A NOTE ON ABSOLUTE GEOMETRY

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Metric axioms have been given in [3] for space euclidean geometry. If we replace the "similarity axiom" by the "congruence axiom", where congruence is defined to be a similarity of ratio one, the resulting structure is absolute geometry. In order to show this we choose a suitable definition for absolute geometry. The Pasch system of axioms, given in an improved formulation by H. S. M. Coxeter in [4], is particularly suitable; the primitive notions are points, betweenness relation, and congruence relation. We can verify that every axiom for the absolute geometry in [4] in a theorem in [3] where the similarity axiom has been replaced by the congruence axiom. The only case for which it is not obvious is axiom 15.15 in [4] which says that if ABC and A'B'C' are two triangles with $BC \equiv B'C'$, $CA \equiv C'A'$, $AB \equiv A'B'$, while D and D' are two further points such that [B, C, D] and [B', C', D'] and $BD \equiv B'D'$, then $AD \equiv A'D'$. In that case we first prove that if two triangles ABC and A'B'C' are such that $AB/A'B' \equiv BC/B'C' \equiv CA/C'A' = 1$ then they are congruent; a proof of this, independent of the similarity axiom, can be found in [2]. The proof of 15.15 in [4] is then obvious. As every axiom in the weakened structure of [3] is a theorem of absolute geometry we have a definition for this geometry.

A system of axioms for space euclidean geometry based on coordinate functions, particularly economical in the number of axioms, has been given by the author in [2]. The same weakening in the similarity axiom gives also a good definition for absolute geometry. In fact the weakened structures of [2] and [3] are equivalent. To show this equivalence we define in [3] the coordinate functions ψ for the elements of an arbitrary bundle of rays. The definitions and notations used here are as in [3]. If, o, a are two non-collinear rays of a bundle, then we define

$$\psi(x) \tilde{=} 0 \quad \text{if} \quad x = o$$

$$\psi(x) \tilde{=} \angle\, ox \quad \text{if} \quad x \in HB_{oa} \quad (HB_{oa} \text{ is the half-bundle } oa),$$

$$\psi(x) \tilde{=} \pi \quad \text{if} \quad x = \bar{o} \quad (\bar{o} \text{ is the ray opposite to } o),$$

$$\psi(x) \tilde{=} -\angle\, ox \quad \text{if} \quad x \in \overline{HB}_{oa} \; (\overline{HB}_{oa} \text{ is the half-bundle opposite to } HB_{oa}).$$

where $a \tilde{=} b$ stands for $a = b \pmod{2\pi}$ (π and 180 are positive real numbers playing the same rôle in [2] and [3]). We shall now prove that axiom CB_1 in [2] is a valid sentence of the weakened structure of [3].

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LEMMA 1. If $H_{oa} = H_{oa'}$, and if $\psi_i(x)$ is a coordinate function for H_{oa} defined with HB_{oa}, and if $\psi_j(x)$ is a coordinate function for H_{oa} defined with $HB_{oa'}$, then

$$\psi_j(x) \cong \pm \psi_i(x) \quad \text{for all} \quad x \in H_{oa}.$$

This is a consequence of the definitions for $\psi_i(x)$, $\psi_j(x)$, and $\overline{HB}_{oa}$.

LEMMA 2. If $H_{oa} = H_{o'a}$, if $\psi_i(x)$ is a coordinate function for H_{oa} defined with HB_{oa}, and if $\psi_j(x)$ is a coordinate function for H_{oa} defined with $HB_{o'a}$, then

$$\psi_j(x) \cong \pm \psi_i(x) \pm \measuredangle oo' \quad \text{for all} \quad x \in H_{oa},$$

the signs being fixed for a given $o' \in H_{oa}$.

Using the property "$[a, b, c]$ implies $\measuredangle ab + \measuredangle bc = \measuredangle ac$", the proof is obtained by considering the following cases.

If $o' = o$, then $\psi_j(x) \cong \psi_i(x)$;

if $o' = \bar{o}$, then $\psi_j(x) \cong - \psi_j(x) + \measuredangle oo'$;

if $o' \in HB_{oa}$ and $[o', a, \bar{o}]$, then $\psi_j(x) \cong + \psi_i(x) - \measuredangle oo'$;

if $o' \in HB_{oa}$ and $[o, a, o']$, then $\psi_j(x) \cong - \psi_i(x) + \measuredangle oo'$;

if $o' \in \overline{HB}_{oa}$ and $[\overline{o'}, a, o]$, then $\psi_j(x) \cong + \psi_i(x) + \measuredangle oo'$;

if $o' \in \overline{HB}_{oa}$ and $[\bar{o}, a, \overline{o'}]$, then $\psi_j(x) \cong - \psi_i(x) - \measuredangle oo'$.

LEMMA 3. If $H_{oa} = H_{o'a'}$, if $\psi_i(x)$ is a coordinate function for H_{oa} defined with HB_{oa}, and if $\psi_j(x)$ is a coordinate function for H_{oa} defined with $HB_{o'a'}$, then

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$$\psi_j(x) \cong \pm \psi_i(x) \pm \angle oo' \quad \text{for all } x \in H_{oa},$$

the signs being fixed for given non-collinear rays $o', a' \in H_{oa}$.

If o and a' are non-collinear then $H_{oa} = H_{oa'} = H_{o'a'}$ and Lemmas 1 and 2 imply the required result. If o and a' are collinear, then there exists a ray a'' in H_{oa}, non-collinear with o and a', and non-collinear with o'. Consequently $H_{oa} = H_{oa''} = H_{o'a''} = H_{o'a'}$, and again Lemmas 1 and 2 imply the required result. Lemma 3 implies immediately that if ψ_i and ψ_j are coordinate functions for a bundle H, then $\psi_i(x) - \psi_i(y) \cong \pm (\psi_j(x) - \psi_j(y))$ for all $x, y \in H$ and we have:

LEMMA 4. *Axiom* CB_1 *in* [2] *is a theorem of the weakened structure of* [3].

All the other axioms of the weakened structure of [2] are valid sentences in the weakened structure of [3], and as all the axioms of the weakened structure of [3] are properties of absolute geometry, it follows that the weakened structures are equivalent. Consequently they are both adequate definitions for the absolute geometry. Furthermore it is well known that absolute geometry has only two models, the euclidean one and the hyperbolic one. Euclidean geometry can be characterized by "the existence of at least one proper similitude" and hyperbolic geometry can be characterized by the negation of this sentence. We have then:

THEOREM. *If in* [2] *or* [3] *the similarity axiom is replaced by the congruence axiom, the resulting structure is absolute geometry. The structure is euclidean if there exists at least one proper similitude and hyperbolic if no proper similitude exists.*

REFERENCES

1. K. Borsuk and W. Szmielew, Foundations of geometry (Amsterdam 1960).

2. R. Brossard, Birkhoff's axioms for space geometry. Amer. Math. Monthly 71 (1964) 593-606.

https://doi.org/10.4153/CMB-1968-086-5 Published online by Cambridge University Press

3. R. Brossard, Metric axioms for space geometry. Amer. Math. Monthly 74 (1967) 777-788.

4. H.S.M. Coxeter, Introduction to geometry (John Wiley 1961).

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