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ON THE RADICAL THEORY OF ANDRUNAKIEVICH VARIETIES

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To Bernhard Neumann on the occasion of his seventy-fifth birthday

In 1978 Anderson and Gardner investigated semisimple classes and recently Buys and Gerber developed the theory of special radicals in Andrunakievich varieties. In this note we continue the study of radical theory in Andrunakievich varieties. Sharpening some of the results of Anderson and Gardner we prove versions of Sands' Theorem characterizing semisimple classes by regularity, coinductivity and being closed under extensions. In the proof we follow a new method which avoids calculations with defining identities of the variety. We generalize van Leeuwen's Theorem characterizing semisimple classes of hereditary radicals as classes being regular and closed under essential extensions and subdirect sums.

1. Preliminaries

We shall work with not necessarily associative algebras over a commutative and associative ring with identity. As usual, for an algebra A we define $A^{(0)} = A$ and $A^{(k)} = A^{(k-1)} \cdot A^{(k-1)}$ for $k = 1, 2, \ldots$. If $M \triangleleft I \triangleleft A$, then let M^* denote the ideal of A generated by M .

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Following [1] a variety $\underline{\underline{V}}$ of algebras is said to be an *Andrunakievich variety of index* n, if $(M^*/M)^{(n)} = 0$ for all $M \lhd I \lhd A \in \underline{\underline{V}}$, and if n is the smallest such integer. An algebra A is *solvable* if there is a natural number k such that $A^{(k)} = 0$. A variety $\underline{\underline{V}}$ of algebras is called an *s-variety* where s is an integer greater than 1, if $I^s \lhd A$ for every $I \lhd A \in \underline{\underline{V}}$. For examples of such varieties we refer to [1]. More about not necessarily associative rings can be found in [16].

Let us recall that a subclass R of a variety $\underline{\underline{V}}$ of algebras is a *radical class* if R is homomorphically closed, inductive (if $I_1 \subseteq \dots \subseteq I_\alpha \subseteq \dots$ is an ascending chain of ideals of $A \in \underline{\underline{V}}$ each of which is in R, then also $\cup I_\alpha \in \mathsf{R}$) and is closed under extensions ($I, A/I \in \mathsf{R}$ imply $A \in \mathsf{R}$). The radical $\mathsf{R}(A)$ of an algebra A is defined as $\mathsf{R}(A) = \sum (I \lhd A : I \in \mathsf{R})$ and $\mathsf{R}(A) \in \mathsf{R}$ always holds. A subclass $\mathsf{C} \subseteq \underline{\underline{V}}$ is said to be *regular* if every nonzero ideal of an algebra $A \in \mathsf{C}$ has a nonzero homomorphic image in C. A class C is *hereditary* if $I \lhd A \in \mathsf{C}$ implies $I \in \mathsf{C}$. Obviously every hereditary class is regular, but not conversely. For a radical class R its semisimple class is defined as the class

$$\mathcal{S}\mathsf{R} = \{A \in \underline{\underline{V}} : \mathsf{R}(A) = 0\} .$$

A semisimple class C is always *closed under subdirect sums* (that is, if $A = \sum_{\text{subdirect}} (A_\alpha : A_\alpha \in \mathsf{C})$, then $A \in \mathsf{C}$) and hence *coinductive* (that is if $I_1 \supseteq \dots \supseteq I_\alpha \supseteq \dots$ is a descending chain of ideals of an algebra $A \in \underline{\underline{V}}$ such that $A/I_\alpha \in \mathsf{C}$ for all α, then $A/\cap I_\alpha \in \mathsf{C}$). It is well-known that semisimple classes are closed under extensions. If C is a regular class, then the class

$$\mathcal{U}\mathsf{C} = \{A \in \underline{\underline{V}} : A \text{ has no nonzero homomorphic image in } \mathsf{C}\}$$

is a radical class. Moreover, for every radical class R and semisimple class C,

$$\mathcal{U}\mathcal{S}\mathsf{R} = \mathsf{R} \quad \text{and} \quad \mathcal{S}\mathcal{U}\mathsf{C} = \mathsf{C}$$

are valid. For details of the general theory of radicals we refer to [15].

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In this section, and in Section 2, S *will always denote a regular and coinductive class which is closed under extensions.*

PROPOSITION 1. *If* $I_n \triangleleft \ldots \triangleleft I_1 \triangleleft I_0 = A$, *and* $I_n \triangleleft A$, *and* $I_k/I_{k+1} \in S$ *for each* $k = 0, 1, \ldots, n-1$, *then* $A/I_n \in S$.

Proof. Since

$$\frac{I_{n-2}/I_n}{I_{n-1}/I_n} \cong I_{n-2}/I_{n-1} \in S$$

and also $I_{n-1}/I_n \in S$, it follows that $I_{n-2}/I_n \in S$ as S is closed under extensions. Iterating this process in $n-1$ steps we arrive at $I_0/I_n \in S$.

PROPOSITION 2. *If* $I_n \triangleleft \ldots \triangleleft I_1 \triangleleft I_0 = A$, *and* $I_k/I_{k+1} \in S$ *for each* $k = 0, 1, \ldots, n$, *then for the radical* $R = \mathcal{U}S$, $R(A) \subseteq R(I_n)$ *holds.*

Proof. For $n = 0$ the assertion is trivial. Suppose that the assertion is valid for $n-1 \geq 0$, that is, $R(A) \subseteq R(I_{n-1})$. Then we have

$$\frac{R(I_{n-1})}{R(I_{n-1}) \cap I_n} \cong \frac{R(I_{n-1})+I_n}{I_n} \triangleleft I_{n-1}/I_n \in S .$$

If $R(I_{n-1})/(R(I_{n-1}) \cap I) \neq 0$ then by the regularity of S it has a non-zero homomorphic image in $S \cap R = 0$ which is impossible.

Hence $R(I_{n-1}) \subseteq I_n$ and also $R(I_{n-1}) \subseteq R(I_n)$ hold. By the induction hypothesis we get $R(A) \subseteq R(I_{n-1}) \subseteq R(I_n)$.

PROPOSITION 3. *If* $S\mathcal{U}S$ *is hereditary and* $A \in S\mathcal{U}S$, *and* $A^k = 0$, *then* $A \in S$.

Proof. By induction. Let $k = 2$ and let us choose an ideal I of A and an ideal M of I being minimal with respect to $A/I \in S$ and $I/M \in S$. Since S is regular and coinductive, such an I and M do exist. Moreover, as $A^2 = 0$, we have $M \triangleleft A$. Therefore

$$\frac{A/M}{I/M} \cong A/I \in S$$

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and $I/M \in \mathrm{S}$ imply $A/M \in \mathrm{S}$, because S is closed under extensions. By the minimality of I we get $I = M$, and hence I has no nonzero homomorphic image in S . Since $I \lhd A \in \mathcal{S}\mathcal{U}\mathrm{S}$, by the definition of $\mathcal{S}\mathcal{U}\mathrm{S}$ we conclude that $I = 0$. Thus $A \cong A/0 \in \mathrm{S}$ has been proved.

Next let us assume the assertion for $k-1 \geq 2$ and let $A \in \mathcal{S}\mathcal{U}\mathrm{S}$ be an algebra such that $A^k = 0$. Again, let us choose I and M as above. Further, let us consider

$$\operatorname{ann} A = \{x \in A : xA = Ax = 0\} .$$

Obviously $\operatorname{ann} A$ and also all subalgebras of $\operatorname{ann} A$ are ideals of A . Setting

$$A' = \frac{A}{M \cap \operatorname{ann} A} , \quad I' = \frac{I}{M \cap \operatorname{ann} A} , \quad M' = \frac{M}{M \cap \operatorname{ann} A}$$

we have

$$A'/I' \in \mathrm{S} \quad \text{and} \quad I'/M' \in \mathrm{S} .$$

Hence Proposition 2 is applicable to the chain

$$M' \lhd I' \lhd A'$$

and the radical $\mathrm{R} = \mathcal{U}\mathrm{S}$ yielding

$$\mathrm{R}(A') \subseteq \mathrm{R}(M') .$$

On the other hand we have

$$\frac{\mathrm{R}(M')}{\mathrm{R}(A')} \lhd \frac{M'}{\mathrm{R}(A')} \lhd \frac{I'}{\mathrm{R}(A')} \lhd \frac{A'}{\mathrm{R}(A')} \in \mathcal{S}\mathcal{U}\mathrm{S} .$$

Since $\mathcal{S}\mathcal{U}\mathrm{S}$ is hereditary, it follows that

$$\mathrm{R}(M')/\mathrm{R}(A') \in \mathrm{R} \cap \mathcal{S}\mathcal{U}\mathrm{S} = 0 ,$$

that is $\mathrm{R}(M') = \mathrm{R}(A')$.

Recalling that

$$A^k = A \cdot A^{k-1} + A^2 \cdot A^{k-2} + \ldots + A^{k-2} \cdot A^2 + A^{k-1} \cdot A ,$$

it follows from $A^k = 0$ that $M^{k-1} \subseteq A^{k-1} \subseteq \operatorname{ann} A$. Hence we have $(M')^{k-1} = 0$ and also

$$\left(M'/\mathrm{R}(A')\right)^{k-1} = \left(M'/\mathrm{R}(M')\right)^{k-1} = 0 .$$

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As $M'/\mathrm{R}(M') \in \mathrm{SUS}$, the induction hypothesis yields $M'/\mathrm{R}(A') \in \mathrm{S}$. Applying Proposition 1 for the chain

$$0 = \frac{\mathrm{R}(A')}{\mathrm{R}(A')} \lhd \frac{M'}{\mathrm{R}(A')} \lhd \frac{I'}{\mathrm{R}(A')} \lhd \frac{A'}{\mathrm{R}(A')}$$

we get $A'/\mathrm{R}(A') \in \mathrm{S}$. Putting $L/(M \cap \operatorname{ann} A) = \mathrm{R}(A')$ we have

$$A/L \cong \frac{A/(M\cap \operatorname{ann}A)}{L/(M\cap \operatorname{ann}A)} = \frac{A'}{\mathrm{R}(A')} \in \mathrm{S} ,$$

and so the minimality of I yields $I \subseteq L$. Hence

$$\frac{I}{M\cap \operatorname{ann}A} \subseteq \frac{L}{M\cap \operatorname{ann}A} = \mathrm{R}(A') \subseteq \mathrm{R}(M') \subseteq M' = \frac{M}{M\cap \operatorname{ann}A} \subseteq \frac{I}{M\cap \operatorname{ann}A}$$

holds implying $I = M$. As at the end of the proof of the case $k = 2$ we conclude $A \in \mathrm{S}$.

For an algebra A let us define

$$A^{(s,0)} = A \quad \text{and} \quad A^{(s,k)} = \left(A^{(s,k-1)}\right)^s$$

for $k = 1, 2, \ldots$.

PROPOSITION 4. *$A^{(s,k)} \subseteq A^{(k)}$ for $k = 0, 1, 2, \ldots$.*

Proof. For $k = 0$ we have $A^{(s,0)} = A = A^{(0)}$. Assuming $A^{(s,k-1)} \subseteq A^{(k-1)}$ for $k > 0$, we have

$$\begin{aligned} A^{(s,k)} = \left(A^{(s,k-1)}\right)^s &= \sum_{i=1}^{s-1} \left(A^{(s,k-1)}\right)^i \left(A^{(s,k-1)}\right)^{s-i} \\ &\subseteq A^{(s,k-1)} \cdot A^{(s,k-1)} \subseteq A^{(k-1)} \cdot A^{(k-1)} = A^{(k)} . \end{aligned}$$

Let us observe that so far everything is valid in any universal class of non-associative algebras.

PROPOSITION 5. *If $\underline{\underline{V}}$ is an s-variety and $H \lhd A$, then $H^{(s,k)} \lhd A$.*

Proof. For $k = 0$, $H^{(s,0)} = H \lhd A$ holds. Supposing the assertion for $k - 1$ we get

$$H^{(s,k)} = \left(H^{(s,k-1)}\right)^s \lhd A .$$

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2. Semisimple classes

In the sequel $\underline{\underline{V}}$ will always denote a variety which is an s-variety and an Andrunakievich variety of index n, and we shall work in a universal class $\underline{\underline{A}} \subseteq \underline{\underline{V}}$. Further S will denote a subclass which is regular, coinductive and closed under extensions.

To be able to prove Theorem 1 and so give a characterization of semi-simple classes, we need to impose the following condition

(*) *to every* $A \in \underline{\underline{A}}$ *there exists a natural number* k *such that* $A^k \subseteq A^{(s,n)}$.

Condition (*) is always satisfied if the rings are associative or if $n = 1$, because then, for $k = s$, $A^k = A^s = A^{(s,1)}$ holds. An example for $\underline{\underline{A}}$ with $n = 1$ and $s = 3$ is the variety of 4-permutable algebras ([1], Propositions 2.2 and 2.3).

THEOREM 1. *Let* $\underline{\underline{A}}$ *satisfy condition* (*). *If* S *is a subclass of* $\underline{\underline{A}}$ *such that* S *is regular, coinductive and closed under extensions, and the semisimple class* $\mathcal{SUS}$ *is hereditary, then* $\mathsf{S} = \mathcal{SUS}$.

Proof. We have to prove the inclusion $\mathcal{SUS} \subseteq \mathsf{S}$ only, as the opposite inclusion is trivially fulfilled. Let us take an algebra $A \in \mathcal{SUS}$ and choose an ideal I of A and an ideal M of I such that they are minimal relative to $A/I \in \mathsf{S}$ and $I/M \in \mathsf{S}$. By the definition of $\mathcal{SUS}$ and the coinductivity of S this is possible. Further, let H denote the ideal of A generated by M. By condition (*) there exists a natural number k such that $M^k \subseteq M^{(s,n)}$ and now Proposition 4 yields

$$M^k \subseteq M^{(s,n)} \subseteq H^{(s,n)} \subseteq H^{(n)} \subseteq M .$$

Further, by Proposition 5, we have $H^{(s,n)} \triangleleft A$. Setting $K = H^{(s,n)}$, $A' = A/K$, $I' = I/K$ and $M' = M/K$ we have $A'/I' \in \mathsf{S}$ and $I'/M' \in \mathsf{S}$. Thus Proposition 2 is applicable for the chain

$$M' \triangleleft I' \triangleleft A'$$

and for the radical $\mathsf{R} = \mathcal{US}$ yielding $\mathsf{R}(A') \subseteq \mathsf{R}(M')$. Furthermore, by $M \subseteq H$, we have

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$$\left(\frac{M'}{\mathrm{R}(A')}\right)^{k} \subseteq \left(\frac{M'}{\mathrm{R}(A')}\right)^{(s,n)} \subseteq \left(\frac{H/K}{\mathrm{R}(A')}\right)^{(s,n)} = \left(\frac{H/H^{(s,n)}}{\mathrm{R}(A')}\right)^{(s,n)} = 0 \; .$$

Since

$$\frac{M'}{\mathrm{R}(A')} \lhd \frac{I'}{\mathrm{R}(A')} \lhd \frac{A'}{\mathrm{R}(A')} \in \mathcal{SUS}$$

and $\mathcal{SUS}$ is hereditary, we have $M'/\mathrm{R}(A') \in \mathcal{SUS}$. Applying Proposition 3 we get $M'/\mathrm{R}(A') \in \mathrm{S}$. Let us consider the chain

$$0 \lhd \frac{M'}{\mathrm{R}(A')} \lhd \frac{I'}{\mathrm{R}(A')} \lhd \frac{A'}{\mathrm{R}(A')} \; ,$$

and apply Proposition 1. Thus we obtain

$$\frac{A/K}{\mathrm{R}(A')} = \frac{A'}{\mathrm{R}(A')} \in \mathrm{S} \; .$$

Putting $L/K = \mathrm{R}(A')$, we have

$$A/L \cong \frac{A/K}{L/K} = \frac{A'}{\mathrm{R}(A')} \in \mathrm{S} \; ,$$

and now the choice of I yields $I \subseteq L$. Since $\mathrm{R}(A') \subseteq \mathrm{R}(M')$, we have

$$I/K \subseteq L/K = \mathrm{R}(A') \subseteq \mathrm{R}(M') \subseteq M' = M/K \subseteq I/K \; ,$$

and therefore $I = M$ holds. Thus I has no nonzero homomorphic image in S , that is, $I \in \mathcal{US}$. Since $I \lhd A \in \mathcal{SUS}$, the hereditariness of $\mathcal{SUS}$ implies $I \in \mathcal{US} \cap \mathcal{SUS} = 0$. Hence $A = A/0 \in \mathrm{S}$ has been proved, and therefore $\mathcal{SUS} \subseteq \mathrm{S}$.

REMARK. In proving characterizations of semisimple classes the usual method is to define mappings and to check their kernels (see for instance [*2*], [*3*], [*4*], [*8*], [*9*], [*11*], [*12*], [*13*], [*14*]). This method fails if the defining identities of the considered variety are too involved, or if the variety is given by other properties (as in the case of Andrunakievich and S-varieties). The proof of Theorem 1 provides an alternative approach for characterizing semisimple classes and avoids calculations involving the defining identities of the variety. Though this method is more complicated (even in the associative case) than the traditional one, it could be applied - with necessary modifications in similar varieties, too.

We say that a radical class is *hypersolvable*, if it contains all solvable algebras of $\underline{\underline{A}}$. A radical class consisting of idempotent

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algebras, is called *subidempotent*. Anderson and Gardner have proved that in an Andrunakievich variety the semisimple class of a hypersolvable radical class is always hereditary ([1], Theorem 3.2). If the variety is also an s-variety, then the semisimple class of any subidempotent radical is hereditary ([1], Theorem 3.7). If the variety consists of algebras over a field, then every radical class is either hypersolvable or subidempotent ([1], Theorem 3.9).

In the proof of Theorem 1 condition (*) enabled us to apply Proposition 3. Nevertheless in that proof instead of Proposition 3 we could have used condition

(**) *if* $B \in \mathcal{SUS}$ *and* $B^{(s,n)} = 0$, *then* $B \in \mathsf{S}$.

PROPOSITION 6. *If* $\mathsf{R} = \mathcal{US}$ *is either hypersolvable or subidempotent, then condition* (**) *is satisfied.*

Proof. Let B be an algebra such that $B \in \mathcal{SUS}$ and $B^{(s,n)} = 0$. Recall that by [1], Theorems 3.2 and 3.7, the semisimple class $\mathcal{SUS}$ is hereditary. If R is hypersolvable, then we have

$$\left(B^{(s,n-1)}\right)^{(s)} \subseteq \left(B^{(s,n-1)}\right)^{s} = B^{(s,n)} = 0$$

and hence $B^{(s,n-1)} \in \mathsf{R}$. As $B^{(s,n-1)} \lhd B \in \mathcal{SUS}$ the hereditariness of $\mathcal{SUS}$ implies $B^{(s,n-1)} \in \mathcal{SUS}$. Thus $B^{(s,n-1)} \in \mathsf{R} \cap \mathcal{SUS} = 0$. Repeating this reasoning in n steps we arrive at $B = 0 \in \mathsf{S}$. Next let R be subidempotent. Now $\mathcal{SUS}$ contains all nilpotent algebras, in particular $B^{(s,k)}/B^{(s,k+1)} \in \mathcal{SUS}$ for every $k = 0, 1, \ldots, n-1$, as $\left(B^{(s,k)}/B^{(s,k+1)}\right)^{s} = 0$. Moreover, Proposition 3 yields $B^{(s,k)}/B^{(s,k+1)} \in \mathsf{S}$. Applying Proposition 1 to the chain

$$0 = B^{(s,n)} \lhd B^{(s,n-1)} \lhd \ldots \lhd B^{(s,1)} \lhd B^{(s,0)} = B$$

we get $B \in \mathsf{S}$.

Thus by Theorem 1, Proposition 6 and [1], Theorems 3.2, 3.7 and 3.9, the following versions of Sands' Theorem are valid (without imposing condition (*) on $\underline{\underline{A}}$).

COROLLARY 1. *A subclass* S *of* $\underline{\underline{A}}$ *is the semisimple class of a hypersolvable radical class if and only if* S *is regular, coinductive,*

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closed under extensions, and does not contain an algebra $A \neq 0$ *such that* $A^2 = 0$.

COROLLARY 2. *A subclass* S *of* $\underline{\underline{A}}$ *is the semisimple class of a subidempotent radical class if and only if* S *is regular, coinductive, closed under extensions and contains all algebras* $A \in \underline{\underline{A}}$ *such that* $A^2 = 0$.

Proof. We still have to prove that $\mathsf{R} = \mathcal{U}\mathsf{S}$ is subidempotent if and only if S contains all algebras A such that $A^2 = 0$. If R is subidempotent, and A is such that $A^2 = 0$, then $\mathsf{R}(A) = 0$, that is $A \in \mathcal{SUS}$. Now Proposition 3 yields $A \in \mathsf{S}$. If R is not subidempotent, then there exists an algebra $A \in \mathsf{R}$ such that $B = A/A^2 \neq 0$. Now $B^2 = 0$, but $B \notin \mathsf{S}$ as $B \in \mathsf{R}$.

COROLLARY 3. *Assume that* $\underline{\underline{A}}$ *consists of algebras over a field. A subclass* S *of* $\underline{\underline{A}}$ *is a semisimple class if and only if* S *is regular, coinductive and closed under extensions.*

Let us notice that these corollaries sharpen the corresponding results of Anderson and Gardner ([*1*], Corollaries 3.3 and 3.8 and Theorem 3.9) inasmuch as for hereditariness only regularity and for being subdirectly closed only coinductivity is required.

3. Essentially closed classes

We can get rid of condition (*) and prove the validity of condition (**) if we impose stronger but natural conditions on the class S . For this end let us remind the reader that an ideal I of an algebra A is called an *essential ideal* of A if $I \cap K \neq 0$ holds for every $0 \neq K \triangleleft A$. This fact will be denoted by $I \triangleleft \cdot A$. A class $\mathsf{C} \subseteq \underline{\underline{V}}$ is said to be *closed under essential extensions* (or *essentially closed*), if $I \triangleleft \cdot A$ and $I \in \mathsf{C}$ imply $A \in \mathsf{C}$.

In this section $\underline{\underline{V}}$ will again stand for an s-variety which is also an Andrunakievich variety of index n . Further S *will always denote a subclass of* $\underline{\underline{V}}$ *which is regular and closed under essential extensions and subdirect sums.* Let us mention that for every hereditary radical R of

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non-associative rings the semisimple class $\mathcal{SR}$ is closed under essential extensions, so the conditions imposed on $\mathcal{S}$ are rather natural.

PROPOSITION 7. *If* $\mathcal{SUS}$ *is hereditary,* $A \in \mathcal{SUS}$ *and* $A^{(k)} = 0$ *then* $A \in \mathcal{S}$.

Proof. If $k = 1$, then $A^2 = A^{(1)} = 0$ then Proposition 3 yields $A \in \mathcal{S}$ as, by [7], $\mathcal{S}$ is closed under extensions. Next let $k > 1$ and assume the assertion for every $l < k$. We have $A^{(1)} = A^2 \triangleleft A$. As is well-known, if B is an ideal of A such that B is maximal relative to $A^{(1)} \cap B = 0$, then

$$A^{(1)} \cong \left(A^{(1)}+B\right)/B \triangleleft \cdot A/B .$$

We claim that also $A^{(1)} + B \triangleleft \cdot A$. Let $I \neq 0$ be any ideal of A . If $I \subseteq B$, then

$$0 \neq I = \left(A^{(1)}+B\right) \cap I .$$

Suppose that $I \nsubseteq B$. Then $B \neq I + B$ and so by the choice of B we have $A^{(1)} \cap (B+I) \neq 0$. Hence there exist elements $a \in A^{(1)}$, $b \in B$, $i \in I$ such that $a = b + i$ and $i \neq 0$. Consequently

$$0 \neq i = a - b \in A^{(1)} + B$$

holds proving

$$\left(A^{(1)}+B\right) \cap I \neq 0 .$$

Thus $A^{(1)} + B \triangleleft \cdot A$. Further, since $\mathcal{SUS}$ is hereditary, by $A^{(1)} \triangleleft A \in \mathcal{SUS}$ we get $A^{(1)} \in \mathcal{SUS}$. Using the induction hypothesis, $\left(A^{(1)}\right)^{(k-1)} = A^{(k)} = 0$ implies $A^{(1)} \in \mathcal{S}$. As $B^{(1)} \triangleleft B \triangleleft \mathcal{SUS}$ and $\mathcal{SUS}$ is hereditary, it follows $B^{(1)} \in \mathcal{SUS}$. Moreover, $B^{(1)} \subseteq A^{(1)} \cap B = 0$ also holds, hence by the hypothesis we get $B \in \mathcal{S}$. Taking into account that $A^{(1)} + B$ is a direct sum and $\mathcal{S}$ is closed under subdirect sums, we get $A^{(1)} + B \in \mathcal{S}$. As $\mathcal{S}$ is essentially closed, $A^{(1)} + B \triangleleft \cdot A$ yields $A \in \mathcal{S}$.

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Let us recall that S is closed under extensions (*cf.* [7]). Using Propositions 4 and 7 a proof similar to that of Theorem 1 yields the following.

THEOREM 2. *If* S *is regular and closed under essential extensions and subdirect sums and* $\mathcal{SUS}$ *is hereditary, then* $\mathsf{S} = \mathcal{SUS}$.

The proofs of [1], Theorems 3.2 and 3.7, give also the following.

PROPOSITION 8. *Let* R *be a hypersolvable or subidempotent radical. If* $I \lhd A \in \underline{\underline{V}}$ *, then also* $\mathsf{R}(I) \lhd A$.

PROPOSITION 9. *Let* R *be a hypersolvable or subidempotent radical class.* R *is hereditary if and only if the semisimple class* $\mathcal{S}\mathsf{R}$ *is closed under essential extensions.*

In view of Proposition 8 the standard "associative" proof works (see for instance [15], Theorem 15.2).

A hereditary hypersolvable radical class will be called *supersolvable*. By Theorem 2 and Proposition 9 we get the following versions of van Leeuwen's Theorem characterizing semisimple classes of hereditary radicals.

COROLLARY 4. S *is the semisimple class of a supersolvable radical if and only if* S *is regular, closed under essential extensions and subdirect sums, and does not contain an algebra* $A \neq 0$ *such that* $A^2 = 0$.

S *is the semisimple class of a hereditary subidempotent radical if and only if* S *is regular, closed under essential extensions and subdirect sums, and contains all algebras* A *such that* $A^2 = 0$.

Let $\underline{\underline{V}}$ *consist of algebras over a field.* S *is the semisimple class of a hereditary radical if and only if* S *is regular, and closed under essential extensions and subdirect sums.*

In [5], Buys and Gerber have developed the theory of special radicals in Andrunakievich varieties of Ω-groups, in particular, of not necessarily associative algebras. In view of the present results we can add to this theory a characterization of special radicals and of semisimple classes of special radicals, as has been done in the associative and alternative case in [6] and [10].

Let us recall that an algebra A is said to be *prime* if $IK = 0$

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implies $I = 0$ or $K = 0$ for any ideals I and K of A. A subclass $\mathbf{M}$ of $\underline{\underline{V}}$ is a *special class*, if $\mathbf{M}$ consists of prime rings, is hereditary and for any prime algebra B the relations $A \lhd B$ and $A \in \mathbf{M}$ imply that also the algebra B is in $\mathbf{M}$. A radical class $\mathbf{R}$ is called a *special radical*, if $\mathbf{R}$ is the upper radical $\mathbf{R} = \mathcal{U}\mathbf{M}$ of a special class $\mathbf{M}$. Let $\mathbf{P}$ denote the class of all prime algebras of $\underline{\underline{V}}$. Now by using Corollary 4 one can prove the following similarly to [*10*], Theorem 14 and to [*6*], Theorem 1.

COROLLARY 5. *A class* $\mathbf{S}$ *is the semisimple class of a special radical if and only if* $\mathbf{S}$ *is regular, closed under essential extensions and subdirect sums, and satisfies condition*

(S) *every algebra* A *of* $\mathbf{S}$ *is a subdirect sum of prime algebras of* $\mathbf{S}$.

A class $\mathbf{R}$ *is a special radical if and only if* $\mathbf{R}$ *is homomorphically closed, hereditary and satisfies condition*

(R) *if every nonzero homomorphic image* B *of an algebra* A *such that* $B \in \mathbf{P}$, *has a nonzero ideal in* $\mathbf{R}$, *then also* $A \in \mathbf{R}$.

References

[*1*] T. Anderson and B.J. Gardner, "Semi-simple classes in a variety satisfying an Andrunakievich lemma", *Bull. Austral. Math. Soc.* 18 (1978), 187-200.

[*2*] T. Anderson and R. Wiegandt, "Semisimple classes of alternative rings", *Proc. Edinburgh Math. Soc.* 25 (1982), 21-26.

[*3*] P.N. Ánh, "On semisimple classes of topological rings", *Ann. Univ. Sci. Budapest* 20 (1977), 59-70.

[*4*] P.N. Ánh and R. Wiegandt, "Semisimple classes of Jordan algebras", preprint.

[*5*] A. Buys and G.K. Gerber, "Special classes in Ω-groups", *Ann. Univ. Sci. Budapest* (to appear).

https://doi.org/10.1017/S0004972700004731 Published online by Cambridge University Press

[6] B.J. Gardner and R. Wiegandt, "Characterizing and constructing special radicals", *Acta Math. Acad. Sci. Hungar.* **40** (1982), 73-83.

[7] L.C.A. van Leeuwen, "Properties of semisimple classes", *J. Natur. Sci. Math.* **15** (1975), 59-67.

[8] L.C.A. van Leeuwen, C. Roos and R. Wiegandt, "Characterizations of semisimple classes", *J. Austral. Math. Soc. Ser. A* **23** (1977), 172-182.

[9] E.R. Puczyłowski, "On semisimple classes of associative and alternative rings", *Proc. Edinburgh Math. Soc.* **27** (1984), 1-5.

[10] Ju.M. Rjabuhin and R. Wiegandt, "On special radicals, supernilpotent radicals and weakly homomorphically closed classes", *J. Austral. Math. Soc. Ser. A* **31** (1981), 152-162.

[11] A.D. Sands, "Strong upper radicals", *Quart. J. Math. Oxford Ser.* (2) **27** (1976), 21-24.

[12] A.D. Sands, "A characterization of semisimple classes", *Proc. Edinburgh Math. Soc.* **24** (1981), 5-7.

[13] B. Terlikowska-Osłowska, "Category with self-dual set of axioms", *Bull. Acad. Polon. Sci.* **25** (1977), 1207-1214.

[14] B. Terlikowska-Osłowska, "Radical and semisimple classes of objects in categories with a self-dual set of axioms", *Bull. Acad. Polon. Sci.* **26** (1978), 7-13.

[15] R. Wiegandt, *Radical and semisimple classes of rings* (Queen's Papers in Pure and Applied Mathematics, 37. Queen's University, Kingston, Ontario, 1974).

[16] K.A. Zhevlakov, A.M. Slin'ko, I.P. Shestakov and A.I. Shirshov, *Rings that are nearly associative* (Academic Press, New York, London, 1982).

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