfrac{3}{2},\tfrac{5}{2}\right). \end{cases}$$

In this case $n=3$, $J_1=[0,1)$, $J_2=[1,\frac{3}{2})$, $J_3=[\frac{3}{2},\frac{5}{2})$, $s=\frac{5}{2}$, and $\xi(\mathrm{d}y)=\frac{2}{5}(\mathbf{1}_{\varphi(J_1)}(y)+\frac{1}{2}\mathbf{1}_{\varphi(J_2)}(y)+\frac{1}{3}\mathbf{1}_{\varphi(J_3)}(y))\,\mathrm{d}y$.

Using the arguments of Asmussen [1, pp. 393–394], we have the following representation for the stationary distribution of V_t^K.

Lemma 2.2. *The stationary distribution V_∞^K of the two-sided reflected Lévy process is given by*

$$\mathrm{P}(V_\infty^K\ge x)=\int_0^a \underbrace{\sum_{k=1}^n \mathrm{P}(X_\tau\ge \hat{A}_\tau^{-z,k}+x)\,p_k(z)\,\xi(\mathrm{d}z)}_{\overline{\pi}_K^z(x)},$$

where $\tau=\inf\{t\ge 0\colon X_t\notin[x-K,\hat{A}_t^{-z,k}+x)\}$, $\hat{A}_t^{-z,k}=-\varphi(h_k(z)-t)$ for $z\in\varphi(J_k)$, and

$$p_k(z)=\mathrm{P}(U\in J_k\mid \varphi(U)=z)=\frac{|h_k'(z)|\,\mathbf{1}_{\varphi(J_k)}(z)}{\sum_{j=1}^n|h_j'(z)|\,\mathbf{1}_{\varphi(J_j)}(z)}.$$

Lemma 2.3. *If $\mathrm{E}\,|X_1|<\infty$ then $\mathrm{E}\,L_t^A<\infty$ and $\mathrm{E}\,L_t^K<\infty$ for each $t\ge 0$.*

Proof. Note that by (1.1) we have $\mathrm{E}\,L_t^K<\infty$ if $\mathrm{E}\,L_t^A<\infty$. The condition $\mathrm{E}\,L_t^A<\infty$ follows from the Wald identity applied to the random walk whose increments are the corrections of X between consecutive visits of the downward barrier (for details, see [2]).

We now need a further slight modification of the Lévy exponent $\kappa(\alpha)$. We will treat large and small jumps separately. Let L be a constant that satisfies $L>\max(K,1)$. Then $\kappa(\alpha)$ can be rewritten as

$$\theta_L\alpha+\frac{\sigma^2\alpha^2}{2}+\int_{-\infty}^{\infty}[\mathrm{e}^{\alpha x}-1-\alpha x\,\mathbf{1}_{\{|x|\le L\}}]\nu(\mathrm{d}x),\qquad \alpha\in\Theta, \tag{2.2}$$

where $\theta_L=\theta+\int_1^L x\nu(\mathrm{d}x)+\int_{-L}^{-1}x\nu(\mathrm{d}x)$.

For any process Y, we will denote its continuous part by $\{Y_t^{\mathrm{c}}\}$ and the jumps by $\Delta Y_s=Y_s-Y_{s-}$.

We split ΔL_t^K into $\underline{\Delta} L_t^K$ and $\overline{\Delta} L_t^K$, corresponding to $\Delta X_s\in[0,L]$ and $\Delta X_s\in(L,\infty)$, respectively, and we split ΔL_t^A into $\underline{\Delta} L_t^A$ and $\overline{\Delta} L_t^A$, corresponding to $\Delta X_s\in[-L,0]$ and $\Delta X_s\in(-\infty,-L)$, respectively. Let

$$\underline{l}_j^K=\mathrm{E}\sum_{0\le s\le 1}\underline{\Delta} L_s^K,\qquad \bar{l}_j^K=\mathrm{E}\sum_{0\le s\le 1}\overline{\Delta} L_s^K,$$

$$\underline{l}_j^A=\mathrm{E}\sum_{0\le s\le 1}\underline{\Delta} L_s^A,\quad\text{and}\quad \bar{l}_j^A=\mathrm{E}\sum_{0\le s\le 1}\overline{\Delta} L_s^A.$$

Then $l_j^K=\underline{l}_j^K+\bar{l}_j^K$ and $l_j^A=\underline{l}_j^A+\bar{l}_j^A$. Finally, let $l_c^K=l^K-l_j^K$ and $l_c^A=l^A-l_j^A$ with $l^A=\mathrm{E}\,L_1^A$.

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Theorem 2.1. *For $\alpha \in \Theta$,*

$$M_t = \kappa(\alpha)\int_0^t \mathrm{e}^{\alpha V_s^K}\,\mathrm{d}s + \mathrm{e}^{\alpha V_0^K} - \mathrm{e}^{\alpha V_t^K} + \alpha\int_0^t \mathrm{e}^{\alpha A_s}\,\mathrm{d}L_s^{A,\mathrm{c}} + \sum_{0\le s\le t}\mathrm{e}^{\alpha A_s}(1-\mathrm{e}^{-\alpha\Delta L_s^A}) - \alpha\mathrm{e}^{\alpha K}L_t^{K,\mathrm{c}} + \mathrm{e}^{\alpha K}\sum_{0\le s\le t}(1-\mathrm{e}^{\alpha\Delta L_s^K}) \tag{2.3}$$

is a zero-mean martingale.

Proof. It is well known that, for an $\mathcal{F}_t$-adapted process $Y_t = \int_0^t \mathrm{d}Y_s^{\mathrm{c}} + \sum_{0\le s\le t}\Delta Y_s$ of locally bounded variation, the process

$$K_t = \kappa(\alpha)\int_0^t \mathrm{e}^{\alpha Z_s}\,\mathrm{d}s + \mathrm{e}^{\alpha x} - \mathrm{e}^{\alpha Z_t} + \alpha\int_0^t \mathrm{e}^{\alpha Z_s}\,\mathrm{d}Y_s^{\mathrm{c}} + \sum_{0\le s\le t}\mathrm{e}^{\alpha Z_s}(1-\mathrm{e}^{-\alpha\Delta Y_s})$$

is a local martingale whenever $\alpha\in\Theta$, where $Z_t = x + X_t + Y_t$. Taking $Y_t = L_t^A - L_t^K$ and using Lemma 2.3 to prove that Y has locally bounded variation, we find that

$$M_t = \kappa(\alpha)\int_0^t \mathrm{e}^{\alpha V_s^K}\,\mathrm{d}s + \mathrm{e}^{\alpha V_0^K} - \mathrm{e}^{\alpha V_t^K} + \alpha\int_0^t \mathrm{e}^{\alpha V_s^K}\,\mathrm{d}L_s^{A,\mathrm{c}} + \sum_{0\le s\le t}\mathrm{e}^{\alpha V_s^K}(1-\mathrm{e}^{-\alpha\Delta L_s^A}) - \alpha\int_0^t \mathrm{e}^{\alpha V_s^K}\,\mathrm{d}L_s^{K,\mathrm{c}} + \sum_{0\le s\le t}\mathrm{e}^{\alpha V_s^K}(1-\mathrm{e}^{\alpha\Delta L_s^K})$$

is a local martingale. Here M_t equals (2.3) since $V_s^K = K$ just after a jump of L_s^K, and $V_s^K = A_s$ just after a jump of L_s^A. To prove that $\{M_t\}$ is a true martingale, it is sufficient to show that $\mathrm{E}\sup_{0\le s\le t} M_s < \infty$. This follows from the following conditions: $V_t^K \le K$, $\mathrm{E}\,L_t^{A,\mathrm{c}}$, $\mathrm{E}\,L_t^{K,\mathrm{c}} < \infty$, $\mathrm{E}\sum_{0\le s\le t}|1-\mathrm{e}^{\alpha\Delta L_s^K}| < \infty$, and $\mathrm{E}\sum_{0\le s\le t}|1-\mathrm{e}^{-\alpha\Delta L_s^A}| < \infty$ (see also the proof of [2, Proposition 3.1]).

Corollary 2.1. *Let $\alpha\in\Theta$. Then l^K satisfies the following equation:*

$$\begin{aligned}\alpha(1-\mathrm{e}^{\alpha K})l^K &= -\kappa(\alpha)\,\mathrm{E}\,\mathrm{e}^{\alpha V_0^K} + \alpha\,\mathrm{E}\,X_1 - \alpha\mathrm{e}^{\alpha K}\bar{l}_j^K + \alpha\bar{l}_j^A + \frac{\alpha^2}{2}\mathrm{E}\sum_{0\le s\le 1}(\underline{\Delta}L_s^K)^2 \\ &\quad + \frac{\alpha^2}{2}\mathrm{E}\sum_{0\le s\le 1}(\underline{\Delta}L_s^A)^2 - \mathrm{e}^{\alpha K}\,\mathrm{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\overline{\Delta}L_s^K}) \\ &\quad - \mathrm{E}\sum_{0\le s\le 1}\mathrm{e}^{\alpha A_s}(1-\mathrm{e}^{-\alpha\overline{\Delta}L_s^A}) - \alpha^2\,\mathrm{E}\int_0^1 A_s\,\mathrm{d}L_s^{A,\mathrm{c}} \\ &\quad - \alpha^2\,\mathrm{E}\sum_{0\le s\le 1}A_s\underline{\Delta}L_s^A + o(\alpha^2).\end{aligned} \tag{2.4}$$

Proof. If we take $t=1$ in M_t and use the stationarity of V_t^K, we obtain

$$0 = \kappa(\alpha)\,\mathrm{E}\,\mathrm{e}^{\alpha V_0^K} + \alpha\,\mathrm{E}\int_0^1 \mathrm{e}^{\alpha A_s}\,\mathrm{d}L_s^{A,\mathrm{c}} + \mathrm{E}\sum_{0\le s\le 1}\mathrm{e}^{\alpha A_s}(1-\mathrm{e}^{-\alpha\Delta L_s^A}) - \alpha\mathrm{e}^{\alpha K}l_c^K + \mathrm{e}^{\alpha K}\,\mathrm{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\Delta L_s^K}).$$

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Moreover, we have

$$\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\Delta L_s^K}) = \sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\underline{\Delta} L_s^K}) + \sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\overline{\Delta} L_s^K}), \tag{2.5}$$

$$\sum_{0\le s\le 1}\mathrm{e}^{\alpha A_s}(1-\mathrm{e}^{-\alpha\Delta L_s^A}) = \sum_{0\le s\le 1}\mathrm{e}^{\alpha A_s}(1-\mathrm{e}^{-\alpha\underline{\Delta} L_s^A}) + \sum_{0\le s\le 1}\mathrm{e}^{\alpha A_s}(1-\mathrm{e}^{-\alpha\overline{\Delta} L_s^A}). \tag{2.6}$$

Applying the expansion

$$\mathrm{e}^{\alpha x} = 1+\alpha x+\frac{(\alpha x)^2}{2}+\frac{(\alpha x)^3}{6}\mathrm{e}^{\theta\alpha x}, \qquad \theta\in[-1,1],$$

to the first terms on the right-hand sides of (2.5) and (2.6), and applying the expansion $\mathrm{e}^{\alpha} = 1+\alpha+o(\alpha)$ to $\alpha\,\mathrm{E}\int_0^1 \mathrm{e}^{\alpha A_s}\,\mathrm{d}L_s^{A,\mathrm{c}}$ completes the proof.

We will also need the following observation.

Lemma 2.4. *We have* $\mathrm{E}\int_0^1 A_s\,\mathrm{d}L_s^A = \mathrm{E}(A_0)l^A$, *where* $\mathrm{E}\,A_0 = \int_0^a y\xi(\mathrm{d}y)$.

Proof. Note that $\mathrm{E}\,L_s^A = s\,\mathrm{E}\,L_1^A = sl^A$, which is a consequence of the fact that L_s^A has independent and stationary increments under the invariant starting position of A. The proof is completed as follows:

$$\begin{aligned}
\mathrm{E}\int_0^1 A_s\,\mathrm{d}L_s^A &= \int_0^a \mathrm{E}\int_0^1 A_s^z\,\mathrm{d}L_s^A\xi(\mathrm{d}z)\\
&= \int_0^a\bigg(\mathrm{E}\,A_s^zL_s^A|_0^1 - \mathrm{E}\int_0^1 L_s^A\,\mathrm{d}A_s^z\bigg)\xi(\mathrm{d}z)\\
&= \int_0^a\bigg(A_1^zl^A - \int_0^1 sl^A\,\mathrm{d}A_s^z\bigg)\xi(\mathrm{d}z)\\
&= \int_0^a\bigg(A_1^zl^A - l^A\bigg(sA_s^z|_0^1 - \int_0^1 A_s^z\,\mathrm{d}s\bigg)\bigg)\xi(\mathrm{d}z)\\
&= \int_0^a l^A\int_0^1 A_s^z\,\mathrm{d}s\xi(\mathrm{d}z)\\
&= l^A\int_0^1\int_0^a A_s^z\xi(\mathrm{d}z)\,\mathrm{d}s\\
&= l^A\,\mathrm{E}\,A_0.
\end{aligned}$$

Now, using Lemma 2.4, we can rewrite (2.4) as follows.

Lemma 2.5. *As* $\alpha\downarrow 0$, *we have*

$$\begin{aligned}
\alpha(1-\mathrm{e}^{\alpha K}+\alpha\,\mathrm{E}\,A_0)l^K &= -\kappa(\alpha)\,\mathrm{E}\,\mathrm{e}^{\alpha V_0^K}+\alpha\,\mathrm{E}\,X_1-\alpha\mathrm{e}^{\alpha K}\bar{l}_j^K+\alpha\bar{l}_j^A\\
&\quad+\frac{\alpha^2}{2}\mathrm{E}\sum_{0\le s\le 1}(\underline{\Delta}L_s^K)^2+\frac{\alpha^2}{2}\mathrm{E}\sum_{0\le s\le 1}(\underline{\Delta}L_s^A)^2\\
&\quad-\mathrm{e}^{\alpha K}\,\mathrm{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\overline{\Delta}L_s^K})-\mathrm{E}\sum_{0\le s\le 1}\mathrm{e}^{\alpha A_s}(1-\mathrm{e}^{-\alpha\overline{\Delta}L_s^A})\\
&\quad+\alpha^2\,\mathrm{E}\,A_0\,\mathrm{E}\,X_1+\alpha^2\,\mathrm{E}\sum_{0\le s\le 1}A_s\overline{\Delta}L_s^A+o(\alpha^2).
\end{aligned}\tag{2.7}$$

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3. Main results

The first main result gives the representation of l^K in terms of the basic characteristics of the process X and lower boundary A.

Theorem 3.1. *Let $\{X_t\}$ be a Lévy process, and let l^K be the loss rate defined in (1.2). If $\int_1^\infty y\nu(\mathrm{d}y) = \infty$ then $l^K = \infty$, otherwise*

$$l^K = \mathrm{E}\, X_1\left(\frac{1}{K-\mathrm{E}\, A_0}\int_0^K x\pi_K(\mathrm{d}x) - \frac{\mathrm{E}\, A_0}{(K-\mathrm{E}\, A_0)}\right) + \frac{\sigma^2}{2(K-\mathrm{E}\, A_0)}$$
$$+ \frac{1}{2(K-\mathrm{E}\, A_0)}\int_0^a\int_z^K\int_{-\infty}^\infty \varphi_K(x,y,z)\nu(\mathrm{d}y)\pi_K^z(\mathrm{d}x)\xi(\mathrm{d}z),$$

where

$$\varphi_K(x,y,z) = \begin{cases} -(x-z)^2 - 2y(x-z) & \text{if } y \le -x+z, \\ y^2 & \text{if } -x+z < y < K-x, \\ 2y(K-x)-(K-x)^2 & \text{if } y \ge K-x. \end{cases}$$

Proof. The first claim follows immediately if we note that, for $\int_1^\infty y\nu(\mathrm{d}y) = \infty$ and $L > K$, we have

$$l^K \ge \int_0^K \pi_K(\mathrm{d}x)\int_L^\infty (y-K+x)\nu(\mathrm{d}y) \ge \int_L^\infty (y-K)\nu(\mathrm{d}y) = \infty.$$

The idea of the proof of the second part of the theorem is based on two steps.

Step 1: expand all the terms on the right-hand side of (2.7). For the first term on the right-hand side of (2.7), we obtain

$$\begin{aligned}\kappa(\alpha)\,\mathrm{E}\,\mathrm{e}^{\alpha V_0^K} &= \int_0^K \mathrm{e}^{\alpha x}\int_{-\infty}^\infty \mathrm{e}^{\alpha y}\,\mathbf{1}_{\{|y|\ge L\}}\,\nu(\mathrm{d}y)\pi_K(\mathrm{d}x) \\ &\quad - \int_{-\infty}^\infty \mathbf{1}_{\{|y|\ge L\}}\,\nu(\mathrm{d}y) + \alpha\left(\theta_L - \int_0^K x\int_{-\infty}^\infty \mathbf{1}_{\{|y|\ge L\}}\,\nu(\mathrm{d}y)\pi_K(\mathrm{d}x)\right) \\ &\quad + \alpha^2\left(\theta_L\int_0^K x\pi_K(\mathrm{d}x) + \frac{\sigma^2}{2} + \int_{-L}^L \frac{y^2}{2}\nu(\mathrm{d}y)\right. \\ &\qquad \left. - \int_0^K \frac{x^2}{2}\int_{-\infty}^\infty \mathbf{1}_{\{|y|\ge L\}}\,\nu(\mathrm{d}y)\pi_K(\mathrm{d}x)\right) + o(\alpha^2).\end{aligned}$$

Similarly,

$$\alpha\,\mathrm{E}\,X_1 = \alpha\theta_L + \alpha\int_{-\infty}^\infty y\,\mathbf{1}_{\{|y|\ge L\}}\,\nu(\mathrm{d}y),$$
$$\frac{\alpha^2}{2}\mathrm{E}\sum_{0\le s\le 1}(\Delta L_s^K)^2 = \frac{\alpha^2}{2}\int_0^K \pi_K(\mathrm{d}x)\int_{K-x}^L (y-K+x)^2\nu(\mathrm{d}y),$$

and

$$\frac{\alpha^2}{2}\mathrm{E}\sum_{0\le s\le 1}(\Delta L_s^A)^2 = \frac{\alpha^2}{2}\int_0^a\int_z^K\int_{-L}^{-x+z}(x+y-z)^2\nu(\mathrm{d}y)\pi_K^z(\mathrm{d}x)\xi(\mathrm{d}z).$$

For $x > 0$, define $\overline{\nu}(x) = \nu((x, \infty))$ and, similarly, for $x < 0$, define $\underline{\nu}(x) = \nu((-\infty, x))$. Then

$$\alpha \bar{l}_j^A = -\alpha \int_{-\infty}^{-L} y\nu(\mathrm{d}y) - \alpha \int_0^K x\pi_K(\mathrm{d}x)\underline{\nu}(-L) + \alpha \int_0^a \int_z^K z\pi_K^z(\mathrm{d}x)\xi(\mathrm{d}z)\underline{\nu}(-L)$$

and

$$\begin{aligned}\alpha \mathrm{e}^{\alpha K} \bar{l}_j^K = {}& \alpha \int_L^\infty y\nu(\mathrm{d}y) + \alpha^2 K \int_L^\infty y\nu(\mathrm{d}y) \\ &+ (\alpha + \alpha^2 K)\Big(\int_0^K x\pi_K(\mathrm{d}x)\overline{\nu}(L) - K\overline{\nu}(L)\Big) + o(\alpha^2).\end{aligned}$$

We also obtain

$$\begin{aligned}\mathrm{e}^{\alpha K}\,\mathrm{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\overline{\Delta}L_s^K}) = {}& \Big(1+\alpha K + \frac{\alpha^2K^2}{2}\Big)\overline{\nu}(L) \\ &- \int_0^K \mathrm{e}^{\alpha x}\int_L^\infty \mathrm{e}^{\alpha y}\nu(\mathrm{d}y)\pi_K(\mathrm{d}x) + o(\alpha^2)\end{aligned}$$

and

$$\mathrm{E}\sum_{0\le s\le 1}\mathrm{e}^{\alpha A_s}(1-\mathrm{e}^{-\alpha\overline{\Delta}L_s^A}) = \mathrm{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\overline{\Delta}L_s^A}) + \alpha^2\,\mathrm{E}\sum_{0\le s\le 1}A_s\overline{\Delta}L_s^A + o(\alpha^2),$$

with

$$\begin{aligned}\mathrm{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\overline{\Delta}L_s^A}) = {}& \underline{\nu}(-L) - \int_0^K \mathrm{e}^{\alpha x}\int_{-\infty}^{-L}\mathrm{e}^{\alpha y}\nu(\mathrm{d}y)\pi_K(\mathrm{d}x) \\ &+ \int_0^a \alpha z\int_z^K \mathrm{e}^{\alpha x}\int_{-\infty}^{-L}\mathrm{e}^{\alpha y}\nu(\mathrm{d}y)\pi_K^z(\mathrm{d}x)\xi(\mathrm{d}z) \\ &- \frac{1}{2}\int_0^a \alpha^2 z^2\int_z^K \mathrm{e}^{\alpha x}\int_{-\infty}^{-L}\mathrm{e}^{\alpha y}\nu(\mathrm{d}y)\pi_K^z(\mathrm{d}x)\xi(\mathrm{d}z) + o(\alpha^2).\end{aligned}$$

Step 2: let $L \to \infty$ and then let $\alpha \downarrow 0$. If we now rearrange all the terms of (2.7) using the above identities and let $L \to \infty$ (note that $\theta_L \to \mathrm{E}\,X_1$ as $L \to \infty$), we obtain

$$\begin{aligned}\alpha(1-\mathrm{e}^{\alpha K}+\alpha\,\mathrm{E}\,A_0)l^K = {}& -\mathrm{E}\,X_1\alpha^2\int_0^K x\pi_K(\mathrm{d}x) - \frac{\sigma^2\alpha^2}{2} \\ &- \frac{\alpha^2}{2}\int_0^a\int_z^K\int_{-x+z}^{K-x} y^2\nu(\mathrm{d}y)\pi_K^z(\mathrm{d}x)\xi(\mathrm{d}z) \\ &+ \frac{\alpha^2}{2}\int_0^a\int_z^K\int_{K-x}^{\infty}((x-K)^2+2y(x-K))\nu(\mathrm{d}y)\pi_K^z(\mathrm{d}x)\xi(\mathrm{d}z) \\ &+ \frac{\alpha^2}{2}\int_0^a\int_z^K\int_{-\infty}^{-x+z}((x-z)^2+2y(x-z))\nu(\mathrm{d}y)\pi_K^z(\mathrm{d}x)\xi(\mathrm{d}z) \\ &+ \alpha^2\,\mathrm{E}\,A_0\,\mathrm{E}\,X_1 + o(\alpha^2).\end{aligned}$$

The proof is completed by dividing both sides of the above equation by $\alpha(1-\mathrm{e}^{\alpha K}+\alpha\,\mathrm{E}\,A_0)$ and sending α to 0.

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Assume now that there exists a $\gamma > 0$ ($\gamma \in \Theta$) such that $\kappa(\gamma) = 0$, and define the new probability measure

$$\frac{\mathrm{dP}^\gamma}{\mathrm{dP}}\bigg|_{\mathcal{F}_t} = \mathrm{e}^{\gamma X_t},$$

for which we have $\mathrm{E}^\gamma X_1 = \kappa'(\gamma) > 0$ since, on P^γ, the process X is a Lévy process with Laplace exponent $\kappa_\gamma(\alpha) = \kappa(\alpha + \gamma)$. We also need two passage times:

$$\tau_z^A(x) = \inf\{t \geq 0 \colon X_t \geq \hat{A}_t^{-z} + x\}, \qquad \tau_{-z}^- = \inf\{t \geq 0 \colon X_t < -z\}.$$

Furthermore, let $\tau^A(x) = \inf\{t \geq 0 \colon X_t \geq \hat{A}_t^\xi + x\}$, where $\hat{A}_t^\xi = \int_0^\infty A_t^{-y} \xi(\mathrm{d}y)$, and let $B^A(x) = X_{\tau^A(x)} - x$. The second main result concerns the asymptotics of l^K as $K \to \infty$.

Theorem 3.2. *Assume that there exists a $\gamma > 0$ ($\gamma \in \Theta$) such that $\kappa(\gamma) = 0$ and $\kappa'(\gamma) < \infty$. Then there exists a random variable $B^A(\infty)$ such that*

$$\lim_{x\to\infty} \mathrm{E}^\gamma \,\mathrm{e}^{-\gamma B^A(x)} = \mathrm{E}^\gamma \,\mathrm{e}^{-\gamma B^A(\infty)}. \tag{3.1}$$

Furthermore, there exists a finite constant D such that

$$l^K \sim D\mathrm{e}^{-\gamma K} \quad \text{as } K \to \infty,$$

where we write $f(K) \sim g(K)$ when $\lim_{K\to\infty} f(K)/g(K) = 1$, and

$$\begin{aligned} D = {} & -\mathrm{E}\, X_1 C_\gamma + \mathrm{E}^\gamma \,\mathrm{e}^{-\gamma B^A(\infty)} \int_0^\infty \mathrm{e}^{\gamma x}\, \mathrm{P}^\gamma(\tau_{-x}^- = \infty) \int_x^\infty (1 - \mathrm{e}^{\gamma(y-x)})\nu(\mathrm{d}y)\mathrm{d}x \\ & + \int_{-\infty}^0 (y + \gamma^{-1}(1 - \mathrm{e}^{\gamma y}))\nu(\mathrm{d}y) \\ & + \int_0^\infty \int_0^{a\wedge x} \mathrm{P}(\tau_z^A(x) < \infty) \int_{-\infty}^{-x+z} (1 - \mathrm{e}^{\gamma(x+y-z)})\nu(\mathrm{d}y)\xi(\mathrm{d}z)\,\mathrm{d}x \end{aligned}$$

with

$$C_\gamma = \mathrm{E}\,\mathrm{e}^{\gamma A_s}.$$

Proof. The proof is based on the observation that

$$l^K = \frac{\mathrm{e}^{\gamma K}}{\mathrm{e}^{\gamma K} - C_\gamma} I_1 + \frac{1}{\mathrm{e}^{\gamma K} - C_\gamma} I_2 + \frac{\mathrm{e}^{\gamma K}\gamma^{-1}}{\mathrm{e}^{\gamma K} - C_\gamma} I_3 + \frac{\gamma^{-1}}{\mathrm{e}^{\gamma K} - C_\gamma} I_4 - \frac{C_\gamma \,\mathrm{E}\, X_1}{\mathrm{e}^{\alpha K} - C_\gamma}, \tag{3.2}$$

where

$$\begin{aligned} I_1 &= \int_0^K \int_{K-x}^\infty (y - K + x)\nu(\mathrm{d}y)\pi_K(\mathrm{d}x), \\ I_2 &= \int_0^a \int_0^K \int_{-\infty}^{-x+z} (x + y - z)\nu(\mathrm{d}y)\pi_K^z(\mathrm{d}x)\xi(\mathrm{d}z), \\ I_3 &= \int_0^K \int_{K-x}^\infty (1 - \mathrm{e}^{\gamma(y-K+x)})\nu(\mathrm{d}y)\pi_K(\mathrm{d}x), \\ I_4 &= \int_0^a \int_0^K \int_{-\infty}^{-x+z} (1 - \mathrm{e}^{\gamma(x+y-z)})\nu(\mathrm{d}y)\pi_K^z(\mathrm{d}x)\xi(\mathrm{d}z). \end{aligned}$$

Indeed, note that from (2.2), taking $\alpha = \gamma$, we obtain

$$0 = \gamma C_\gamma l_c^A - \gamma \mathrm{e}^{\gamma K} l_c^K + C_\gamma \operatorname{E} \sum_{0 \le s \le 1} (1 - \mathrm{e}^{-\gamma \Delta L_s^A}) + \mathrm{e}^{\gamma K} \operatorname{E} \sum_{0 \le s \le 1} (1 - \mathrm{e}^{\gamma \Delta L_s^K}),$$

where we have used the fact that $\operatorname{E} \int_0^1 \mathrm{e}^{\gamma A_s} \, \mathrm{d} L_s^A = C_\gamma l_c^A$. Let $\varepsilon > 0$. We split ΔL_t^K into $\underline{\Delta}^\varepsilon L_t^K$ and $\overline{\Delta}^\varepsilon L_t^K$, corresponding to $\Delta X_s \in [0, \varepsilon]$ and $\Delta X_s \in (\varepsilon, \infty)$, respectively, and we split ΔL_t^A into $\underline{\Delta}^\varepsilon L_t^A$ and $\overline{\Delta}^\varepsilon L_t^A$, corresponding to $\Delta X_s \in [-\varepsilon, 0]$ and $\Delta X_s \in (-\infty, -\varepsilon)$, respectively. Now we have

$$\mathrm{e}^{\gamma K} \operatorname{E} \sum_{0 \le s \le 1} (1 - \mathrm{e}^{\gamma \underline{\Delta}^\varepsilon L_s^K}) = \mathrm{e}^{\gamma K} \bigg(-\gamma (l_j^K - \bar{l}_j^K) - \frac{\gamma^2}{2} \operatorname{E} \sum_{0 \le s \le 1} (\underline{\Delta}^\varepsilon L_s^K)^2 \bigg) + o(\varepsilon^2),$$

$$\operatorname{E} \sum_{0 \le s \le 1} (1 - \mathrm{e}^{-\gamma \underline{\Delta}^\varepsilon L_s^A}) = \gamma (l_j^A - \bar{l}_j^A) - \frac{\gamma^2}{2} \operatorname{E} \sum_{0 \le s \le 1} (\underline{\Delta}^\varepsilon L_s^A)^2 + o(\varepsilon^2).$$

Thus,

$$\begin{aligned} 0 = {} & \gamma C_\gamma l_c^A - \gamma \mathrm{e}^{\gamma K} l_c^K + C_\gamma \operatorname{E} \sum_{0 \le s \le 1} (1 - \mathrm{e}^{-\gamma \overline{\Delta}^\varepsilon L_s^A}) + \mathrm{e}^{\gamma K} \operatorname{E} \sum_{0 \le s \le 1} (1 - \mathrm{e}^{\gamma \overline{\Delta}^\varepsilon L_s^K}) \\ & + \gamma C_\gamma (l_j^A - \bar{l}_j^A) - \frac{\gamma^2}{2} C_\gamma \operatorname{E} \sum_{0 \le s \le 1} (\underline{\Delta}^\varepsilon L_s^A)^2 - \mathrm{e}^{\gamma K} \gamma (l_j^K - \bar{l}_j^K) - \frac{\gamma^2}{2} \operatorname{E} \sum_{0 \le s \le 1} (\underline{\Delta}^\varepsilon L_s^K)^2 \\ & + o(\varepsilon^2). \end{aligned}$$

Using the fact that $l^A = l^K - \operatorname{E} X_1$, we have

$$\begin{aligned} l^K (C_\gamma - \mathrm{e}^{\gamma K}) \gamma = {} & \gamma C_\gamma \operatorname{E} X_1 + \gamma C_\gamma \bar{l}_j^A - \gamma \mathrm{e}^{\gamma K} \bar{l}_j^K - C_\gamma \operatorname{E} \sum_{0 \le s \le 1} (1 - \mathrm{e}^{-\gamma \overline{\Delta}^\varepsilon L_s^A}) \\ & - \mathrm{e}^{\gamma K} \operatorname{E} \sum_{0 \le s \le 1} (1 - \mathrm{e}^{\gamma \overline{\Delta}^\varepsilon L_s^K}) + \frac{\gamma^2}{2} C_\gamma \operatorname{E} \sum_{0 \le s \le 1} (\underline{\Delta}^\varepsilon L_s^A)^2 \\ & + \frac{\gamma^2}{2} \operatorname{E} \sum_{0 \le s \le 1} (\underline{\Delta}^\varepsilon L_s^K)^2 + o(\varepsilon^2). \end{aligned}$$

If we send $\varepsilon \to 0$, we obtain

$$l^K (C_\gamma - \mathrm{e}^{\gamma K}) \gamma = \gamma C_\gamma \operatorname{E} X_1 - \gamma C_\gamma I_2 - \gamma \mathrm{e}^{\gamma K} I_1 - C_\gamma I_4 - \mathrm{e}^{\gamma K} I_3.$$

Now (3.2) follows by dividing by $(C_\gamma - \mathrm{e}^{\gamma K}) \gamma$.

Note that I_1 and I_3 are the same as those in [2, Theorem 3.2], and that I_2 and I_4 have only an additional integral over $\xi(\mathrm{d}z)$, for which we should take $x - z$ instead of x under the integral signs. Thus, using the same arguments as in the proof of [2, Theorem 4.1] completes the proof, once we prove weak convergence (3.1). To prove (3.1), we can use classical renewal arguments applied to the process $\{X_{\tau^A(an)}, \ n \in N\}$ (by considering *ladder height lines* $\hat{A}_t$ starting from an invariant measure shifted by a from the previous position of the ladder process).

Example 3.1. For a stable M/M/1 queue, that is, for $X_t = \sum_{i=1}^{N_t} \sigma_i - t$ with $\{\sigma_i\}_{\{i \ge 1\}}$ being independent and identically exponentially distributed random variables with intensity μ, and

N_t being a Poisson process with intensity $\lambda < \mu$, we have $\nu(\mathrm{d}x) = \mu\lambda \mathrm{e}^{-\mu x}\,\mathrm{d}x$ and $\mathrm{P}^{\gamma}(\tau_{-x}^{-} = \infty) = 1 - \mathrm{e}^{-\gamma x}$, where $\gamma = \mu - \lambda$, since considering X on P^{γ} is equivalent to exchanging the intensities of arrival and service processes. Moreover, choosing the saw-like lower boundary given in (2.1) by the lack of memory of exponential distributions on P^{γ} we have $B^A(\infty) = e_1 - Y$, where Y has uniform distribution $\xi(\mathrm{d}x) = \mathrm{d}x/a$ $(x \in [0, a])$ and e_1 is an exponential random variable with intensity λ. This gives

$$D = \frac{1}{a}(\mathrm{e}^{a(\mu-\lambda)} - 1)\frac{\mu - \lambda}{\mu}\frac{\lambda}{\mu}.$$

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