TTF-CLASSES OVER PERFECT RINGS

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For a ring R with unit, let ${}_R\mathscr{M}$ denote the category of unitary left R-modules. Following S. E. Dickson [3], a (non-empty) class $\mathscr{P}$ of R-modules is a *torsion class* in ${}_R\mathscr{M}$ if $\mathscr{P}$ is closed under factors, extensions, and direct sums. If, in addition, $\mathscr{P}$ is closed under submodules, then $\mathscr{P}$ is said to be *hereditary*.

An example of a hereditary torsion class is the class $\mathscr{T} = \{A | \operatorname{Hom}_R(A, E(R)) = 0\}$, where $E(R)$ denotes the injective envelope of R. This torsion has been studied in [8], [9], and [11] and, as noted in [11], coincides with usual class of torsion R-modules whenever R is a commutative integral domain. In [9], J. P. Jans has established that if R is a right perfect ring in the sense of H. Bass [2], then $\mathscr{T}$ is closed under direct products (i.e., $\mathscr{T}$ is a *TTF-class* in Jans' terminology). The main purpose of this note is to show that if R is right perfect then every hereditary torsion class is a *TTF*-class (Corollary 1.6). To further point out the analogy between the class $\mathscr{T}$ and the usual torsion modules, we show that among commutative rings with non-essential singular ideal, integral domains are characterized by the property that $\mathscr{T}$ is the unique maximal element in the lattice of hereditary torsion classes (Theorem 2.2).

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Let $\mathscr{C}$ be any class of R-modules and let $T(\mathscr{C})$ be the class of all R-modules M such that every non-zero homomorphic image of M has a nonzero submodule isomorphic to a member of $\mathscr{C}$. It is readily verified that $T(\mathscr{C})$ is a torsion class which contains $\mathscr{C}$ whenever $\mathscr{C}$ is closed under factors. In addition, if $\mathscr{C}$ is closed under cyclic submodules, $T(\mathscr{C})$ is then hereditary.

Let $\mathscr{S}$ be a representative set of non-isomorphic simple R-modules. In [3], Dickson defined the class $\mathscr{D} = T(\mathscr{S})$ (see also [4]). By the previous statements $\mathscr{D}$ is a hereditary torsion class containing $\mathscr{S}$. We shall say a (hereditary) torsion class $\mathscr{P}$ is of *simple type* if $\mathscr{P} = T(\mathscr{C})$ for some subset $\mathscr{C}$ of $\mathscr{S}$.

Before proceeding, we recall from [3] that if $\mathscr{P}$ is a torsion class, every R-module A has a unique maximal $\mathscr{P}$-submodule, $P(A)$ and the factor module $A/P(A)$ has only the zero submodule lying in P; i.e., the module $A/P(A)$ is *$\mathscr{P}$-torsion-free.*

PROPOSITION 1.1. *A hereditary torsion class $\mathscr{P}$ is of simple type if and only if $\mathscr{P} \subseteq \mathscr{D}$.*

PROOF. If $\mathscr{P} \subseteq \mathscr{D}$, let $\mathscr{C} = \mathscr{P} \cap \mathscr{S}$. Then clearly $T(\mathscr{C}) \subseteq \mathscr{P}$. Let $M \in \mathscr{P}$ and let N be the maximum $T(\mathscr{C})$-submodule of M. Then since $M \in \mathscr{D}$, $M/N = 0$ or else M/N has a nonzero simple submodule S. If the latter holds, then $S \in \mathscr{C} \subseteq T(\mathscr{C})$ and this contradicts M/N being $T(\mathscr{C})$-torsion-free. Hence $M = N \in T(\mathscr{C})$. The reverse implication is clear.

COROLLARY 1.2. *Every hereditary torsion class in ${}_R\mathscr{M}$ is of simple type if and only if every non-zero R-module has a non-zero simple submodule. In particular, if R is right perfect, then all hereditary torsion classes in ${}_R\mathscr{M}$ are of simple type.*

We now investigate the internal structure of modules which are torsion with respect to a torsion class of simple type. The method used in the next two lemmas and the resulting theorem is essentially that of Jans [9, Theorem 3.1]. As in [9], a hereditary torsion class closed under direct products will be called a *TTF-class.*

For an R-module M let $s(M)$ denote the socle of M. Thus $s(M) = 0$ if M has no nonzero simple submodules, or else $s(M)$ is the sum of all nonzero simple submodules of M. For each ordinal α, define $s^\alpha(M)$ as follows:

(1) $s^0(M) = 0$ and $s^1(M) = s(M)$.

(2) If $\alpha = \beta + 1$ is not a limit ordinal, define $s^\alpha(M)$ by $s^\alpha(M)/s^\beta(M) = s(M/s^\beta(M))$.

(3) If α is a limit ordinal, $s^\alpha(M) = \bigcup_{\beta<\alpha} s^\beta(M)$. The simple modules appearing in $s^{\alpha+1}(M)/s^\alpha(M)$ for some ordinal α are called the *composition factors* of M.

LEMMA 1.3. *Let $\mathscr{C}$ be a set of simple R-modules and $\mathscr{P} = T(\mathscr{C})$. Then $M \in \mathscr{P}$ if and only if $s^\alpha(M) = M$ for some ordinal α and each composition factor of M is isomorphic to a member of $\mathscr{C}$.*

PROOF. Let $M \in \mathscr{P}$. Then $M/s^\beta(M) \neq 0$ implies $s(M/s^\beta(M)) \neq 0$ and hence $s^\alpha(M) = M$ for some ordinal α. Since $M/s^\beta(M) \in \mathscr{P}$ for any ordinal β, we have $s^{\beta+1}(M)/s^\beta(M) \in \mathscr{P}$ since $\mathscr{P}$ is hereditary. Thus each composition factor of M is a member of $\mathscr{P}$ and so is isomorphic to a member of $\mathscr{C}$.

Conversely, suppose $s^\alpha(M) = M$ for some α and all composition factors of M are isomorphic to members of $\mathscr{C}$. Let $P(M)$ denote the maximum P-torsion submodule of M and suppose $M/P(M) \neq 0$. Choose β minimal with respect to $s^\beta(M) \nsubseteq P(M)$ and note that β is not a limit ordinal. Thus $s^{\beta-1}(M) \subseteq P(M)$ and so there is a non-zero homomorphism f mapping $s^\beta(M)/s^{\beta-1}(M)$ into $M/P(M)$. Let S be a simple submodule of $s^\beta(M)/s^{\beta-1}(M)$ with $f(S) \neq 0$. Then $f(S) \cong S$ and so $M/P(M)$ has a submodule isomorphic to a member of $\mathscr{C}$, contradicting $P(M/P(M)) = 0$. Thus $M = P(M)$ and hence $M \in \mathscr{P}$.

LEMMA 1.4. *Let $M \in \mathscr{D}$ and let S be a simple R-module. Suppose there is an idempotent $e \in R$ such that $eS \neq 0$ and $eS' = 0$ for all simple modules S' not isomorphic to S. Then S is a composition factor of M if and only if $eM \neq 0$.*

PROOF. If S is a composition factor of M then for some α, $S \subseteq s^{\alpha+1}(M)/s^{\alpha}(M)$, hence $es^{\alpha+1}(M) \nsubseteq s^{\alpha}(M)$ and so $eM \neq 0$.

On the other hand if $eM \neq 0$ then $ex \neq 0$ for some $x \in M$. Choose α minimal relative to $ex \in s^{\alpha}(M)$. Then α is not a limit ordinal, so $\alpha = \beta + 1$ and $ex \notin s^{\beta}(M)$. If S is not isomorphic to a submodule of $s^{\beta+1}(M)/s^{\beta}(M)$, then since $s^{\beta+1}(M)/s^{\beta}(M)$ is a direct sum of simples all non-isomorphic to S, we have $es^{\beta+1}(M) \subseteq s^{\beta}(M)$. But $e^2x = ex \in s^{\beta+1}(M)$ and $ex \notin s^{\beta}(M)$. Thus an isomorphic copy of S must occur in $s^{\beta+1}(M)/s^{\beta}(M)$ and hence S is a composition factor of M.

THEOREM 1.5. *Let $\mathscr{P}$ be a torsion class of simple type, $\{S_\alpha : \alpha \in A\}$ a representative set of non-isomorphic simple R-modules, and let $B = \{\beta \in A : S_\beta$ is $\mathscr{P}$-torsion free$\}$. Suppose for each $\beta \in B$ there is an idempotent $e_\beta \in R$ such that $e_\beta S_\beta \neq 0$ and $e_\beta S_\alpha = 0$ for all $\alpha \in A$, $\alpha \neq \beta$. Then if $\mathscr{D}$ is a TTF-class, $\mathscr{P}$ is a TTF-class.*

PROOF. By Lemma 1.3, $M \in \mathscr{P}$ if and only if $s^{\delta}(M) = M$ for some ordinal δ and all composition factors of M are isomorphic to a member of $\{S_\alpha : \alpha \in A - B\}$. Let $\{M_i : i \in I\} \subseteq P$, and $M = \prod_{i \in I} M_i$. Since D is closed under direct products and $\mathscr{P} \subseteq \mathscr{D}$, $M \in \mathscr{D}$ and so $s^{\delta}(M) = M$ for some ordinal δ. Since each $M_i \in \mathscr{P}$, no S_β is a composition factor of M_i for any $\beta \in B$. Thus by Lemma 1.4, $e_\beta M_i = 0$ for all $\beta \in B$, $i \in I$. Clearly then $e_\beta M = 0$ for all $\beta \in B$ and so again by Lemma 1.4, each composition factor of M is isomorphic to a member of $\{S_\alpha : \alpha \in A - B\}$. Thus $M \in \mathscr{P}$ and so $\mathscr{P}$ is closed under direct products.

COROLLARY 1.6. *If R is semi-perfect and $\mathscr{D}$ is a TTF-class, then every hereditary torsion class of simple type is a TTF-class. In particular, if R is right perfect, then every hereditary torsion class in ${}_R\mathscr{M}$ is a TTF-class.*

PROOF. Let R be semi-perfect with Jacobson radical N. Then R/N is Artinian semisimple and so there are only finitely many non-isomorphic simple R/N-modules $S_1, \cdots, S_n$. Choose idempotents $f_1, \cdots, f_n \in R/N$ satisfying $f_i S_i \neq 0$ and $f_i S_j = 0$ if $i \neq j$. The idempotents $f_1, \cdots, f_n$ can be lifted to idempotents $e_1, \cdots, e_n$ of R[2], and because of the correspondences between simple R/N-modules and simple R-modules, $S_1, \cdots, S_n$ are the simple R-modules and satisfy $e_i S_i \neq 0$ and $e_i S_j \neq 0$ for $i \neq j$. The corollary now follows from the theorem.

It is not in general true that $\mathscr{D}$ closed under direct products implies that all $\mathscr{P}$ of simple type are TTF-classes. For let $R_i = Z_p$, $i = 1, 2, \cdots$, where Z_p denotes the ring of integers modulo a prime p, and let $K = \prod_{i=1}^{\infty} R_i$. Now let R be the subring of K generated by $A = \sum_{i=1}^{\infty} R_i$ together with the unity of K. It is easily checked that every nonzero R-module has a non-zero simple submodule. Thus, in view of Corollary 1.2, $\mathscr{D} = {}_R\mathscr{M}$ and $\mathscr{D}$ is therefore closed under direct products. Now assume that $\mathscr{P} = T(\{R_i : i = 1, 2, \cdots,\})$ is closed under products. Then $K \in \mathscr{P}$ and thus the simple submodule R/A of K/A belongs to $\mathscr{P}$. Hence R/A is isomorphic to some R_i, a contradiction to the fact that A is an essential ideal and R_i is a direct summand of R. Thus $\mathscr{P}$ is not closed under direct products.

Corollary 1.7. *If $\mathscr{D}$ is a TTF-class and $\mathscr{P}$ is a torsion class of simple type such that all $\mathscr{P}$-torsion-free simple modules are projective, then $\mathscr{P}$ is a TTF-class.*

Proof. For each projective simple module S, there is an idempotent $e \in R$ such that $S \cong Re$. It is easily checked that these idempotents satisfy the condition of Theorem 1.5.

The next statement gives a necessary condition in order that $\mathscr{D}$ be a proper *TTF*-class. Write

$$I_{\mathscr{D}} = \cap \{L | L \text{ is a left ideal of } R \text{ and } R/L \in \mathscr{D}\},$$

and observe that if $\mathscr{D}$ is a *TTF*-class then $R/I_{\mathscr{D}} \in \mathscr{D}$. Thus we have the following

Lemma 1.8. *If $\mathscr{D}$ is a TTF-class then $I_{\mathscr{D}}$ has no maximal submodules.*

Corollary 1.9. (i) *If $I_{\mathscr{D}}$ is a finitely generated R-module then $\mathscr{D}$ is a TTF-class if and only if $\mathscr{D} = {}_R\mathscr{M}$.*

(ii) *If R is left noetherian then $\mathscr{D}$ is a TTF-class if and only if R is left artinian.*

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For an R-module A let $Z(A)$ denote its singular submodule [10] and let $\mathscr{Z} = \{A | Z(A) = A\}$. In general $\mathscr{Z}$ is not a torsion class, since it is not closed under extensions. However, if $Z(R) = 0$ then E. Gentile has shown in [8] that $\mathscr{Z} = \mathscr{T}$; in fact V. Dlab [5] establishes that $\mathscr{Z} = \mathscr{T}$ if and only if $Z(R) = 0$ and thus $\mathscr{Z}$ is a torsion class if and only if it coincides with $\mathscr{T}$. Thus for commutative integral domains, $\mathscr{Z} = \mathscr{T}$ is the class of usual torsion modules. The smallest torsion class containing $\mathscr{Z}$ will be denoted by $\mathscr{G}$; i.e., $\mathscr{G} = T(\mathscr{Z})$ (cf. [1] and [6]).

Lemma 2.1. [5]. *For any ring R, $\mathscr{T} \subseteq \mathscr{Z}$.*

Proof. Suppose $A \in \mathscr{T}$ and $Z(A) \neq A$. If $x \notin Z(A)$ then $(0:x) \cap I = 0$ for some left ideal $I \neq 0$ of R. Then we have the exact sequence $0 \to I \to Rx$ with $Rx \in \mathscr{T}$. Hence $0 \neq I \in \mathscr{T}$, contrary to R being $\mathscr{T}$-torsion-free.

Theorem 2.2. *Let R be a commutative ring with $G(R) \neq R$. Then $\mathscr{T}$ contains every proper hereditary torsion class if and only if R is an integral domain.*

Proof. If R is an integral domain and $\mathscr{P}$ is any proper hereditary torsion class then $R \notin \mathscr{P}$. Thus if $A \in \mathscr{P}$ and $a \in A$ then $(0:a) \neq 0$ otherwise $R \cong Ra \in \mathscr{P}$. Hence every module in $\mathscr{P}$ is a torsion R-module and so $\mathscr{P} \subseteq \mathscr{T}$. For the converse we note that since $G(R) \neq R$, $\mathscr{G}$ is a proper torsion class, and so $\mathscr{Z} \subseteq \mathscr{G} \subseteq \mathscr{T}$. By Lemma 2.1, $\mathscr{T} \subseteq \mathscr{Z}$ and hence $\mathscr{T} = \mathscr{Z}$. Thus by Dlab's result [5], $Z(R) = 0$ and since R is commutative this means R has no nonzero nilpotent elements. Therefore, for any non-zero ideal I of R, necessarily

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$$\{x : x \in I \text{ and } (0 : x) \supseteq I\} = \{0\}.$$

Hence the torsion class $\mathscr{P}$ generated by all R-modules M such that $IM = 0$ satisfies $P(I) = 0$. As a consequence $\mathscr{P} \subseteq \mathscr{T} \subseteq \mathscr{Z}$. Hence

$$R/I = P(R/I) = Z(R/I),$$

i.e., I is essential in R. Thus R is an integral domain as required.

We remark that the condition $G(R) \neq R$ is needed. For let R be a (commutative) local ring R with nilpotent maximal ideal. Evidently R has only two hereditary torsion classes ${}_R\mathscr{M}$ and 0; $\mathscr{G} = {}_R\mathscr{M}$ and $\mathscr{T} = 0$.

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