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Mixed perverse sheaves for schemes over number fields

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Abstract. This paper generalizes the definition of mixed perverse sheaves to schemes of finite type over a number field. Their basic properties, e.g., characterization of simple objects, are shown.

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Introduction

The aim of this note is to present a theory of mixed perverse sheaves for varieties over number fields. The great model is of course the theory for varieties over finite fields developed by Beilinson, Bernstein and Deligne in [BBD]. At the time that paper was written there was not yet a published construction of a 'derived' category of l-adic sheaves, so they could only treat the more special case of finite or algebraically closed base fields. Meanwhile, Ekedahl's construction [E] has filled this gap.

As he points out his formalism allows easily the application of the methods of [BBD] over any field. In doing this we would indeed get a perverse t-structure on the 'derived' category of l-adic sheaves for varieties over an arbitrary field. However, this obvious approach does not allow to define a well-behaved notion of weights. This was the point I was interested in most. In fact, the conjectures on motivic sheaves ask for mixed sheaves i.e. ones allowing a formalism of weights. The corresponding theory in the Hodge setting are Saito's Hodge modules.

It turns out a different approach solves the problem without major difficulty. The main idea is to make sure that during the whole construction all l-adic complexes live not only on a variety over $\mathbb{Q}$ but extend to some model. The precise definition is given in 1.2. We use the arguments of [BBD] and the existence of Ekedahl's categories for one dimensional base schemes.

We obtain thus for any scheme of finite type over a number field a new triangulated category of horizontal l-adic sheaves which allows to define

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- a perverse t-structure for the middle perversity (cf. Sect. 2) compatible with the one for finite base field or analytic spaces;
- a notion of purity, weights and mixedness (cf. Sect. 3) drawn from the one for finite base field.

The subcategories of mixed complexes are stable under the six Grothendieck functors. We can control the behaviour of weights under the functors Rf^*, Rf_*, $Rf^!$, $Rf_!$, $R\ \underline{\mathrm{Hom}}$, $\otimes$ in the same way as in the case of finite base fields. Any perverse sheaf F is finite extension of simple perverse sheaves. If F is mixed, then these simple perverse sheaves will be pure. However, it has to be mentioned that our approach still falls short of providing a weight filtration on F.

The morphisms in the new category are given by a direct limit over continuous étale cohomology groups. In the particular case of the base $B = \mathrm{Spec}\ K$ where K is a number field, we have

$$H^i_{\mathrm{hor}}(B, F) = R^i\Gamma(B_{\mathrm{hor}}, F) = \varinjlim H^i(G_S, F) \quad \text{(Lemma 4.3)}.$$

(F a Galois-module unramified almost everywhere; the direct system runs through all finite sets of primes of K.)

1. Derived categories of constructible sheaves

We fix an irreducible regular noetherian 1-dimensional connected scheme U_0. Let $B = \mathrm{Spec}\ K$ be its generic point. Let $\mathcal{U}$ be the category dual to the category of all open subschemes of U_0.

We call a U_0-scheme horizontal if it is flat and of finite type over U_0. If X is a horizontal U_0-scheme, then we will denote by $\mathcal{U}X$ the direct system of schemes $X \times_{U_0} U$ for U in $\mathcal{U}$. The limit of $\mathcal{U}X$ is representable by X_B. In fact there is even an equivalence of categories

$$\begin{gathered}\{\text{schemes of finite type over } B\}\\ \updownarrow \\ \{\text{direct limits of } \mathcal{U}X \text{ for horizontal } X \text{ over } U_0\}.\end{gathered}$$

We will say that $\mathcal{U}X$ has some property **P** if all X_U have **P** for sufficiently small U. The same convention will be used for morphisms. Let l be a prime invertible on U_0.

We consider the category of constructible $\mathbb{Z}_l$-sheaves on a horizontal scheme X ([SGA 5] VI 1.1.1) i.e. objects are projective systems $(F_n)_{n\in\mathbf{N}}$ where

- F_n is an étale $\mathbb{Z}/l^{n+1}$-torsion sheaf;
- the maps $F_n/l^{k+1}F_n \to F_k$ are isomorphisms for all $k \leqslant n$;
- all F_n are finite constructible.

Morphisms of constructible $\mathbb{Z}_l$-sheaves are morphisms of projective systems. A constructible $\mathbb{Z}_l$-sheaf is lisse ('constant tordue' in [SGA 5] VI 1.2.1) if all F_n are locally constant. The category of constructible $\mathbb{Q}_l$-sheaves is obtained by tensoring the morphisms with $\mathbb{Q}$. A lisse $\mathbb{Q}_l$-sheaf is given by a continuous representation of $\pi_1(X)$ on a finite dimensional $\mathbb{Q}_l$-vector space.

LEMMA 1.1 (Jouanolou). *In a projective system as above, F_n is constructible if and only if F_0 is. There is a stratification of X by locally closed subschemes such that F becomes lisse on these subschemes.*

Proof. This is [SGA 5] VI 1.1.1 and [SGA 5] VI Proposition 1.2.6. □

Let $D^b_c(X, \mathbb{Z}_l)$ be the bounded 'derived' category of constructible $\mathbb{Z}_l$-sheaves on X as defined by Ekedahl [E] Theorem 6.3. It has a canonical t-structure whose heart is the category of constructible $\mathbb{Z}_l$-sheaves. By $D^b_c(X, \mathbb{Q}_l)$ we denote the localization of $D^b_c(X, \mathbb{Z}_l)$ at the full subcategory of objects that are annihilated by some power of l. It inherits the t-structure and has as heart the category of constructible $\mathbb{Q}_l$-sheaves.

Remark. In the situation of Section 3, i.e. when X is a horizontal scheme over the ring of integers of a number field, a more accessible construction of the same $D^b_c(X, \mathbb{Z}_l)$ was already given before Ekedahl. It is due to Deligne in [Weil II] 1.1.2 and more extensively in [BBD] 2.2.14–2.2.16.

We will use the same convention as in [BBD] and denote the functors on derived categories by f_*, $\underline{\mathrm{Hom}}$ etc. instead of Rf_*, $R\,\underline{\mathrm{Hom}}$ etc.

DEFINITION 1.2. For a horizontal U_0-scheme X, let $D^b_c(\mathcal{U}X, \mathbb{Z}_l)$ be the direct 2-limit of the categories $D^b_c(X', \mathbb{Z}_l)$ for X' in the category $\mathcal{U}X$. Let $D^b_c(\mathcal{U}X, \mathbb{Q}_l)$ be the category obtained by tensoring all morphism in $D^b_c(\mathcal{U}X, \mathbb{Z}_l)$ by $\mathbb{Q}$.

Objects in $D^b_c(\mathcal{U}X, \mathbb{Z}_l)$ are objects in some $D^b_c(X', \mathbb{Z}_l)$. Morphisms are given by

$$\mathrm{Hom}_{D^b_c(\mathcal{U}X, \mathbb{Z}_l)}(M, N) = \varinjlim_{U \in \mathcal{U}} \mathrm{Hom}_{D^b_c(X_U, \mathbb{Z}_l)}(M|_{X_U}, N|_{X_U}).$$

Hence two objects are identified if they become equal after restriction to a smaller base U.

Remark. As the transition functors are exact, the new category $D^b_c(\mathcal{U}X, \mathbb{Z}_l)$ trivially inherits the structure of a triangulated category and a canonical t-structure whose heart is the 2-limit of the categories of constructible $\mathbb{Z}_l$-sheaves on X_U's. All properties listed in [E] Theorem 6.3 carry over.

There is of course a natural functor

$$\eta^*\colon D^b_c(\mathcal{U}X, \mathbb{Z}_l) \to D^b_c(X_B, \mathbb{Z}_l),$$

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induced by $\eta\colon B \to U_0$. It is in general not an equivalence of categories. This is in fact the whole point of this article.

Counterexample. For a number field K and U_0 its ring of S-integers the morphism

$$\mathrm{Hom}_{D^b_c(\mathcal{U},\mathbb{Q}_l)}(\mathbb{Q}_l, \mathbb{Q}_l(1)[1]) \to \mathrm{Hom}_{D^b_c(B,\mathbb{Q}_l)}(\mathbb{Q}_l, \mathbb{Q}_l(1)[1])$$

is injective but not surjective as worked out in [J2] 6.8.4 (ii). There is an extension of $\mathbb{Q}_l$ by $\mathbb{Q}_l(1)$ over K which does not extend to any ring in $\mathcal{U}$.

PROPOSITION 1.3. *The heart of the canonical t-structure on $D^b_c(\mathcal{U}X, \mathbb{Z}_l)$ is equivalent to the full subcategory of constructible $\mathbb{Z}_l$-sheaves on X_B that extend to some model of X_B. We will call these sheaves horizontal.*

Proof. The equivalence is induced by η^*. If $a\colon X \to U_0$ is the structural morphism, then we can express morphisms of horizontal $\mathbb{Z}_l$-sheaves on X_U by $\mathrm{Hom} = \Gamma R^0 a_* H^0\, \underline{\mathrm{Hom}}$. We can assume that $R^0 a_* H^0\, \underline{\mathrm{Hom}}$ is lisse. The question is reduced to $X = U \in \mathcal{U}$. Here horizontal objects in the heart are representations of $\pi_1(U)$. Morphisms are not changed by passing to smaller $U' \in \mathcal{U}$ or even to B. $\square$

2. Perverse t-structures

A horizontal stratification of a horizontal U_0-scheme X is a stratification by locally closed subschemes which are smooth over some U in $\mathcal{U}$. We always assume that the closure of a stratum is again union of strata.

LEMMA 2.1. (a) *Let X be a U_0-scheme of finite type. Then up to restriction to an appropriate U in $\mathcal{U}$, X admits a horizontal stratification.*

(b) *Let F be a constructible $\mathbb{Z}_l$-sheaf on a horizontal U_0-scheme X. Then up to base change to a smaller U in $\mathcal{U}$, X admits a horizontal stratification* **S** *such that F is lisse on the strata.*

Proof. Without loss of generality we can assume that X is reduced. Let Y be the set of points in X which are smooth over U_0. It is open and non-empty. If $Y_B = X_B$, then $Y_U = X_U$ for sufficiently small U in $\mathcal{U}$ and we have won. Else replace X by $X \backslash Y$ and proceed by noetherian induction.

For part (b) we start off with the stratification (possibly by singular schemes) obtained in Lemma 1.1. Without loss of generality the strata are integral. After base change by some $U \in \mathcal{U}$, we can assume their generic point is over B. Using part (a) we can refine the strata up to change of base to obtain a horizontal stratification. $\square$

A lisse $\mathbb{Z}_l$-sheaf F on a smooth horizontal scheme X is called irreducible if F_0 is irreducible as a representation of $\pi_1(X)$. Note that this notion is stable in $\mathcal{U}X$.

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As the fibres of F_0 are finite, any lisse $\mathbb{Z}_l$-sheaf on X is extension of irreducible ones.

Let X a horizontal U_0-scheme. We consider pairs $(\mathbf{S}, L)$ as follows:

(a) Let $\mathbf{S}$ be a horizontal stratification of X.
(b) Let L be the following data: for each $S \in \mathbf{S}$ a collection of irreducible lisse $\mathbb{Z}_l$-sheaves on S.

A sheaf on X is called $(\mathbf{S}, L)$-constructible if it becomes lisse on the strata in $\mathbf{S}$ and is isomorphic to a finite extensions of objects in $L(S)$ there. We impose the condition

(c) For S in $\mathbf{S}$ and F in $L(S)$, let s be the inclusion of S into X. We assume that all $R^n s_* F$ are $(\mathbf{S}, L)$-constructible.

This is nothing but the imitation of [BBD] 2.2.10 for our situation.

DEFINITION 2.2. For a pair $(\mathbf{S}, L)$ as in (a), (b) and (c), let $D^b_{(\mathbf{S},L)}(X, \mathbb{Z}_l)$ be the full subcategory of those objects in $D^b_c(X, \mathbb{Z}_l)$ whose cohomology sheaves are $(\mathbf{S}, L)$-constructible.

PROPOSITION 2.3. *$D^b_c(\mathcal{U}X, \mathbb{Z}_l)$ is equivalent to the 2-limit of the categories $D^b_{(\mathbf{S},L)}(X_U, \mathbb{Z}_l)$ for varying $X_U \in \mathcal{U}X$ and all pairs $(\mathbf{S}, L)$ satisfying* (a), (b) *and* (c).

Proof. Let C be an object in some $D^b_c(X_U, \mathbb{Z}_l)$. By Lemma 2.1 (b) we can find a pair $(\mathbf{S}, L)$ satisfying (a) and (b) such that all $H^i(C)$ are $(\mathbf{S}, L)$-constructible if we restrict to a smaller base U. All $L(S)$ can be assumed to be finite. As in [BBD] 2.2.10 (basically using the constructibility result [SGA 4 1/2] Th. finitude 1.1) we can refine the stratification to get one that also satisfies (c). We might have to restrict to smaller base in $\mathcal{U}$ in order to get rid of the strata over closed points in U_0 . $\square$

Remark. The constructibility quote is the only place where we use that our base is 1-dimensional. Note, however, that [SGA 4 1/2]. Th. finitude 1.9 would allow to use a more general base scheme U_0 if we wanted.

By the dimension of a horizontal stratum we mean its relative dimension over U_0. Let $\mathbf{S}$ be a horizontal stratification of a horizontal U_0-scheme X. Let

$$\begin{aligned} p\colon \mathbf{S} &\to \mathbb{Z}, \\ S &\mapsto -\dim(S) \end{aligned}$$

be the middle perversity.

LEMMA 2.4. ([BBD] 2.1.2 and 2.1.3). *There is a perverse t-structure on $D^b_{(\mathbf{S},L)}(X, \mathbb{Z}_l)$ where ${}^pD^{b,\leqslant 0}_{(\mathbf{S},L)}$* (resp. ${}^pD^{b,\geqslant 0}_{(\mathbf{S},L)}$) *is the full subcategory of objects*

C such that for each stratum S in **S** *with inclusion $s\colon S \to X$ we have $H^n s^* C = 0$ for $n > p(S)$* (resp. $H^n s^! C = 0$ for $n < p(S)$). *The objects in the heart are called $(\mathbf{S}, L)$-perverse sheaves on X.*

Proof. As in [BBD] 2.1.3 this follows from [BBD] Theorem 1.4.10. □

THEOREM 2.5. ([BBD] 2.2.11). *The definition in the previous lemma induces a t-structure on $D^b_c(\mathcal{U}X, \mathbb{Z}_l)$. The objects in its heart are called perverse $\mathbb{Z}_l$-sheaves on $\mathcal{U}X$.*

Proof. By Proposition 2.3 we have to check the same thing as in [BBD] 2.2.11 i.e. that the t-structures are compatible when passing to the limit over pairs $(\mathbf{S}, L)$. As in loc. cit. the essential ingredient is purity ([SGA 4] XVI 3.7). It allows to check the conditions of [BBD] 2.1.14. Cf. the discussion in loc. cit. 2.1.13. □

Remark. It is because we wanted to quote purity in this proof that we arranged carefully all strata to be smooth over the base U_0.

By tensoring all morphisms with $\mathbb{Q}$, we also get the corresponding theory of $\mathbb{Q}_l$-perverse sheaves. In this case we have an even better characterization.

For $a\colon X \to U_0$ let the duality functor $\mathbb{D}$ on $D^b(X, \mathbb{Q}_l)$ be given by $\underline{\mathrm{Hom}}(\cdot, a^! \mathbb{Q}_l)$ (cf. [E] Thm 6.3 (iii)). If X is smooth of dimension d over U_0 and C a complex whose cohomology is lisse, then

$$H^i \mathbb{D} C = (H^{-i-2d} C)^\vee \otimes \mathbb{Q}_l(d).$$

COROLLARY 2.6. *Let C be an object in $D^b_c(\mathcal{U}X, \mathbb{Q}_l)$ that is represented by an object in $D^b_{(\mathbf{S},L)}(X_U, \mathbb{Q}_l)$. Then*

(a) $C \in {}^pD_c^{\leqslant 0}(\mathcal{U}X, \mathbb{Q}_l)$ iff $s^* H^i(C) = 0$ for all $s \in \mathbf{S}$ and $i > -\dim(S)$.
(b) $C \in {}^pD_c^{\geqslant 0}(\mathcal{U}X, \mathbb{Q}_l)$ iff $s^* H^i(\mathbb{D}C) = 0$ for all $s \in \mathbf{S}$ and $i > -\dim(S)$.

Proof. This is the analogue of the discussion in [BBD] 2.1.16. Recall that in the algebraic setting we have $p^*(S) = -p(S) - 2\dim(S)$. □

Let $X \to Y$ be a morphism of horizontal U_0-schemes, then

$$f_!, f_*\colon D^b_c(\mathcal{U}X, \mathbb{Q}_l) \to D^b_c(\mathcal{U}Y, \mathbb{Q}_l),$$
$$f^!, f^*\colon D^b_c(\mathcal{U}Y, \mathbb{Q}_l) \to D^b_c(\mathcal{U}X, \mathbb{Q}_l)$$

have the exactness properties for the perverse t-structure deduced in [BBD] 4.1. and 4.2. We assemble a number of them for the convenience of the reader:

- If j is an immersion of horizontal schemes, then $j_!$ and j^* are t-right exact; $j^!$ and j_* are t-left exact (loc. cit. 2.1.6).
- If f is affine, then f_* is t-right exact (loc. cit. 4.1.1) and $f_!$ t-left exact (loc. cit. 4.1.2).
- If f is quasi-finite and affine, then f_* and $f_!$ are t-exact (loc. cit. 4.1.3).

- If f is proper and of fibre-dimension $\leqslant d$, then the cohomological amplitude of $f_* = f_!$ is $[-d, d]$ (loc. cit. 4.2.4).
- If f is smooth of relative dimension d, then $f^*[d] = f^![-d](-d)$ is t-exact (loc. cit. 4.2.4).

THEOREM 2.7. ([BBD] 4.3.1]). (i) *The category of perverse sheaves in $D_c^b(\mathcal{U}X, \mathbb{Q}_l)$ is artinian and noetherian; any object is of finite length.*

(ii) *Let $j: V \to X$ be an immersion of horizontal U_0-schemes where V is smooth over U_0 in $\mathcal{U}$. Let L be an irreducible lisse $\mathbb{Q}_l$-sheaf on V. Then the intermediate direct image $j_{!*}(L[\dim(V)])$ is a simple perverse $\mathbb{Q}_l$-sheaf on $\mathcal{U}X$.*

(iii) *All simple perverse $\mathbb{Q}_l$-sheaves are of the type described in* (ii).

Proof. The argument is the same as in [BBD] 4.3.1–4.3.4. Note that the analogues of loc. cit. 2.1.9 and 1.4.25 hold (there are misprints in the references in 4.3.3 and 4.3.4). □

We only get the full formalism between the limit categories. However, we can get partial results for sheaves on horizontal schemes. This can be very useful for explicit applications.

DEFINITION 2.8. Let $f: X \to Y$ be a morphism of horizontal U_0-schemes. Let $(\mathbf{S}, L)$ and $(\mathbf{T}, M)$ be data satisfying (a), (b) and (c) on X and Y respectively. Then f_* is called $(\mathbf{S}, L)$-to-$(\mathbf{T}, M)$-admissible if f_* maps $D^b_{(\mathbf{S},L)}(X, \mathbb{Z}_l)$ to $D^b_{(\mathbf{T},M)}(Y, \mathbb{Z}_l)$.

We define the same notion mutis mutanda with respect to f_*, $f_!$ or $f^!$.

LEMMA 2.9. *f_* is $(\mathbf{S}, L)$-to-$(\mathbf{T}, M)$-admissible if and only if for all $S \in \mathbf{S}$ and $F \in L(S)$ we get $(\mathbf{T}, M)$-constructible sheaves $R^i f_* F$. Similar criteria hold for f_*, $f_!$ or $f^!$.*

Proof. Long exact sequence for cohomology. □

Remark. We have the ‘same’ exactness properties on the level of $D^b_{(\mathbf{S},L)}$ and admissible functors that we have in full generality for the limit categories.

We want to give some examples of geometric situations that give fairly general criteria for admissibility.

PROPOSITION 2.10. *Let f be smooth and proper. Assume $\mathbf{S} = \{X\}$, $\mathbf{T} = \{Y\}$ and L* resp. *M are the categories of all irreducible lisse $\mathbb{Z}_l$-sheaves on X* resp. *Y. Then f_* is admissible.*

Proof. [SGA 4] Exp. XVI Corollary 2.2. □

PROPOSITION 2.11. *Let Y be a smooth U_0-scheme. Let $\overline{f}: \overline{X} \to Y$ be smooth and proper. Let $j: X \to \overline{X}$ be the complement of a strict divisor with normal crossings relative to Y* ([SGA 1] XIII 2.1). *Let $f = \overline{f} \circ j$.*

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Then f_* is $(\mathbf{S}, L)$-to-$(\mathbf{T}, M)$-admissible where $\mathbf{T} = \{Y\}$, M the set of irreducible lisse $\mathbb{Z}_l$-sheaves on Y, $\mathbf{S} = \{X\}$ and L the set of j^*F for lisse and irreducible F on $\overline{X}$.

Proof. Let D be the complement of X in $\overline{X}$. We have $D = \Sigma D_i$ where all D_i and their mutual multiple intersections are smooth and proper over Y. The cohomology objects of j_*F are extensions of sheaves that (a) have support on all of $\overline{X}$ or on one of these subschemes and (b) are lisse on their support. This follows by arguments as in [SGA 4 1/2] Th. finitude App. 1.3.3. Applying [SGA 4] Exp. XVI Corollary 2.2 to these supports we get the result. □

Finally let us remark that we could have replaced $\mathbb{Z}_l$ by the valuation ring V_λ in some finite extension E_λ of $\mathbb{Q}_l$ (resp. $\mathbb{Q}_l$ by E_λ or even $\overline{\mathbb{Q}}_l$) in the whole discussion.

We could also define $\mathbb{Z}/l^n$-perverse sheaves on $\mathcal{U}X$ in the same way. Note however that in this case $D^b_c(\mathcal{U}X, \mathbb{Z}/l^n)$ is equivalent to $D^b_c(X_B, \mathbb{Z}/l^n)$ hence we would get nothing new. On the other hand there is also a perverse t-structure for the middle perversity on $D^b_c(X_B, \mathbb{Z}_l)$. Clearly the functors

$$\eta^*: D^b_c(\mathcal{U}X, \mathbb{Z}_l) \to D^b_c(X_B, \mathbb{Z}_l),$$
$$\overline{\eta}^*: D^b_c(\mathcal{U}X, \mathbb{Z}_l) \to D^b_c(X_{\overline{B}}, \mathbb{Z}_l)$$

are t-exact.

LEMMA 2.12. *The functors η^* and $\overline{\eta}^*$ are faithful on $\mathbb{Q}_l$-perverse sheaves.*

Proof. We only have to check faithfulness for simple perverse sheaves. By 2.7 is suffices to consider lisse sheaves and in this case the assertion is clear. □

For a closed point $u \in U_0$ we have also the base change

$$u^*: D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l) \to D^b_c(X_u, \mathbb{Q}_l)$$

and the right-hand side can also be equipped with the perverse t-structure for the middle perversity.

LEMMA 2.13. *u^* is t-exact for the perverse t-structures.*

Proof. We use the characterization of 2.6 for the perverse t-structures. Note that the U_0-dimension of a stratum S is equal to the dimension of S_u unless the fibre is empty. If s is its inclusion into X, it is enough to check that s^*C and $s^*\mathbb{D}C$ commute with base change to u for $C \in D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$. For s^* this is simply functoriality. We claim that $\mathbb{D}$ also commutes with this base change. First suppose that i is a closed immersion of X into a smooth scheme $\tilde{X}$ of dimension d. Let x be the structural morphism of X. We have

$$\begin{aligned} \mathbb{D}(C) &= \underline{\mathrm{Hom}}(C, x^!\mathbb{Q}_l) \cong \underline{\mathrm{Hom}}(C, i^!\mathbb{Q}_l(d)_{\tilde{X}}[2d]) \\ &\cong i^*\underline{\mathrm{Hom}}(i_*C, \mathbb{Q}_l(d)_{\tilde{X}}[2d]). \end{aligned}$$

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This obviously commutes with base change by u. The general case follows by the Čech-spectral sequence because Zariski-locally X is a closed subscheme of a smooth scheme. □

Of course the reverse functor u_* has no chance of being t-exact for any horizontal stratification on X.

3. Weights

We now restrict to the more special situation of a number field K and $U_0 = \operatorname{Spec} \mathcal{O}_K[1/l]$. Hence the residue fields of the closed points of U_0 are finite. Also we concentrate on working with $\mathbb{Q}_l$-sheaves rather than $\mathbb{Z}_l$-sheaves. The same theory could also be developed for E_λ- (finite extension of $\mathbb{Q}_l$) or $\overline{\mathbb{Q}}_l$-coefficients.

We want to generalize the theory in [BBD] Ch. 5 to our situation.

DEFINITION 3.1. ([BBD] 5.1.5, [Weil II] 1.2.1, 1.2.2). Let F be a $\mathbb{Q}_l$-sheaf on a horizontal U_0-scheme X. It is called pointwise pure of weight $w \in \mathbb{Z}$ if the following holds: For all closed points $x \in X$, with residue field $\mathbb{F}_q$, the operation of the Frobenius automorphism Fr of $\mathbb{F}_q$ on F_x has as $\overline{\mathbb{Q}}_l$-eigenvalues algebraic integers of absolute value $q^{w/2}$ for all embeddings of $\overline{\mathbb{Q}}_l$ into $\mathbb{C}$.

F is called mixed if it admits a filtration whose quotients are pointwise pure of some weight.

Let $D^b_m(\mathcal{U}X, \mathbb{Q}_l)$ be the subcategory of those objects C in $D^b_c(\mathcal{U}X, \mathbb{Q}_l)$ such that all $H^i(C)$ become mixed after base change by appropriate $U \in \mathcal{U}$.

Remark. A $\mathbb{Q}_l$-sheaf F on X can be pulled back to X_u for almost all $u \in U_0$. If F is mixed, then almost all F_u are mixed. The converse seems wrong.

PROPOSITION 3.2. ([BBD] 5.1.6 and 5.1.7). (0) $D^b_m(\mathcal{U}X, \mathbb{Q}_l)$ *is stable under the functors* $f^*, f^!, f_!, f_*, \otimes$ *and* $\underline{\mathrm{Hom}}$.

(i) *The perverse* t*-structure induces a* t*-structure on* $D^b_m(\mathcal{U}X, \mathbb{Q}_l)$.
(ii) *Any subquotient of a mixed perverse* $\mathbb{Q}_l$*-sheaf on* $\mathcal{U}X$ *is mixed.*

Proof. The hard part is (0). This is [Weil II] Corollary 6.1.11. The rest follows easily as in the proof of [BBD] 5.1.7. □

Remark. In the sequel we develop the theory for the limit categories only. We have similar results for categories $D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ with pure $F \in L(S)$ as long as all constructions are carried out using admissible functors only.

DEFINITION 3.3. ([BBD] 5.1.8). An object C in $D^b_m(\mathcal{U}X, \mathbb{Q}_l)$ has weights $\leqslant w$ if the pointwise weights of $H^i(C)$ are $\leqslant w + i$ for all i. C has weights $\geqslant w$ if its Verdier dual $\mathbb{D}C$ has weights $\leqslant -w$. The corresponding categories are denoted $D^b_{\leqslant w}(\mathcal{U}X, \mathbb{Q}_l)$ and $D^b_{\geqslant w}(\mathcal{U}X, \mathbb{Q}_l)$ respectively. C is called pure of weight w if it is both of weights $\geqslant w$ and $\leqslant w$.

Note that a pure $\mathbb{Q}_l$-sheaf is not necessarily pure as a mixed complex. However, if F is lisse and pure of weight w on a smooth horizontal scheme X of dimension d, then the corresponding perverse object $F[d]$ is pure of weight $w + d$ as a mixed complex.

Remark. If C is mixed, then it has weights $\leqslant w$ ($\geqslant w$) if and only if for almost all $u \in U_0$ the pull-backs C_u have weights $\leqslant w$ ($\geqslant w$).

The stability properties of [BBD] 5.1.14 and 5.1.15 (i) hold. In particular $f_!$ and f^* respect $D^b_{\leqslant w}$ and $f^!$ and f_* respect $D^b_{\geqslant w}$ etc.

Warning. The analogue of loc. cit. 5.1.15 (ii) is wrong. I.e. there are non-trivial morphisms in $D^b_m(\mathcal{U}X, \mathbb{Q}_l)$ from C with weights $\leqslant w$ to L with weights $> w$. A counterexample is constructed in [J2] 6.8.4 (i). This will have consequences for the weight filtrations.

PROPOSITION 3.4. ([BBD] 5.3.1). *If F is a perverse sheaf on $\mathcal{U}X$ which is mixed of weights* $\geqslant w$ (resp. $\leqslant w$), *then any subquotient of F is again mixed of weights* $\geqslant w$ (resp. $\leqslant w$).

Proof. By the remark after 3.3 this follows from [BBD] 5.3.1. □

COROLLARY 3.5. ([BBD] 5.3.2). *Let $j: V \to X$ a horizontal affine immersion. If F is a mixed perverse sheaf of weights $\leqslant w$ ($\geqslant w$) on $\mathcal{U}V$, then the perverse sheaf $j_{!*}F$ is again mixed of weights $\leqslant w$ ($\geqslant w$) on $\mathcal{U}X$. In particular if F is pure, then so is $j_{!*}F$.*

Proof. Again this is a consequence of the finite field case [BBD] 5.3.2. □

COROLLARY 3.6. ([BBD] 5.3.4). *A simple mixed perverse sheaf is pure.*

Proof. The proof of [BBD] 5.3.4 can be repeated. The necessary properties of simple horizontal sheaves have been checked. □

Remark. A morphism between mixed perverse sheaves of disjoint weights vanishes.

DEFINITION 3.7. An ascending filtration W_* on a perverse sheaf F is called weight filtration if $Gr^W_i F$ is pure of weight i.

LEMMA 3.8. *A weight filtration is unique if it exists. A morphism between perverse sheaves equipped with a weight filtration is necessarily strict.*

Proof. These are immediate consequences of the last remark. □

Remark. The category of perverse sheaves with weight filtration is abelian. However, it is not closed under extensions in the category of mixed perverse sheaves cf. the above warning.

PROPOSITION 3.9. *Let $f: X \to Y$ be a smooth and proper morphism. Let F be a mixed perverse sheaf on X equipped with a weight filtration. Then all ${}^pH^i(f_*F)$ carry a weight filtration.*

Proof. f_* preserves weights. The long exact sequence for perverse cohomology proves the result. □

4. Morphisms in $D^b(\mathcal{U}X, \mathbb{Z}_l)$

Let us return to the general U_0 of Section 1 for a moment.

LEMMA 4.1. *Let X be a scheme of finite type over some regular scheme of dimension* 0 *or* 1. *Let F be a constructible $\mathbb{Z}_l$-sheaf on X. Then*

$$H^i_{\text{cont}}(X, F) = \mathrm{Hom}_{D^b_c(X,\mathbb{Z}_l)}(\mathbb{Z}_l, F[i]),$$

where the left-hand side is continuous étale cohomology ([J1] Ch. 3).

Proof. Let $\hat{F}$ be the normalization of F in the sense of [E] Section 2. By loc. cit. Proposition 2.7 we compute the right-hand side as

$$\mathrm{Hom}_{D^b(X,\mathbb{Z}_l)}(\mathbb{Z}_l, F[i]) = \mathrm{Hom}_{D^*_{\text{norm}}(\text{et}^{\mathbf{N}}_X - \mathbb{Z}_l)}(\mathbb{Z}_l, \hat{F}[i])$$

with the notations of loc. cit. The latter is a full subcategory of the (true) derived category $D(\text{et}^{\mathbf{N}}_X - \mathbb{Z}_l)$ of the abelian category of projective systems of étale sheaves on X ringed by $(\mathbb{Z}/l^n)_n$. Hence $\mathrm{Hom}_{D(\text{et}^{\mathbf{N}}_X - \mathbb{Z}_l)}(\mathbb{Z}_l, \cdot[i])$ is the ith derived functor of $\varprojlim(\Gamma(X, \cdot\ _n))$. By definition this means

$$\mathrm{Hom}_{D(\text{et}^{\mathbf{N}}_X - \mathbb{Z}_l)}(\mathbb{Z}_l, (\cdot_n)[i]) = H^i_{\text{cont}}(X, \cdot).$$

It remains to show that

$$\mathrm{Hom}_{D(\text{et}^{\mathbf{N}}_X - \mathbb{Z}_l)}(\mathbb{Z}_l, \hat{F}[i]) = \mathrm{Hom}_{D(\text{et}^{\mathbf{N}}_X - \mathbb{Z}_l)}(\mathbb{Z}_l, F[i]).$$

By loc. cit. Proposition 3.4. (iii) the map $H^0(\hat{F}) \to F$ has essentially zero kernel and cokernel. The argument given there also shows that all $H^i(\hat{F})$ for $i \neq 0$ are essentially zero. As our X is noetherian, essentially zero means Mittag–Leffler zero. $H^i_{\text{cont}}(X, \cdot)$ vanishes on those. This completes the proof. □

COROLLARY 4.2. *Let X be a horizontal U_0-scheme. Let F be a constructible $\mathbb{Z}_l$-sheaf on X. Then*

$$\mathrm{Hom}_{D^b_c(\mathcal{U}X,\mathbb{Z}_l)}(\mathbb{Z}_l, F[i]) = \varinjlim_{\mathcal{U}} H^i_{\text{cont}}(X_U, F|_U).$$

Proof. Trivial. □

From now on let again U_0 be as in Section 3 i.e. l-integers of an algebraic number field. We understand the resulting groups even better.

LEMMA 4.3. *Let F be a lisse horizontal sheaf on $U \in \mathcal{U}$. Then*

$$\mathrm{Hom}_{D^b_c(U,\mathbb{Z}_l)}(\mathbb{Z}_l, F[i]) = H^i_{\mathrm{cont}}(G_S, F) = \varprojlim_n H^i_{\mathrm{cont}}(G_S, F_n),$$

$$\mathrm{Hom}_{D^b_c(\mathcal{U},\mathbb{Z}_l)}(\mathbb{Z}_l, F[i]) = \varinjlim_S H^i_{\mathrm{cont}}(G_S, F) = \varinjlim_S \varprojlim_n H^i_{\mathrm{cont}}(G_S, F_n),$$

where S is the set of primes in Spec O_K *which are not in U and G_S is the Galois group of the maximal extension of K unramified outside S.*

Proof. By the previous lemma we have to compute continuous étale cohomology of F. Let K_S be the maximal extension of K which is unramified outside the finite set of places S. Let U_S be the normalization of U in K_S. This is a pro-étale U-scheme. It is Galois with group G_S over U. We apply the Hochschild–Serre spectral sequence for continuous étale cohomology [J1] 3.3

$$E_2^{pq} = H^p_{\mathrm{cont}}\left(G_S, (H^q_{\mathrm{et}}(U_S, \pi^* F_n))_{n\in\mathbf{N}}\right) \Rightarrow H^{p+q}_{\mathrm{cont}}(U, F).$$

By the proof of [M] II 2.9

$$H^q_{\mathrm{et}}(U_S, F_n) = \begin{cases} (F_n)_{\overline{\eta}} & \text{if } q = 0 \\ 0 & \text{else.} \end{cases}$$

The spectral sequence collapses. This proves the first equality.

By [M] II 2.9 we have also

$$H^q_{\mathrm{et}}(U, F_n) = H^q_{\mathrm{cont}}(G_S, (F_n)_{\overline{\eta}}).$$

It is finite by [M] I Corollary 4.15. Hence continuous étale cohomology is given by the projective limit over étale cohomology groups. □

COROLLARY 4.4. *If $l \neq 2$ or if K is purely imaginary, then the cohomological dimension of $D^b_c(\mathcal{U}, \mathbb{Z}_l)$ is* 2. *For all l and K the cohomological dimension of $D^b_c(\mathcal{U}, \mathbb{Q}_l)$ is* 2.

Proof. The first assertion is [M] I 4.10 (c). In the second case we only have to consider free $\mathbb{Z}_l$-modules whose continuous G_S-cohomology is torsion for $q > 2$. □

5. Remarks

As always we can define a cohomology theory using the sheaves in $D^b_c(\mathcal{U}X, \mathbb{Z}_l)$.

DEFINITION 5.1. Let X_B be a B-scheme. For any horizontal $\mathbb{Z}_l$-sheaf F on X_B we can define its horizontal étale cohomology

$$H^i_{\text{hor}}(X_B, F) = \text{Hom}_{D^b_c(\mathcal{U}X,\mathbb{Z}_l)}(\mathbb{Z}_l, F[i]),$$

where X is any horizontal model of X_B over an open part of U_0.

Remark. Note that $H^i_{\text{hor}}(X_B, F)$ is independent of the models we choose for X_B and F.

The most important case is $F = \mathbb{Q}_l(j)$. With the usual methods we get cohomology, cohomology with compact support, homology etc.

Let $a\colon X_B \to B$ and $j\colon X_B \to \overline{X}_B$ a compactification

$$\begin{aligned}
H^i_{\text{hor}}(X_B, j) &= \text{Hom}_{D^b_c(\mathcal{U}X)}(\mathbb{Q}_l, \mathbb{Q}_l(j)[i]) \\
H^i_{\text{hor},c}(X_B, j) &= \text{Hom}_{D^b_c(\mathcal{U}\overline{X})}(\mathbb{Q}_l, j_!\mathbb{Q}_l(j)[i]) \\
&= \text{Hom}_{D^b_c(\mathcal{U})}(\mathbb{Q}_l, a_!\mathbb{Q}_l(j)[i]) \\
H_i(X_B, j) &= \text{Hom}_{D^b_c(\mathcal{U}\overline{X})}(j_!\mathbb{Q}_l(j)[i], \mathbb{Q}_l) \\
&= \text{Hom}_{D^b_c(\mathcal{U})}(a_!\mathbb{Q}_l, \mathbb{Q}_l(-j)[-i]).
\end{aligned}$$

These satisfy all axioms of [G] Definition 1.1 and 1.2 e.g. long exact sequences, Poincaré duality between cohomology and homology. The necessary representation of the cohomology theory by a presheaf can be achieved by the methods of [H] Ch. 9.

By [G] Theorem 2.2 or [H] 18.2.6–7 we get Chern class morphisms

$$c_j\colon \varinjlim K_{-i+2j}(X_U) \to H^i_{\text{hor}}(X_B, j).$$

Jannsen's conjectures [J3] p. 317 and p. 325 imply in our setting:

CONJECTURE 5.2. Let $a\colon X \to U$ be smooth and proper for $U \in \mathcal{U}$. Then

$$H^2_{\text{hor}}(B, R^i a_* \mathbb{Q}_l(n)) = 0 \quad \text{if} \quad \begin{array}{ll} \text{(a)} & i+1 < n, \quad \text{or} \\ \text{(b)} & i+1 > 2n. \end{array}$$

The regulator map c_n induces an isomorphism

$$\varinjlim K_{2n-i-1}(X_U)^{(n)}_{\mathbb{Q}_l} \to H^1_{\text{hor}}(B, R^i a_* \mathbb{Q}_l(n)),$$

for $i + 1 < n$.

Applying the Leray spectral sequence, the conjectures could also be expressed in terms of $H^i_{\text{hor}}(X_B, \mathbb{Q}_l(n))$.

For each embedding of K into $\mathbb{C}$ we get a natural exact forgetful functor from horizontal perverse $\mathbb{Z}_l$-sheaves on X_B to perverse $\mathbb{Z}_l$-sheaves on the classical site of $X_B \times \mathbb{C}$. This allows to compare them to Saito's Hodge modules [S1], [S2]. We can define the abelian category of mixed realization sheaves on X_B in the same way mixed realizations on B itself were defined ([J2], [H]).

DEFINITION 5.3. A mixed realization sheaf on X_B is given by the following data:

(1) for each prime l a horizontal mixed perverse sheaf F_l on X_B equipped with a weight filtration;
(2) for each embedding σ of K into $\mathbb{C}$ an algebraic Hodge module on the analytic space $X_B \otimes_\sigma \mathbb{C}(\mathbb{C})$;
(3) comparison isomorphisms between them in the category of filtered perverse $\mathbb{Q}_l$-sheaves on $X_B \otimes_\sigma \mathbb{C}(\mathbb{C})$.

They can be seen as a mixed realization variant of motivic sheaves on X_B.

Remark. It is a non-trivial question to construct the surrounding triangulated category for these realization sheaves.

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References

[BBD] Beilinson, A. A., Bernstein, J., Deligne, P.: 'Faisceaux pervers', in B. Teissier, J. L. Verdier, 'Analyse et Topologie sur les Espaces singuliers' (I), *Astérisque 100, Soc. Math.* France 1982.
[E] Ekedahl, T.: 'On the Adic Formalism', in P. Cartier et al. (eds.), 'The Grothendieck Festschrift', Volume II, Birkhäuser 1990, 197–218.
[G] Gillet, H.: 'Riemann–Roch theorems for higher algebraic K-theory', *Adv. in Math.* 40 (1981) 203–289.
[H] Huber, A.: 'Mixed Motives and Their Realization in Derived Categories', LNM 1604, Springer 1995.
[J1] Jannsen, U.: 'Continuous étale cohomology', *Math. Ann.* 280 (1988) 207–245.
[J2] Jannsen, U.: 'Mixed Motives and Algebraic K-Theory', LNM 1400, Springer-Verlag 1990.
[J3] Jannsen, U.: 'On the l-adic Cohomology of Varieties over Number Fields and its Galois Cohomology', in 'Galois Groups over $\mathbb{Q}$', eds. Y. Ihara, K. Ribet, J.-P. Serre, Springer 1989, 315–360.
[M] Milne, J. S.: 'Artithmetic Duality Theorems', *Persp. in Math.*, Vol. 1, Academic Press, Boston 1986.
[S1] Morihiko Saito: 'Modules de Hodge Polarisables', *Publ. RIMS, Kyoto Univ.* 24 (1988) 849–995.
[S2] Morihiko Saito: 'Mixed Hodge Modules', *Publ. RIMS, Kyoto Univ.* 26 (1990) 221–333.
[Weil II] Deligne, P.: 'La Conjecture de Weil. II', *Publ. Math. IHES* 52 (1980) 313–428.
[SGA 1] Grothendieck, A., Raynaud, M.: 'Revêtements étale et groupe fondamental', Springer LN 224, Springer 1971.

[SGA 4] Artin, M., Grothendieck, A., Verdier, J. L., et al.: 'Théorie des Topos et Cohomologie Etale des Schémas', Springer LNM 269, 270, 305, Springer 1972, 1973.

[SGA 4 1/2] Deligne P., et al.: 'Cohomologie Etale', Springer LNM 569, Springer 1977.

[SGA 5] Grothendieck, A., et al.: 'Cohomologie l-adique et Fonctions L', Springer LNM 589, Springer 1977.