# ISOMETRIC IMMERSIONS OF ALMOST HERMITIAN MANIFOLDS

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The Lefschetz theorem on hyperplane sections, as proved by Andreotti and Frankel (**1**), depends upon the following result.

THEOREM. *If $M$ is a non-singular affine algebraic variety of real dimension $2k$ of complex $n$-space, then*

$$H_i(M, \mathbf{Z}) = 0 \quad \textit{for } i > k.$$

This theorem, which is interesting in itself, has been strengthened by Milnor (**7**), who showed that $M$ has the homotopy type of a $k$-dimensional CW-complex.

In this paper we generalize the above theorem in two directions. First, we replace complex $n$-space by some other complete simply connected Riemannian manifold $\bar{M}$ which either has non-positive curvature or is a compact symmetric space. Secondly, we allow $M$ and $\bar{M}$ to be quasi-Kählerian (see below) instead of Kählerian.

We first introduce some notation. Let $M$ and $\bar{M}$ be $C^\infty$ Riemannian manifolds with $M$ isometrically immersed in $\bar{M}$. Denote by $\langle\ ,\ \rangle$ the metric tensor of either $M$ or $\bar{M}$. Let $\mathfrak{X}(M)$ and $\bar{\mathfrak{X}}(M)$ denote the Lie algebras of vector fields on $M$ and the restrictions to $M$ of vector fields on $\bar{M}$, respectively. Then we may write $\bar{\mathfrak{X}}(M) = \mathfrak{X}(M) \oplus \mathfrak{X}(M)^\perp$. The *configuration tensor* is the function $T\colon \mathfrak{X}(M) \times \bar{\mathfrak{X}}(M) \to \bar{\mathfrak{X}}(M)$ defined by the formulas

$$T_X Y = \bar{\nabla}_X Y - \nabla_X Y, \qquad T_X Z = P\bar{\nabla}_X Z$$

for $X$, $Y \in \mathfrak{X}(M)$, $Z \in \mathfrak{X}(M)^\perp$. Here, $\nabla$ and $\bar{\nabla}$ are the Riemannian connections of $M$ and $\bar{M}$, respectively, and $P\colon \bar{\mathfrak{X}}(M) \to \mathfrak{X}(M)$ is the orthogonal projection. Then (**3**) $T_X Y$ is symmetric in $X$ and $Y$ for $X$, $Y \in \mathfrak{X}(M)$ and for $X \in \mathfrak{X}(M)$, $T_X$ is a skew-symmetric linear operator.

Let $J$ be an almost complex structure on $\bar{M}$, i.e., a linear map $J\colon \mathfrak{X}(\bar{M}) \to \mathfrak{X}(\bar{M})$ with $J^2 = -I$. We say that $M$ is an *almost complex submanifold* of $\bar{M}$ (with respect to $J$) provided $JX \in \mathfrak{X}(M)$ for all $X \in \mathfrak{X}(M)$. Thus, $J$ induces an almost complex structure on $M$ which we continue to denote by $J$.

If $\langle JX, JY\rangle = \langle X, Y\rangle$ for all $X$, $Y \in \mathfrak{X}(M)$, we say that $M$ is *almost Hermitian* (with respect to the given almost complex structure $J$ and metric

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tensor $\langle\ ,\ \rangle$). If this is the case, then $M$ is *Kählerian* if and only if $\nabla_X(J)(Y) = 0$ for all $X, Y \in \mathfrak{X}(M)$. However, this condition is unnecessarily strong for our purposes.

*Definition.* An almost Hermitian manifold $M$ is *quasi-Kählerian* if and only if

$$\nabla_X(J)(Y) + \nabla_{JX}(J)(JY) = 0$$

for all $X, Y \in \mathfrak{X}(M)$.

The *Kähler form* $F$ of an almost Hermitian manifold $M$ is the 2-form defined by $F(X, Y) = \langle JX, Y\rangle$ for $X, Y \in \mathfrak{X}(M)$. In (**3**) it is shown that either of the conditions $dF = 0$ or $\nabla_X(J)(X) = 0$ for all $X \in \mathfrak{X}(M)$ is sufficient that $M$ be quasi-Kählerian. Furthermore, the almost complex structure of a quasi-Kähler manifold $M$ is integrable if and only if $M$ is Kählerian. A necessary condition that $M$ be quasi-Kählerian is that the coderivative $\delta F = 0$.

LEMMA 1. *Let $\bar{M}$ be a quasi-Kähler manifold with almost complex structure $J$, and suppose that $M$ is an almost complex submanifold of $\bar{M}$. Then*

(i) *$M$ is quasi-Kählerian (with respect to the naturally induced almost complex structure);*

(ii) *we have that*

$$T_X Y + T_{JX} JY = 0 \quad \textit{for all } X, Y \in \mathfrak{X}(M); \tag{1}$$

(iii) *$M$ is a minimal variety of $\bar{M}$.*

This lemma is proved in (**3**).

Let $M_p$ and $\bar{M}_p$ denote the tangent spaces to $M$ and $\bar{M}$ at a point $p \in M$. The configuration tensor $T$ gives rise to a tensor on $M_p$ which we denote by $T_x y$ for $x, y \in M_p$. We denote by $R_{xy}$ ($x, y \in M_p$) and $\bar{R}_{zw}$ ($z, w \in \bar{M}_p$) the curvature operators of $M$ and $\bar{M}$, respectively. If $\|x \wedge y\| \neq 0 \neq \|z \wedge w\|$, we write $K_{xy} = \|x \wedge y\|^{-2}\langle R_{xy}x, y\rangle$ and $\bar{K}_{zw} = \|z \wedge w\|^{-2}\langle \bar{R}_{zw}z, w\rangle$ for the sectional curvatures of $M$ and $\bar{M}$, respectively. For $q \in \bar{M}$, let $U(q, b)$ be the closed "geodesic" neighbourhood consisting of all points whose distance to $q$ is less than or equal to $b$.

In the following lemma, (i) is essentially due to Hermann (**7**).

LEMMA 2. *Suppose that $\bar{M}$ is a completely simply connected Riemannian manifold, and let $M$ be a closed isometrically immersed submanifold of dimension $n$. Suppose that $k$ is an integer such that for all $p \in M$ and all $z \in M_p^{\perp}$ at least $k$ of the eigenvalues $\kappa$ of $x \to T_x z$ (counted according to multiplicity) satisfy $\kappa \leqq 0$.*

(i) *If $\sup \bar{K} \leqq 0$, then $M$ has the homotopy type of a* CW-*complex with no cells of dimension greater than $n - k$.*

(ii) *If $\bar{M}$ is a compact symmetric space and $0 < b \leqq \frac{1}{2}\pi(\max \bar{K})^{-1/2}$, then for almost all $q \in \bar{M}$, $M \cap U(q, b)$ has the homotopy type of a* CW-*complex with no cells of dimension greater than $n - k$.*

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*Proof.* Let $\rho$ denote the distance function of $\bar{M}$. Choose a point $m_0 \in \bar{M}$, $m_0 \notin M$ such that the real-valued function $f$ on $M$ defined by $f(m) = \rho(m, m_0)$ has no degenerate critical points; this is possible by Sard's theorem; see (**2**, p. 225). (Note that $f$ is differentiable off the cut locus of $m_0$). Let $m \in M$ not lie in the cut locus of $m_0$, and assume that $m$ is a critical point of $f$. In (i) there is no cut locus and in (ii) the cut locus and the conjugate locus coincide. Hence, in either case there exists a unique unit speed geodesic $\sigma$: $[0, b] \to \bar{M}$ from $m_0$ to $m$. We denote by $\sigma'$ the velocity of $\sigma$ and by $Z'$ the covariant derivative of a vector field $Z$ along $\sigma$. For $u, v \in \bar{M}_m$ let $U$ and $V$ denote the unique Jacobi vector fields along $\sigma$ (i.e., $U'' = \bar{R}_{U\sigma'}\sigma'$) such that $U(0) = V(0) = 0$ and $U(1) = u$, $V(1) = v$. Define

$$(2) \qquad Q(u, v) = \int_0^b \{\langle U', V'\rangle - \langle \bar{R}_{\sigma' U}\sigma', V\rangle\}(t)\, dt = \langle U', V\rangle(b).$$

Then (**2**, p. 219) the Hessian $H_f$ of $f$ is given by the formula

$$(3) \qquad H_f(x, y) = Q(x, y) + \langle T_x y, z\rangle$$

for $x, y \in M_m$, where $z = \sigma'(b) \in M_m{}^\perp$.

If $\sup \bar{K} \leqq 0$, then from (2) and (3) we conclude that

$$(4) \qquad H_f(x, x) \geqq - \langle T_x z, x\rangle$$

for all $x \in M_m$. The hypotheses of the lemma now imply that any subspace of $M_m$ on which $H_f$ is negative-definite must have dimension less than or equal $n - k$. Since $f$ is obviously bounded from below, (i) now follows from (**8**, Theorem 3.5).

For (ii) we let $q = m_0$. The set of all such $q$ (such that the function $f$ given by $f(m) = \rho(m, q)$ has no degenerate critical points) constitutes almost all of $\bar{M}$ by Sard's theorem. We observe that if $\bar{M}$ is a compact simply connected symmetric space, (a) the eigenvectors of $w \to \bar{R}_{zw}z$ diagonalize $Q$ and (b) if $w$ is an eigenvector of $w \to \bar{R}_{zw}z$ with $\langle w, z\rangle = 0$, $||w|| = 1$ corresponding to the eigenvalue $\lambda = \bar{K}_{zw}$, then

$$Q(w, w) = ||w||^2 \sqrt{\lambda} \cot(\sqrt{\lambda}\, b).$$

This is proved in (**6**). Hence, (4) holds, and the proof of the remainder of (ii) is the same as that of (i).

We now use Lemmas 1 and 2 to prove the main results of this paper.

THEOREM 1. *Let $\bar{M}$ be a complete simply connected quasi-Kähler manifold with non-positive sectional curvature. If $M$ is a closed isometrically immersed almost complex submanifold of real dimension $2k$, then $M$ has the homotopy type of a* CW-*complex with no cells of dimension greater than $k$.*

*Proof.* From (1) it follows that if $\kappa$ is an eigenvalue of $x \to T_x z$, then $-\kappa$ is also an eigenvalue. Hence, at least $k$ of the eigenvalues are less than or equal to zero. Now, Theorem 1 follows from Lemma 2 (i).

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In exactly the same way, the following theorem follows from Lemma 2 (ii).

THEOREM 2. *Let $\bar{M}$ be a compact simply connected symmetric space which is also a quasi-Kähler manifold. If $M$ is a compact isometrically immersed almost complex submanifold of real dimension $2k$ and $0 < b \leqq \frac{1}{2}\pi(\max \bar{K})^{-1/2}$, then for almost all $q \in \bar{M}$, $M \cap U(q, b)$ has the homotopy type of a* CW-*complex with no cells of dimension greater than $k$.*

The most important class of manifolds $M$ to which Theorems 1 and 2 apply are, of course, Kähler manifolds. In fact, I do not know of any examples of non-Kähler quasi-Kähler manifolds of non-positive curvature. It seems probable, however, that some examples could be constructed along the lines of (**4**); these would be 6-dimensional almost complex manifolds contained in $R^8$.

The sphere $S^6$ is a compact symmetric space with a quasi-Kähler almost complex structure (with respect to the natural metric). However, Theorem 2 yields no new information about $S^6$, since in (**5**) it is shown that $S^6$ has no 4-dimensional almost complex submanifolds, not even locally. On the other hand, Theorem 2 does yield new information about the symmetric space $S^7 \times S^7$; this has a quasi-Kählerian almost complex structure $J$ with respect to the natural metric since $S^7 \times S^7 = \mathrm{Spin}(8)/G_2$. The almost complex structure $J$ is derived (**10**) from the triality automorphism of Spin(8), which is an outer automorphism of order 3.

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