4
The energy–momentum relation

If the external forces vanish, the equations of motion must have a solution, in which the particle travels at constant velocity $\boldsymbol{v}$ in the company of its electromagnetic fields. There seems to be no accepted terminology for this object. Since it will be used as a basic building block later on, we need a short descriptive name and we call this particular solution a *charge soliton*, or simply soliton, at velocity $\boldsymbol{v}$, in analogy to solitons of nonlinear wave equations. The soliton has an energy and a momentum which are linked through the energy–momentum relation.

For the Lorentz model, by Lorentz invariance, it suffices to determine the four-vector of total momentum in the rest frame, where it is of the form $(m_\mathrm{s}, 0)$, m_s being the rest mass of the soliton. m_s depends on $|\boldsymbol{\omega}_\mathrm{E}|$. Through a Lorentz boost one obtains the charge soliton moving with velocity $\boldsymbol{v}$ and, of course, the relativistic energy–momentum relation. No such argument is available for the Abraham model and one simply has to compute its energy–momentum relation, which can be achieved along two equivalent routes. The first one is dynamic, as alluded to above, while the second one is static and directly determines the minimal energy at fixed total momentum. The minimizer is the charge soliton.

In the following two sections we compute the conserved energy and momentum, the charge solitons, and the energy–momentum relation for both the Abraham and the Lorentz model. $\phi_\mathrm{ex} = 0,\ \boldsymbol{A}_\mathrm{ex} = 0$ is assumed throughout.

4.1 The Abraham model

The mechanical momentum of the particle is given by

$$m_\mathrm{b}\gamma\boldsymbol{v} \tag{4.1}$$

and the momentum of the field by

$$\mathcal{P}_\mathrm{f} = \int \mathrm{d}^3x \big(\boldsymbol{E}(\boldsymbol{x}) \times \boldsymbol{B}(\boldsymbol{x})\big)\,. \tag{4.2}$$

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Thus we set the total momentum

$$\mathcal{P} = m_{\mathrm{b}}\gamma \boldsymbol{v} + \mathcal{P}_{\mathrm{f}} \tag{4.3}$$

as a functional on $\mathcal{M}$. It is easily checked that $\mathcal{P}$ is conserved by the coupled Maxwell and Newton equations (2.39)–(2.41). To ensure that $\mathcal{P}$ corresponds physically to the total momentum we note that the Lagrangian (2.43) of the Abraham model is invariant under spatial translations. By Noether's theorem, this symmetry is linked with a conserved quantity which turns out to be $\mathcal{P}$.

We want to minimize the energy at fixed total momentum. One eliminates $\boldsymbol{v}$ from (2.44) and (4.3) and thus has to minimize

$$\Big(m_{\mathrm{b}}^2 + \Big(\mathcal{P} - \int \mathrm{d}^3x(\boldsymbol{E}\times\boldsymbol{B})\Big)^2\Big)^{1/2} + \frac{1}{2}\int \mathrm{d}^3x(\boldsymbol{E}^2 + \boldsymbol{B}^2) \tag{4.4}$$

at fixed $\mathcal{P}$ and subject to the constraints $\nabla\cdot\boldsymbol{E} = e\varphi$, $\nabla\cdot\boldsymbol{B} = 0$. By translation invariance we may center φ at an arbitrary $\boldsymbol{q}\in\mathbb{R}^3$. For $\boldsymbol{q} = 0$, say, the minimizer is unique and given by

$$\begin{aligned}\boldsymbol{E}_{v}(\boldsymbol{x}) &= -\nabla\phi_{v\varphi}(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{v}\cdot\nabla\phi_{v\varphi}(\boldsymbol{x}))\,,\\ \boldsymbol{B}_{v}(\boldsymbol{x}) &= -\boldsymbol{v}\times\nabla\phi_{v\varphi}(\boldsymbol{x})\end{aligned} \tag{4.5}$$

with $\boldsymbol{v}\in\mathbb{V} = \{\boldsymbol{v}|\,|\boldsymbol{v}| < 1\}$. Here

$$\widehat{\phi}_{v}(\boldsymbol{k}) = e[\boldsymbol{k}^2 - (\boldsymbol{v}\cdot\boldsymbol{k})^2]^{-1}\,, \tag{4.6}$$

or in physical space

$$\phi_{v}(\boldsymbol{x}) = e(4\pi)^{-1}(\gamma^{-2}\boldsymbol{x}^2 + (\boldsymbol{v}\cdot\boldsymbol{x})^2)^{-1/2}\,, \tag{4.7}$$

and $\phi_{v\varphi}$ is shorthand for the convolution $\phi_v * \varphi$, i.e. $\widehat{\phi}_{v\varphi}(\boldsymbol{k}) = (2\pi)^{3/2}\widehat{\varphi}(\boldsymbol{k})\widehat{\phi}_v(\boldsymbol{k})$. $\boldsymbol{v}$ has to be adjusted such that $\mathcal{P} = \boldsymbol{P}_{\mathrm{s}}(\boldsymbol{v})$ with

$$\begin{aligned}\boldsymbol{P}_{\mathrm{s}}(\boldsymbol{v}) &= m_{\mathrm{b}}\gamma\boldsymbol{v} + e^2\int \mathrm{d}^3k|\widehat{\varphi}(\boldsymbol{k})|^2\big([\boldsymbol{k}^2 - (\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-1}\boldsymbol{v}\\ &\qquad - \gamma^{-2}[\boldsymbol{k}^2 - (\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-2}(\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{k}\big)\\ &= \boldsymbol{v}\big(m_{\mathrm{b}}\gamma + m_{\mathrm{f}}|\boldsymbol{v}|^{-3}\big[-|\boldsymbol{v}| + (1+\boldsymbol{v}^2)\mathrm{arctanh}|\boldsymbol{v}|\big]\big)\,,\end{aligned} \tag{4.8}$$

where m_{f} is the electrostatic energy of the charge distribution $e\varphi$,

$$m_{\mathrm{f}} = \frac{1}{2}e^2\int \mathrm{d}^3x\,\mathrm{d}^3x'\varphi(\boldsymbol{x})\,\varphi(\boldsymbol{x}')(4\pi|\boldsymbol{x}-\boldsymbol{x}'|)^{-1}\,. \tag{4.9}$$

The map $\mathbb{V}\ni\boldsymbol{v}\mapsto\boldsymbol{P}_{\mathrm{s}}(\boldsymbol{v})\in\mathbb{R}^3$ is one-to-one and therefore $\mathcal{P} = \boldsymbol{P}_{\mathrm{s}}(\boldsymbol{v})$ has a unique solution. The minimizing energy is given by

$$\begin{aligned} E_{\mathrm{s}}(\boldsymbol{v}) &= m_{\mathrm{b}}\gamma + \frac{1}{2}e^2 \int \mathrm{d}^3k |\widehat{\varphi}(\boldsymbol{k})|^2 \, [\boldsymbol{k}^2 - (\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-2} \big((1+\boldsymbol{v}^2)\boldsymbol{k}^2 \\ &\quad - (3-\boldsymbol{v}^2)(\boldsymbol{v}\cdot\boldsymbol{k})^2\big) \\ &= m_{\mathrm{b}}\gamma + m_{\mathrm{f}}\, |\boldsymbol{v}|^{-1}\big[-|\boldsymbol{v}| + 2\,\mathrm{arctanh}|\boldsymbol{v}|\big]\,. \end{aligned} \tag{4.10}$$

Eliminating now $\boldsymbol{v}$ from E_{s} and $\boldsymbol{P}_{\mathrm{s}}$ yields the *energy–momentum relation*

$$E_{\mathrm{eff}}(\boldsymbol{p}) = E_{\mathrm{s}}(\boldsymbol{v}(\boldsymbol{p})) \tag{4.11}$$

with $\boldsymbol{v}(\boldsymbol{P}_{\mathrm{s}})$ the function inverse to $\boldsymbol{P}_{\mathrm{s}}(\boldsymbol{v})$. It is emphasized that E_{eff} depends on the charge distribution only through its electrostatic energy.

We note that

$$\boldsymbol{P}_{\mathrm{s}}(\boldsymbol{v}) = \nabla_{\boldsymbol{v}} T(\boldsymbol{v})\,, \tag{4.12}$$

where

$$\begin{aligned} T(\boldsymbol{v}) &= -m_{\mathrm{b}}\gamma^{-1} + \frac{1}{2}e^2\,\gamma^{-2} \int \mathrm{d}^3k |\widehat{\varphi}(\boldsymbol{k})|^2 \, [\boldsymbol{k}^2 - (\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-1} \\ &= -m_{\mathrm{b}}\gamma^{-1} - m_{\mathrm{f}}\, |\boldsymbol{v}|^{-1}(1-|\boldsymbol{v}|^2)\,\mathrm{arctanh}|\boldsymbol{v}|\,, \end{aligned} \tag{4.13}$$

and that

$$E_{\mathrm{s}}(\boldsymbol{v}) = \boldsymbol{P}_{\mathrm{s}}(\boldsymbol{v})\cdot\boldsymbol{v} - T(\boldsymbol{v})\,. \tag{4.14}$$

This suggests that T will play the role of the inertial term in an effective Lagrangian and E_{s} the role of an effective Hamiltonian as our notation in (4.11) indicates already. In particular,

$$\boldsymbol{v} = \nabla_{\boldsymbol{p}}\, E_{\mathrm{eff}}(\boldsymbol{p}) \tag{4.15}$$

and, equivalently,

$$\frac{\mathrm{d}\boldsymbol{P}_{\mathrm{s}}(\boldsymbol{v})}{\mathrm{d}\boldsymbol{v}}\,\boldsymbol{v} = \nabla_{\boldsymbol{v}}\, E_{\mathrm{s}}(\boldsymbol{v}) \tag{4.16}$$

which implies that $\boldsymbol{v}$ is to be interpreted as a velocity and $\mathrm{d}\boldsymbol{P}_{\mathrm{s}}/\mathrm{d}\boldsymbol{v}$, regarded as a 3×3 matrix, as the velocity-dependent mass.

For a relativistic theory one expects that

$$E_{\mathrm{s}}(\boldsymbol{v}) = (m_{\mathrm{b}} + m_{\mathrm{f}})\gamma\,, \quad \boldsymbol{P}_{\mathrm{s}}(\boldsymbol{v}) = (m_{\mathrm{b}} + m_{\mathrm{f}})\gamma\boldsymbol{v}\,. \tag{4.17}$$

Since the Abraham model is semirelativistic, there is no reason for such a property to be satisfied. Still, as in the relativistic case, the energy–momentum relation depends on the charge distribution $e\varphi$ only through m_{f}.

To gain a feeling for the field contributions to the mass we define

$$m_{\mathrm{f}}(v) = \frac{\mathrm{d}(\boldsymbol{P}_{\mathrm{s}} - m_{\mathrm{b}}\gamma \boldsymbol{v})}{\mathrm{d}\boldsymbol{v}} = m_{\mathrm{l}}(\boldsymbol{v})\widehat{\boldsymbol{v}} \otimes \widehat{\boldsymbol{v}} + m_{\mathrm{t}}(\boldsymbol{v})(\mathbb{1} - \widehat{\boldsymbol{v}} \otimes \widehat{\boldsymbol{v}}) , \qquad (4.18)$$

where $\widehat{\boldsymbol{v}}$ is the unit vector along $\boldsymbol{v}$; $m_{\mathrm{l}}(\boldsymbol{v})$ is the longitudinal and $m_{\mathrm{t}}(\boldsymbol{v})$ is the transverse field mass. Using (4.8) one obtains

$$m_{\mathrm{l}}(\boldsymbol{v}) = m_{\mathrm{f}}|\boldsymbol{v}|^{-3}\big(2|\boldsymbol{v}|(1 - |\boldsymbol{v}|^2)^{-1} - 2\,\mathrm{arctanh}|\boldsymbol{v}|\big) , \qquad (4.19)$$

$$m_{\mathrm{t}}(\boldsymbol{v}) = m_{\mathrm{f}}|\boldsymbol{v}|^{-3}\big(- |\boldsymbol{v}| + (1 + |\boldsymbol{v}|^2)\mathrm{arctanh}|\boldsymbol{v}|\big) , \qquad (4.20)$$

and by expanding in small $\boldsymbol{v}$, i.e. small $|\boldsymbol{v}|/c$,

$$m_{\mathrm{l}}(\boldsymbol{v}) = \frac{4}{3}m_{\mathrm{f}}\Big(1 + \frac{6}{5}|\boldsymbol{v}|^2 + \frac{9}{7}|\boldsymbol{v}|^4 + \cdots\Big) , \qquad (4.21)$$

$$m_{\mathrm{t}}(\boldsymbol{v}) = \frac{4}{3}m_{\mathrm{f}}\Big(1 + \frac{2}{5}|\boldsymbol{v}|^2 + \frac{9}{35}|\boldsymbol{v}|^4 + \cdots\Big) . \qquad (4.22)$$

In particular one has

$$E_{\mathrm{s}}(\boldsymbol{v}) - E_{\mathrm{s}}(0) \cong \frac{1}{2}\Big(m_{\mathrm{b}} + \frac{4}{3}m_{\mathrm{f}}\Big)\boldsymbol{v}^2, \quad \boldsymbol{P}_{\mathrm{s}}(\boldsymbol{v}) = \Big(m_{\mathrm{b}} + \frac{4}{3}m_{\mathrm{f}}\Big)\boldsymbol{v} . \qquad (4.23)$$

Thus the effective mass in the nonrelativistic approximation is

$$m_{\mathrm{eff}} = m_{\mathrm{b}} + \frac{4}{3}m_{\mathrm{f}} . \qquad (4.24)$$

We compare (4.19)–(4.22) with a relativistic particle for small $\boldsymbol{v}$ and of the same mass. Then

$$m^{\mathrm{rel}} = m_{\mathrm{l}}^{\mathrm{rel}}\widehat{\boldsymbol{v}} \otimes \widehat{\boldsymbol{v}} + m_{\mathrm{t}}^{\mathrm{rel}}(\mathbb{1} - \widehat{\boldsymbol{v}} \otimes \widehat{\boldsymbol{v}}) \qquad (4.25)$$

with

$$m_{\mathrm{l}}^{\mathrm{rel}}(\boldsymbol{v}) = \Big(m_{\mathrm{b}} + \frac{4}{3}m_{\mathrm{f}}\Big)\gamma^3 = \Big(m_{\mathrm{b}} + \frac{4}{3}m_{\mathrm{f}}\Big)\Big(1 + \frac{3}{2}|\boldsymbol{v}|^2 + \frac{9}{8}|\boldsymbol{v}|^4 + \cdots\Big) , \qquad (4.26)$$

$$m_{\mathrm{t}}^{\mathrm{rel}}(\boldsymbol{v}) = \Big(m_{\mathrm{b}} + \frac{4}{3}m_{\mathrm{f}}\Big)\gamma = \Big(m_{\mathrm{b}} + \frac{4}{3}m_{\mathrm{f}}\Big)\Big(1 + \frac{1}{2}|\boldsymbol{v}|^2 + \frac{3}{8}|\boldsymbol{v}|^4 + \cdots\Big) . \qquad (4.27)$$

If one sets the bare mass to zero, $m_{\mathrm{b}} = 0$, even for $|\boldsymbol{v}| = 0.5$ the error in the velocity-dependent mass is less than 5%. Only at speeds $|\boldsymbol{v}| > 0.5$ will the Abraham model lose its empirical validity. The model could be partially saved by declaring the Compton wavelength as the characteristic size of the charge distribution. Then $m_{\mathrm{f}}/m_{\mathrm{b}} \cong 0.01$ and the relativistic dispersion would be violated only for speeds very close to one.

The energy minimizer has a simple dynamical interpretation. We look for a solution of (2.39)–(2.41) traveling at constant velocity. Let us first define

$$S_{\boldsymbol{q},\boldsymbol{v}} = (\boldsymbol{E}_{\boldsymbol{v}}(\boldsymbol{x}-\boldsymbol{q}), \boldsymbol{B}_{\boldsymbol{v}}(\boldsymbol{x}-\boldsymbol{q}), \boldsymbol{q}, \boldsymbol{v}) \tag{4.28}$$

with $\boldsymbol{v} \in \mathbb{V}$, $\boldsymbol{q} \in \mathbb{R}^3$, and $\boldsymbol{B}_{\boldsymbol{v}}$, $\boldsymbol{E}_{\boldsymbol{v}}$ from (4.5). Then the solution traveling at constant velocity is

$$Y(t) = S_{\boldsymbol{q}+\boldsymbol{v}t,\boldsymbol{v}} . \tag{4.29}$$

The particular state (4.28) will play an important role and is called a *charge soliton*, labeled by its center $\boldsymbol{q}$ and its velocity $\boldsymbol{v}$. It has the energy $\mathcal{E}(S_{\boldsymbol{q},\boldsymbol{v}}) = E_{\mathrm{s}}(\boldsymbol{v})$ and momentum $\mathcal{P}(S_{\boldsymbol{q},\boldsymbol{v}}) = \boldsymbol{P}_{\mathrm{s}}(\boldsymbol{v})$. The set of all charge solitons is

$$\mathcal{S} = \{S_{\boldsymbol{q},\boldsymbol{v}} | \, \boldsymbol{v} \in \mathbb{V}, \boldsymbol{q} \in \mathbb{R}^3\} \subset \mathcal{M} . \tag{4.30}$$

Sometimes we use the same words and symbols for the field configuration only.

There is an instructive alternate way to represent the charge soliton. We consider the inhomogeneous Maxwell–Lorentz equations (2.39) and prescribe the initial data at time τ. We require that the particle travels along the straight line $\boldsymbol{q} = \boldsymbol{v}t$. If we let $\tau \to -\infty$ and consider the solution at time $t = 0$, then in (2.16), (2.17) the initial fields will have escaped to infinity and only the retarded fields survive. Using (2.16), (2.17) this leads to

$$\begin{aligned} \boldsymbol{E}_{\boldsymbol{v}}(\boldsymbol{x}) = -\int_{-\infty}^{0} \mathrm{d}t \int \mathrm{d}^3 y \, \big(\nabla G_{-t}\,(\boldsymbol{x}-\boldsymbol{y})\, e\varphi(\boldsymbol{y}-\boldsymbol{v}t) \\ + \partial_t \, G_{-t}\,(\boldsymbol{x}-\boldsymbol{y}) \boldsymbol{v} e\varphi(\boldsymbol{y}-\boldsymbol{v}t)\big) , \end{aligned} \tag{4.31}$$

$$\boldsymbol{B}_{\boldsymbol{v}}(\boldsymbol{x}) = \int_{-\infty}^{0} \mathrm{d}t \int \mathrm{d}^3 y \, \nabla \times G_{-t}\,(\boldsymbol{x}-\boldsymbol{y})\, \boldsymbol{v} e\varphi(\boldsymbol{y}-\boldsymbol{v}t) , \tag{4.32}$$

which can be checked either in Fourier space or as being a solution of the Maxwell equations traveling at constant velocity $\boldsymbol{v}$.

4.2 The Lorentz model

We fix a Lorentz frame, $\mathcal{F}_{\mathrm{L}}$, and seek a solution with $\mathbf{q}(\tau) = 0$, $\mathbf{w}(\tau) = \mathbf{w}$ for all τ. The corresponding four-current is

$$\mathbf{j}(\mathbf{x}) = e\varphi_{\mathrm{r}}(|\mathbf{x}|)\boldsymbol{\Omega} \cdot \mathbf{x} \tag{4.33}$$

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and provides the source for the electromagnetic vector potential. The inhomogeneous Maxwell equations yield

$$\phi_{\mathbf{0},\omega}(\boldsymbol{x}) = \int \mathrm{d}^3x' \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} e\varphi(\boldsymbol{x}'), \tag{4.34}$$

$$\boldsymbol{A}_{\mathbf{0},\omega}(\boldsymbol{x}) = \int \mathrm{d}^3x' \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} \boldsymbol{\omega}_{\mathrm{E}} \times \boldsymbol{x}' e\varphi(\boldsymbol{x}'), \tag{4.35}$$

the index $\mathbf{0}$ standing for $\boldsymbol{v} = \mathbf{0}$.

Outside the support of the charge distribution, $\phi_{\mathbf{0},\omega}$ is the Coulomb potential,

$$\phi_{\mathbf{0},\omega}(\boldsymbol{x}) = \frac{e}{4\pi|\boldsymbol{x}|}, \quad |\boldsymbol{x}| \geq R_\varphi, \tag{4.36}$$

and $\boldsymbol{A}_{\mathbf{0},\omega}$ is the vector potential generated by the magnetic moment

$$\boldsymbol{\mu} = \frac{1}{2}\int \mathrm{d}^3x \boldsymbol{x} \times (\boldsymbol{\omega}_{\mathrm{E}} \times \boldsymbol{x}) e\varphi(\boldsymbol{x}) = \mu \boldsymbol{\omega}_{\mathrm{E}} \text{ with } \mu = \frac{1}{3} e \int \mathrm{d}^3x \varphi(\boldsymbol{x}) \boldsymbol{x}^2, \tag{4.37}$$

which means

$$\boldsymbol{A}_{\mathbf{0},\omega}(\boldsymbol{x}) = \frac{\boldsymbol{\mu} \times \boldsymbol{x}}{4\pi|\boldsymbol{x}|^3}, \quad |\boldsymbol{x}| \geq R_\varphi. \tag{4.38}$$

To check the Lorentz force and torque we note that a well-defined momentum and angular momentum requires the equator to have subluminal speed, i.e.

$$\omega R_\varphi \leq 1, \quad \omega = |\boldsymbol{\omega}_{\mathrm{E}}|. \tag{4.39}$$

Inserting the fields (4.34), (4.35) in Eqs. (2.92), (2.95) we indeed find $\mathbf{f}(\tau) = 0$, $\mathbf{t}(\tau) = 0$ and thus (2.86), (2.87) are satisfied.

The family of charge solitons is obtained from (4.34), (4.35) through a Lorentz boost with velocity $\mathbf{u} = (\gamma, \gamma\boldsymbol{v})$. They are labeled by their center at $t = 0$, set equal to zero here, by the velocity $\boldsymbol{v}$, and by their angular velocity $\boldsymbol{\omega}$. Explicitly we have

$$\phi(\boldsymbol{x}, t) = \phi_{v,\omega}(\boldsymbol{x} - \boldsymbol{v}t), \quad \boldsymbol{A}(\boldsymbol{x}, t) = A_{v,\omega}(\boldsymbol{x} - \boldsymbol{v}t). \tag{4.40}$$

Because of the convolution structure $\phi_{v,\omega}$, $\boldsymbol{A}_{v,\omega}$ are more easily written in Fourier space, where

$$\widehat{\phi}_{v,\omega}(\boldsymbol{k}) = \frac{e}{\boldsymbol{k}^2 - (\boldsymbol{k}\cdot\boldsymbol{v})^2}\left[\widehat{\varphi}(D^{-1}\boldsymbol{k}) + \boldsymbol{v}\cdot(\boldsymbol{\omega} \times \mathrm{i}\nabla_{\boldsymbol{k}}\widehat{\varphi})(D^{-1}\boldsymbol{k})\right], \tag{4.41}$$

$$\begin{aligned}\widehat{\boldsymbol{A}}_{v,\omega}(\boldsymbol{k}) = \frac{e}{\boldsymbol{k}^2 - (\boldsymbol{k}\cdot\boldsymbol{v})^2}\Big[&\boldsymbol{v}\widehat{\varphi}(D^{-1}\boldsymbol{k}) + \frac{1}{\gamma}(\boldsymbol{\omega} \times \mathrm{i}\nabla_{\boldsymbol{k}}\widehat{\varphi})(D^{-1}\boldsymbol{k}) \\ &+ \frac{\gamma\boldsymbol{v}}{1+\gamma}\boldsymbol{v}\cdot(\boldsymbol{\omega} \times \mathrm{i}\nabla_{\boldsymbol{k}}\widehat{\varphi})(D^{-1}\boldsymbol{k})\Big]\end{aligned} \tag{4.42}$$

with $D^{-1}\boldsymbol{k} = \boldsymbol{k} - (\gamma^{-1} - 1)(\widehat{\boldsymbol{v}} \cdot \boldsymbol{k})\widehat{\boldsymbol{v}}$. We note that (4.40), (4.42) coincide with (4.5), (4.6) for $\boldsymbol{\omega} = 0$ and $D = 1$. Put differently, (4.40) and (4.42) properly incorporate the Lorentz contraction of the charge distribution and the extra fields due to the nonvanishing magnetic moment. To obtain the energy–momentum relation we only have to compute the energy of the soliton in its rest frame. By rotation invariance, this energy depends on $\boldsymbol{\omega}$ through its absolute value $\omega = |\boldsymbol{\omega}|$. From (2.89) the bare gyrational mass of the particle is given by

$$\begin{aligned} m_{\mathrm{g}}(\omega) &= m_{\mathrm{b}} \int \mathrm{d}^3 x \varphi(\boldsymbol{x})(1 - |\boldsymbol{\omega}_{\mathrm{E}} \times \boldsymbol{x}|^2)^{-1/2} \\ &= m_{\mathrm{b}} \int_0^\infty \mathrm{d}r 4\pi r^2 \varphi_{\mathrm{r}}(r) \frac{1}{\omega r} \operatorname{arctanh} \omega r \,. \end{aligned} \tag{4.43}$$

The field energy is defined through

$$m_{\mathrm{f}} = \frac{1}{2} \int \mathrm{d}^3 x (\boldsymbol{E}^2 + \boldsymbol{B}^2) \,. \tag{4.44}$$

Inserting from (4.34), (4.35) results in

$$m_{\mathrm{f}}(\omega) = \frac{1}{2} e^2 \int \mathrm{d}^3 k |\widehat{\varphi}|^2 \frac{1}{\boldsymbol{k}^2} + \frac{1}{3} \omega^2 e^2 \int \mathrm{d}^3 k |\nabla_{\boldsymbol{k}} \widehat{\varphi}|^2 \frac{1}{\boldsymbol{k}^2} \,. \tag{4.45}$$

Thus the charge soliton carries the energy

$$m_{\mathrm{s}}(\omega) = m_{\mathrm{g}}(\omega) + m_{\mathrm{f}}(\omega) \tag{4.46}$$

and its energy–momentum relation is necessarily relativistic,

$$E = (\boldsymbol{p}^2 + m_{\mathrm{s}}^2)^{1/2} \,. \tag{4.47}$$

The rotational degrees of freedom are handled in the same spirit. The charge distribution carries the magnetic moment defined in (4.37). μ sets the rotational coupling to the electromagnetic field. Like the charge, it is not renormalized through the interaction with the field. According to (2.93), (2.94), the bare angular momentum of the particle is

$$\mathbf{s}_{\mathrm{b}} = I_{\mathrm{b}}(\omega) \mathbf{w}_{\mathrm{E}} \,, \tag{4.48}$$

where

$$I_{\mathrm{b}}(\omega) = m_{\mathrm{b}} \int_0^\infty \mathrm{d}r 4\pi r^2 \varphi_{\mathrm{r}}(r) \frac{1}{2\omega^2} \Big(-1 + \frac{1 + \omega^2 r^2}{\omega r} \operatorname{arctanh} \omega r \Big) \,. \tag{4.49}$$

In addition, the soliton carries a field angular momentum defined by

$$\boldsymbol{s}_{\mathrm{f}} = \int \mathrm{d}^3 x \boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{B}) \tag{4.50}$$

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with $\boldsymbol{E}$, $\boldsymbol{B}$ in their rest frame inserted from (4.34), (4.35). One obtains

$$\mathbf{s}_{\mathrm{f}} = I_{\mathrm{f}}\mathbf{w}_{\mathrm{E}}\,, \quad I_{\mathrm{f}} = \frac{2}{3}e^2 \int \mathrm{d}^3k |\nabla_{\boldsymbol{k}}\widehat{\varphi}|^2 \frac{1}{\boldsymbol{k}^2}\,. \tag{4.51}$$

Thus the charge soliton carries the spin

$$\mathbf{s}_{\mathrm{s}} = \mathbf{s}_{\mathrm{b}} + \mathbf{s}_{\mathrm{f}} = (I_{\mathrm{b}}(\omega) + I_{\mathrm{f}})\mathbf{w}_{\mathrm{E}}\,. \tag{4.52}$$

4.3 The limit of zero bare mass

The bare mass seems to be an artifact of the theory, since there is no way to determine its value through experiments involving only electromagnetic forces (unless the charge distribution could be probed). Thus a natural and conceptually attractive proposal is to take $m_{\mathrm{b}} = 0$, thereby declaring all mass to be of electromagnetic origin. We discuss here the limit $m_{\mathrm{b}} \to 0_+$ on the level of the energy–momentum relation, whereas the correct procedure would be to study this limit on the level of a solution to the evolution equations. The problem remains unexplored, since for the equations of motion zero bare mass is rather singular.

(i) *Abraham model*. Since m_{s} is additive, the only choice is simply to set $m_{\mathrm{b}} = 0$. In particular, the kinetic energy equals $\frac{1}{2}(\frac{4}{3}m_{\mathrm{f}})\boldsymbol{v}^2$ for small velocities. If we equate $4m_{\mathrm{f}}/3$ with m_{exp}, the experimental mass of the electron, we conclude that $R_\varphi \cong r_{\mathrm{cl}} = 3 \times 10^{-13}$ cm with a prefactor which depends on the choice of the form factor $\widehat{\varphi}$.

(ii) *Lorentz model*. Since m_{g} depends on ω, the Lorentz model offers more variety. We recall Eq. (4.43). If the integral is bounded, which in particular is the case for φ bounded and $\omega R_\varphi \le 1$, then m_{g} vanishes in the limit $m_{\mathrm{b}} \to 0$. We conclude that $m_{\mathrm{s}} = m_{\mathrm{f}}(\omega)$ and $I_{\mathrm{s}} = I_{\mathrm{f}}$. A novel situation occurs if the integral in (4.43) can be made to diverge, for which we must choose φ to be well concentrated at the sphere with radius R_φ. To be concrete let us set $R = R_\varphi$ and $\varphi(\boldsymbol{x}) = \delta(|\boldsymbol{x}| - R)(4\pi R^2)^{-1}$. We also reintroduce c. Then the integral in (4.43) becomes

$$m_{\mathrm{g}}(\omega) = m_{\mathrm{b}}\frac{c}{\omega R}\ \mathrm{arctanh}\ \frac{\omega R}{c}\,. \tag{4.53}$$

We let $\omega R/c \to 1$ and $m_{\mathrm{b}} \to 0$ such that

$$m_{\mathrm{b}}\ \mathrm{arctanh}\frac{\omega R}{c} \to \bar{m} \tag{4.54}$$

with $\bar{m} \ge 0$ still at our choice. Note that in this limit the equator rotates with the speed of light. For the mass, moment of inertia, and magnetic moment of the

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soliton, one obtains, respectively,

$$m_{\mathrm{s}} = \bar{m} + \frac{11}{18}\frac{e^2}{c^2 R}\,, \quad I_{\mathrm{s}} = \frac{2}{3}\bar{m}cR + \frac{2}{9}\frac{e^2}{c}\,, \quad \mu = \frac{1}{3}eR\,, \tag{4.55}$$

which leaves us with R and $\bar{m}$ as free parameters. They can be fitted through the experimentally determined mass and gyromagnetic ratio of the electron. While for the mass we simply set $m_{\mathrm{s}} = m_{\mathrm{exp}}$, the g-factor requires a more elaborate discussion which will be taken up in section 10.1.

Notes and references

Section 4.1

Abraham (1905) computes the energy–momentum relation in essence along the same lines as outlined here (except for the variational characterization). Sommerfeld (1905) uses the expansion of the exact self-force, as will be explained in chapter 7. Lorentz (1904a) proposes a model charge which relativistically contracts parallel to its momentary velocity. Thus provisionally we replace the charge distribution $e\varphi(\boldsymbol{x})$ by its Lorentz contracted version

$$\varphi_{\mathrm{L}}(\boldsymbol{x}) = \gamma\varphi_{\mathrm{r}}([\boldsymbol{x}^2 + \gamma^2(\boldsymbol{x}\cdot\boldsymbol{v})^2]^{1/2})\,, \quad \widehat{\varphi}_{\mathrm{L}}(\boldsymbol{k}) = \widehat{\varphi}_{\mathrm{r}}([\boldsymbol{k}^2 - (\boldsymbol{v}\cdot\boldsymbol{k})^2]^{1/2})\,. \tag{4.56}$$

This expression is substituted in (4.5) and gives the electromagnetic fields comoving with the charge at velocity $\boldsymbol{v}$. Their energy and momentum are computed as before with the result

$$\boldsymbol{P}_{\mathrm{L}}(\boldsymbol{v}) = \boldsymbol{v}\Big(m_{\mathrm{b}}\gamma(\boldsymbol{v}) + \frac{4}{3}m_{\mathrm{f}}\gamma(\boldsymbol{v})\Big)\,, \tag{4.57}$$

$$E_{\mathrm{L}}(\boldsymbol{v}) = m_{\mathrm{b}}\gamma(\boldsymbol{v}) + m_{\mathrm{f}}\gamma(\boldsymbol{v})\Big(1 + \frac{1}{3}\boldsymbol{v}^2\Big)\,. \tag{4.58}$$

The momentum has the anticipated form, except for the factor 4/3 which should be 1. The energy has an unwanted $\boldsymbol{v}^2/3$. In particular the relation (4.16) does not hold, which implies that the power equation $\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{L}}(\boldsymbol{v})$ differs from the force equation $\boldsymbol{v}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P}_{\mathrm{L}}(\boldsymbol{v})$. We refer to Yaghjian (1992) for a thorough discussion, which however somehow misses step zero, namely to specify a relativistically covariant model for an extended charge, as, e.g., in section 2.5. Schott (1912, 1915) employs a deformable elastic medium as a model charge. To compute the velocity-dependent mass he uses essentially the same method as Sommerfeld, an exact self-force and an expansion in the charge diameter. Schott considers also electron models different from those of Abraham and Lorentz. Reviews are Neumann (1914) and Richardson (1916).

There have been various attempts to improve on the oversimplistic version (4.56) of the Lorentz model. Fermi (1922) argues that in a relativistic theory energy and momentum have to be redefined. His argument has been rediscovered several times and is explained in Rohrlich (1990). Poincaré (1906) takes the elastic stresses into account. We refer to Rohrlich (1960) and Yaghjian (1992), and the instructive example by Schwinger (1983).

Section 4.2

Since the Lorentz model is defined through a Lagrangian, the total energy and momentum are determined from Noether's theorem for space-time translations. The transformation as a four-vector is then automatically guaranteed, a property which we used in the computation of the soliton mass.

Section 4.3

The limit of zero bare mass is discussed in Appel and Kiessling (2001).