

AUTOMORPHISMS OF NORMAL TRANSFORMATION SEMIGROUPS

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1. Introduction and preliminaries

Let X be an infinite set, \mathcal{G}_X be the group of all bijections of X and S be a semigroup of total transformations of X with the composition of transformations f and g in S defined by the formula

$$fg(x) = f(g(x)), \quad \text{where } x \in X.$$

We say that S is a \mathcal{G}_X -normal semigroup if

$$hSh^{-1} = S, \quad \text{for all } h \in \mathcal{G}_X.$$

The full transformation semigroup T_X , the semigroups of all 1–1 and all onto transformations and the group \mathcal{G}_X itself, are examples of \mathcal{G}_X -normal semigroups.

If S is a \mathcal{G}_X -normal semigroup, then for each $h \in \mathcal{G}_X$, the map ϕ of S given by

$$\phi(f) = hfh^{-1} \quad (f \in S)$$

is an automorphism of S , specifically an *inner automorphism* of S . Our purpose is to prove the following:

Theorem 1.1. *Every automorphism of a \mathcal{G}_X -normal semigroup is inner.*

The subject of this paper was suggested to the author by G. R. Wood.

The question of whether inner automorphisms exhaust all automorphisms of a \mathcal{G}_X -normal semigroup has attracted the attention of a number of authors. In 1937 Schreier [10] was the first to give a positive answer for T_X . Then Malcev [6] extended this result to every ideal of T_X . Next Sullivan [12] generalized this work and confirmed that if a semigroup contains all constant transformations (in particular if a \mathcal{G}_X -normal semigroup contains a constant transformation) then it possesses only inner automorphisms, while Fitzpatrick and Symons [3] showed this for a semigroup containing \mathcal{G}_X . Schein [8, 9] discovered that a \mathcal{G}_X -normal semigroup of 1–1 transformations has only inner automorphisms (see [4] for the special case of Baer–Levi semigroups).

Our result subsumes all previously stated results for \mathcal{G}_X -normal semigroups and describes completely all automorphisms of every \mathcal{G}_X -normal transformation semigroup.

In this paper we use a technique which differs from those used by Sullivan [12] and Schein [8, 9]. The essence is the production of certain maximal right (Section 2) and left (Section 3) ideals. We note a remarkable duality between properties of these right and left ideals.

- For the purpose of our proof we partition all \mathcal{G}_X -normal semigroups into three types:
1. Semigroups containing a constant map; and constant-free semigroups into:
 2. Semigroups of 1–1 transformations; and
 3. Constant-free semigroups containing a transformation which is not 1–1.

All automorphisms of semigroups of the first type are inner [12, Theorem 1], so we can restrict our attention to constant-free semigroups.

We begin with some general notes on \mathcal{G}_X -normal semigroups.

For a function $f: X \rightarrow X$ we denote the *range* of f by $R(f)$ ($= f(X)$) and the *partition* of f by $\pi(f)$ ($= \{f^{-1}(x): x \in R(f)\}$).

If S is an arbitrary semigroup of transformations, let

$$R(S) = \{R(f): f \in S\} \quad \text{and} \quad \pi(S) = \{\pi(f): f \in S\}.$$

We say that $R(S)$ ($\pi(S)$) is *normal* if for each $h \in \mathcal{G}_X$

$$h(R(S)) = R(S) \quad (h(\pi(S)) = \pi(S)),$$

(by $h(R(S))$ we mean $\{h(A): A \in R(S)\}$ and by $h(\pi(S))$ we mean $\{h(\mathcal{A}): \mathcal{A} \in \pi(S)\}$, where $h(\mathcal{A}) = \{h(A): A \in \mathcal{A}\}$).

Lemma 1.2. *If S is a \mathcal{G}_X -normal semigroup, then $R(S)$ and $\pi(S)$ are normal.*

The proof is straightforward. □

We say that a semigroup S is *trivial* if $S = \{\Delta_X\}$, where Δ_X is the identity transformation of X . In what follows S is non-trivial.

Result 1.3. *Every \mathcal{G}_X -normal semigroup S is transitive.*

Proof. Take arbitrary x, y in X . We construct f in S such that $f(x) = y$.

Firstly let x and y be distinct and suppose there exists a $g \in S$ with $g(x) = z \neq x$. If $z = y$ we let $f = g$, otherwise $(y, z)g(y, z)$ is the required f ((y, z) denotes the transposition interchanging y and z). To construct g , observe that since S is non-trivial there exists a $q \in S$ together with distinct u and v in X such that $q(u) = v$. If $u = x$ we let $g = q$, otherwise $g = (u, x)q(u, x)$.

Now suppose $y = x$, choose any p in S and let $p(x) = w$. If $w = x$ we let $f = p$. Otherwise choose $t \in S$ with $t(w) = x$ (using the first part of the proof), then $f = tp$ takes x to x as required. □

Remark 1.4. We exclude from our consideration \mathcal{G}_X -normal subsemigroups of \mathcal{G}_X , since they are all subgroups of \mathcal{G}_X , and hence have only inner automorphisms [11].

2. \mathcal{G}_X -normal semigroups of 1–1 transformations

In this section S denotes a \mathcal{G}_X -normal semigroup of 1–1 transformations.

Definition 2.1. Let $x \in X$ and

$$\mathcal{R}_x = \{r \in S : x \in X \setminus R(r)\}.$$

Then \mathcal{R}_x is a right ideal of S , which we call a *point right ideal*. □

We will use the following observation based on the normality of $R(S)$ (Lemma 1.2) and the fact that S is not a subsemigroup of \mathcal{G}_X , that is $R(S)$ contains proper subsets of X .

Remark 2.2. Given $x, y \in X$ with $x \neq y$ there exists an A in $R(S)$ with $x \in X \setminus A$ and $y \in A$. □

Lemma 2.3. Given $x, y \in X$ the following three statements are equivalent:

- (i) $\mathcal{R}_x \subseteq \mathcal{R}_y$;
- (ii) $x = y$;
- (iii) $\mathcal{R}_x = \mathcal{R}_y$.

Proof. Implications (ii) \Rightarrow (iii) and (iii) \Rightarrow (i) are trivial. We show (i) \Rightarrow (ii). Suppose $x \neq y$ and choose an $A \in R(S)$ with $x \in X \setminus A$, $y \in A$ (Remark 2.2). If $f \in S$ with $R(f) = A$, then $f \in \mathcal{R}_x \setminus \mathcal{R}_y$, so $\mathcal{R}_x \not\subseteq \mathcal{R}_y$, proving (i) \Rightarrow (ii). □

Define a map $\theta: X \rightarrow \{\mathcal{R}_x : x \in X\}$ via $\theta(x) = \mathcal{R}_x$, each $x \in X$.

Lemma 2.4. θ is a bijection.

Proof. Clearly θ is onto and Lemma 2.3 ensures θ is 1–1. □

Definition 2.5. Given distinct $f_1, f_2 \in S$ let

$$\mathcal{R}_{f_1, f_2} = \{r \in S : f_1 r = f_2 r\}.$$

Then \mathcal{R}_{f_1, f_2} is a right ideal of S (possibly empty), which we call a *function right ideal*. □

We will show (Result 2.8) that there always exist distinct f_1, f_2 in S such that \mathcal{R}_{f_1, f_2} is non-empty. However \mathcal{R}_{f_1, f_2} may be empty. Observe that given f_1 and f_2 ,

$$r \in \mathcal{R}_{f_1, f_2} \text{ iff } R(r) \subseteq \{x \in X : f_1(x) = f_2(x)\}.$$

Hence if we choose f_1 and f_2 which are never equal, then $\mathcal{R}_{f_1, f_2} = \Phi$.

Let S , for example, be the Baer–Levi semigroup of type $(|X|, |X|)$ [2], that is the semigroup of all 1–1 transformations f such that $|R(f)| = |X \setminus R(f)| = |X|$. Note that S is

\mathcal{G}_X -normal and choose $f_1 \in S$, then $X \setminus R(f_1) \in R(S)$ (Lemma 1.2). If $f_2 \in S$ with $R(f_2) = X \setminus R(f_1)$, then $\mathcal{R}_{f_1, f_2} = \Phi$.

The following notation applies to an arbitrary \mathcal{G}_X -normal semigroup S .

Notation 2.6. Let f_1, f_2 be distinct transformations in S . Then

$$\mathcal{D}_{f_1, f_2} = \{x \in X : f_1(x) \neq f_2(x)\}$$

and

$$D_{f_1, f_2} = \{\{f_1(x), f_2(x)\} : x \in \mathcal{D}_{f_1, f_2}\}. \quad \square$$

Returning to semigroups of 1–1 transformations, we now derive relationships between point right ideals and function right ideals.

Result 2.7. Let $f_1, f_2 \in S$ with $\mathcal{R}_{f_1, f_2} \neq \Phi$. Then

$$\mathcal{R}_{f_1, f_2} = \bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_x.$$

Proof. Let $r \in \mathcal{R}_{f_1, f_2}$, that is $f_1 r = f_2 r$. If $x \in \mathcal{D}_{f_1, f_2}$, or $f_1(x) \neq f_2(x)$, then $x \in X \setminus R(r)$, so $r \in \mathcal{R}_x$, and since this is true for each $x \in \mathcal{D}_{f_1, f_2}$ we conclude

$$r \in \bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_x$$

or

$$\mathcal{R}_{f_1, f_2} \subseteq \bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_x.$$

Conversely, if

$$r \in \bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_x,$$

then for each y in $R(r)$ we have $y \in X \setminus \mathcal{D}_{f_1, f_2}$, or $f_1(y) = f_2(y)$ and hence $f_1 r = f_2 r$, that is $r \in \mathcal{R}_{f_1, f_2}$, so

$$\bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_x \subseteq \mathcal{R}_{f_1, f_2},$$

which proves the desired equality. □

Result 2.8. Given $x \in X$ there exist $f_1, f_2 \in S$ such that $\mathcal{R}_x = \mathcal{R}_{f_1, f_2}$.

Proof. On account of Result 2.7 it is sufficient to construct f_1, f_2 such that $\mathcal{D}_{f_1, f_2} = \{x\}$.

Observe that there exists an f in S with

$$|X \setminus R(f)| \geq 2.$$

(For an arbitrary f in $S \setminus \mathcal{G}_x$

$$|X \setminus R(f^2)| = |X \setminus R(f)| + |X \setminus R(f)|$$

and we replace f with f^2).

Using the normality of $R(S)$ (Lemma 1.2) choose an f in S with

$$x \in X \setminus R(f) \quad \text{and} \quad |X \setminus R(f)| \geq 2.$$

Let $f(x) = y$ and $z \in X \setminus R(f)$, $z \neq x$. If

$$g = (x, z)f(x, z)$$

then $g(z) = y$ and $z \in X \setminus R(g)$. We let

$$h = (y, z), f_1 = gf \quad \text{and} \quad f_2 = hgh^{-1}f.$$

Then for each $u \neq x$:

$$\begin{aligned} f_1(u) &= gf(u) = gh^{-1}f(u), && \text{since } f(u) \neq y \text{ for } u \neq x \\ & && \text{and } z \notin R(f); \\ & && \text{since } gh^{-1}f(u) \neq y \\ &= hgh^{-1}f(u), && \text{for } f(u) \neq y \\ & && \text{and } z \notin R(g); \\ &= f_2(u). \end{aligned}$$

However

$$f_1(x) = gf(x) = g(y)$$

while

$$f_2(x) = hgh^{-1}f(x) = hgh^{-1}(y) = hg(z) = h(y) = z \neq g(y),$$

since $z \in X \setminus R(g)$. Hence $f_1(x) \neq f_2(x)$ and $\mathcal{D}_{f_1, f_2} = \{x\}$. □

Result 2.9. Given f_1 and f_2 in S , \mathcal{R}_{f_1, f_2} is a maximal function right ideal if and only if $|\mathcal{D}_{f_1, f_2}| = 1$.

Proof. Suppose \mathcal{R}_{f_1, f_2} is a maximal function right ideal, while $x, y \in \mathcal{D}_{f_1, f_2}$, $x \neq y$.

Then

$$\begin{aligned} \mathcal{R}_{f_1, f_2} &= \bigcap_{z \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_z \quad (\text{Result 2.7}) \\ &\subseteq \mathcal{R}_x \cap \mathcal{R}_y \\ &\subsetneq \mathcal{R}_x \quad (\text{Lemma 2.3}). \end{aligned}$$

It follows from Result 2.8 that there exist g_1 and g_2 with

$$\mathcal{R}_{g_1, g_2} = \mathcal{R}_x,$$

and so

$$\mathcal{R}_{f_1, f_2} \subsetneq \mathcal{R}_x = \mathcal{R}_{g_1, g_2},$$

a contradiction to the maximality of \mathcal{R}_{f_1, f_2} . Hence $|\mathcal{D}_{f_1, f_2}| = 1$.

For the converse, suppose $\mathcal{D}_{f_1, f_2} = \{x\}$, some $x \in X$, while there exist $g_1, g_2 \in S$ such that

$$\mathcal{R}_{g_1, g_2} \supsetneq \mathcal{R}_{f_1, f_2}.$$

Since

$$\mathcal{R}_{g_1, g_2} = \bigcap_{y \in \mathcal{D}_{g_1, g_2}} \mathcal{R}_y \quad (\text{Result 2.7})$$

we have

$$\bigcap_{y \in \mathcal{D}_{g_1, g_2}} \mathcal{R}_y = \mathcal{R}_{g_1, g_2} \supsetneq \mathcal{R}_{f_1, f_2} = \mathcal{R}_x \quad (\text{Result 2.7 again}),$$

and so Lemma 2.3 ensures $\mathcal{D}_{g_1, g_2} = \{x\}$, that is

$$\mathcal{R}_{g_1, g_2} = \mathcal{R}_x = \mathcal{R}_{f_1, f_2}. \quad \square$$

Corollary 2.10. *Given f_1 and f_2 in S , \mathcal{R}_{f_1, f_2} is a maximal function right ideal if and only if $\mathcal{R}_{f_1, f_2} = \mathcal{R}_x$, some $x \in X$.*

Proof. Follows from Results 2.7 and 2.9. □

We show now that each automorphism ϕ of S permutes point right ideals.

Result 2.11. *Given $x \in X$,*

$$\phi(\mathcal{R}_x) = \mathcal{R}_y,$$

for some $y \in X$.

Proof. Choose f_1 and f_2 in S such that $\mathcal{R}_{f_1, f_2} = \mathcal{R}_x$ (Result 2.8), then

$$\begin{aligned} \phi(\mathcal{R}_x) &= \phi(\mathcal{R}_{f_1, f_2}) = \phi(\{r: f_1 r = f_2 r\}) \\ &= \{\phi(r): \phi(f_1 r) = \phi(f_2 r)\} \\ &= \{\phi(r): \phi(f_1)\phi(r) = \phi(f_2)\phi(r)\} \\ &= \{r': \phi(f_1)r' = \phi(f_2)r'\} \\ &= \mathcal{R}_{\phi(f_1), \phi(f_2)}. \end{aligned}$$

Now Corollary 2.10 ensures \mathcal{R}_{f_1, f_2} is a maximal function right ideal, hence $\mathcal{R}_{\phi(f_1), \phi(f_2)} (= \phi(\mathcal{R}_{f_1, f_2}))$ is a maximal function right ideal, so there exists $y \in X$ such that

$$\mathcal{R}_{\phi(f_1), \phi(f_2)} = \mathcal{R}_y \quad (\text{Corollary 2.10})$$

and thus

$$\phi(\mathcal{R}_x) = \mathcal{R}_{\phi(f_1), \phi(f_2)} = \mathcal{R}_y. \quad \square$$

Define a map

$$\eta: \{\mathcal{R}_x: x \in X\} \rightarrow \{\mathcal{R}_x: x \in X\}$$

via $\eta(\mathcal{R}_x) = \phi(\mathcal{R}_x)$, each $\mathcal{R}_x \subseteq S$.

Lemma 2.12. η is a bijection.

Proof. That η is a mapping is the content of Result 2.11. Similarly by considering the automorphism ϕ^{-1} we define a map

$$\zeta: \{\mathcal{R}_x: x \in X\} \rightarrow \{\mathcal{R}_x: x \in X\}$$

via $\zeta(\mathcal{R}_x) = \phi^{-1}(\mathcal{R}_x)$, each $\mathcal{R}_x \subseteq S$.

Certainly, ζ is the inverse of η and so η is a bijection. □

We now define a map

$$h: X \rightarrow X \quad \text{via} \quad h(x) = y, \quad \text{where} \quad \eta(\mathcal{R}_x) = \mathcal{R}_y, \quad \text{each} \quad x \in X.$$

It is clear, that

$$h = \theta^{-1} \eta \theta,$$

and so Lemmas 2.4 and 2.12 ensure h is a bijection of X . We call h the *bijection associated with ϕ* .

Lemma 2.13. *Given $f \in S$,*

$$R(\phi(f)) = h(R(f)).$$

Proof. Observe that to show $R(\phi(f)) = h(R(f))$ it is sufficient to show that

$$X \setminus R(\phi(f)) = h(X \setminus R(f)),$$

because for the bijection h , $h(X \setminus R(f)) = X \setminus h(R(f))$.

Now if $x \in X \setminus R(f)$, that is $f \in \mathcal{R}_x$, then $\phi(f) \in \eta(\mathcal{R}_x) = \mathcal{R}_{h(x)}$, so $h(x) \in X \setminus R(\phi(f))$, or

$$h(X \setminus R(f)) \subseteq X \setminus R(\phi(f)).$$

To show the reverse inclusion is true, observe that $h^{-1} = \theta^{-1}\eta^{-1}\theta$ is the bijection associated with ϕ^{-1} and so the first part of the proof implies that given $g \in S$,

$$h^{-1}(X \setminus R(g)) \subseteq X \setminus R(\phi^{-1}(g)).$$

In particular taking $g = \phi(f)$ we have $h^{-1}(X \setminus R(\phi(f))) \subseteq X \setminus R(\phi^{-1}(\phi(f)))$, or

$$h(X \setminus R(f)) \supseteq X \setminus R(\phi(f)),$$

and the equality follows. □

We complete our study of automorphisms of \mathcal{G}_X -normal semigroups of 1–1 transformations, that is, semigroups of Type 2, by presenting the following result.

Result 2.14. *Let S be a \mathcal{G}_X -normal semigroup of 1–1 transformations ($S \not\subseteq \mathcal{G}_X$). Then each automorphism ϕ of S is inner, that is, for some $h \in \mathcal{G}_X$*

$$\phi(f) = hfh^{-1}, \text{ for each } f \in S.$$

Proof. Consider the bijection h associated with ϕ as defined prior to Lemma 2.13. Take an arbitrary $f \in S$, $x \in X$ and let $f(x) = y$. Choose A in $R(S)$ with $A \neq X$ and $x \in A$. Let $z \in X \setminus A$ and $B = (A \setminus \{x\}) \cup \{z\} \in R(S)$ (Lemma 1.2). Choose p and q in S such that $R(p) = A$ and $R(q) = B$.

Now $R(p) \setminus R(q) = A \setminus B = \{x\}$, thus $R(fp) \setminus R(fq) = \{f(x)\} = \{y\}$. Using Lemma 2.13 we have:

$$R(\phi(p)) \setminus R(\phi(q)) = \{h(x)\}$$

and

$$R(\phi(fp)) \setminus R(\phi(fq)) = \{h(y)\}.$$

However

$$\begin{aligned} R(\phi(fp)) \setminus R(\phi(fq)) &= R(\phi(f)\phi(p)) \setminus R(\phi(f)\phi(q)) \\ &= \{\phi(f)h(x)\}, \end{aligned}$$

so

$$\phi(f)h(x) = h(y) = hf(x), \quad \text{that is}$$

$$\phi(f) = hf h^{-1}. \quad \square$$

Remark 2.15. The fact that every \mathcal{G}_X -normal semigroup of 1-1 transformations possesses only inner automorphisms was first established by B. M. Schein [8,9]. We understand that his proof, based on the study of ordered sets of ranges, is quite different from ours. □

3. \mathcal{G}_X -normal constant-free semigroups containing a transformation which is not 1-1

Let S be a \mathcal{G}_X -normal constant-free semigroup containing a transformation which is not 1-1. We prove that all automorphisms of S are inner. We start by showing that $R(S)$ contains only sets of cardinality $|X|$.

Lemma 3.1. *If S is a \mathcal{G}_X -normal constant-free semigroup, then $|R(f)| = |X|$, each $f \in S$.*

Proof. Suppose there is an f in S with $|R(f)| = \alpha < |X|$, that is $|\pi(f)| = |R(f)| = \alpha$. We show that there exists an $A \in \pi(f)$ with $|A| \geq \alpha$. The result is clear when α is finite. Hence assume α is infinite and denote by α^+ the cardinal successor of α . Then either $\alpha^+ = |X|$ (and so $|X|$ is regular [7, 21.14]) or there exists $\beta < |X|$, $\beta = \alpha^+$ (and so β is regular [7, 21.14]). The assumption that each $A \in \pi(f)$ has a cardinality less than α implies that $|\cup \pi(f)| < |X|$ or $|\cup \pi(f)| < \beta < |X|$ respectively [7, 21.18], a contradiction. Hence we can choose an $A \in \pi(f)$ with $|A| \geq \alpha$ and a $B \in R(S)$ with $B \subseteq A$ and $|B| = \alpha$ (Lemma 1.2) together with a $g \in S$ such that $R(g) = B$. Then $|R(fg)| = 1$, so that fg is a constant map in S , a contradiction which proves $|R(f)| = |X|$. □

Let \mathcal{P}_2 be the set of all doubletons in X .

Definition 3.2. Given $A \in \mathcal{P}_2$, $A = \{a_1, a_2\}$, let

$$\mathcal{L}_A = \{l \in S : l(a_1) = l(a_2)\}.$$

Then \mathcal{L}_A is a left ideal of S which we call a *point left ideal*. □

Lemma 3.3. *For each $A \in \mathcal{P}_2$, $\mathcal{L}_A \neq \Phi$.*

Proof. Choose a map f in S which is not 1-1, say $f(x) = f(y)$ for distinct $x, y \in X$. If $h \in \mathcal{G}_X$ is such that $\{h(x), h(y)\} = A$ then $hf h^{-1} \in \mathcal{L}_A$. □

Lemma 3.4. *Given $A, B \in \mathcal{P}_2$, the following three statements are equivalent:*

- (i) $\mathcal{L}_A \subseteq \mathcal{L}_B$;
- (ii) $A = B$;
- (iii) $\mathcal{L}_A = \mathcal{L}_B$.

Proof. Implications (ii) \Rightarrow (iii) and (iii) \Rightarrow (i) are trivial. We show (i) \Rightarrow (ii).

Let $B = \{b_1, b_2\}$ and suppose $A \neq B$, say $b_1 \in B \setminus A$. Choose an $l \in \mathcal{L}_A$ (Lemma 3.3) and let $x \in R(l) \setminus l(A \cup B)$ (note: $|X| = |R(l)| > |l(A \cup B)|$, Lemma 3.1). If $y \in X$ is such that $l(y) = x$, let $h = (b_1, y)$ and $f = hlh^{-1}$. We show $f \in \mathcal{L}_A \setminus \mathcal{L}_B$. That $f \in \mathcal{L}_A$ follows from the fact that h moves only points b_1 and y , which are not in A . To show that $f \notin \mathcal{L}_B$, observe that $f(b_1) = hlh^{-1}(b_1) = hl(y) = h(x)$, while $f(b_2) = hlh^{-1}(b_2) = hl(b_2)$, because $b_2 \neq y$ (else $x = l(y) = l(b_2) = l(B)$, contrary to the choice of x). Hence $f(b_2) \neq h(x)$, because $l(b_2) = l(B) \neq x$. Thus $f(b_1) \neq f(b_2)$ and $f \in \mathcal{L}_B$.

Define a map $\delta: \mathcal{P}_2 \rightarrow \{\mathcal{L}_A : A \in \mathcal{P}_2\}$ via $\delta(A) = \mathcal{L}_A$, each $A \in \mathcal{P}_2$.

Lemma 3.5. *δ is a bijection.*

Proof. Clearly δ is onto and Lemma 3.4 ensures δ is 1–1. □

Definition 3.6. Given distinct $f_1, f_2 \in S$ let

$$\mathcal{L}_{f_1, f_2} = \{l \in S : lf_1 = lf_2\}.$$

Then \mathcal{L}_{f_1, f_2} is a left ideal of S (possibly empty, see Example 3.7 below), which we call a *function left ideal*. □

We will show (Result 3.10) that for each \mathcal{G}_X -normal constant-free semigroup S containing a transformation which is not 1–1 there exist $f_1, f_2 \in S$ with $\mathcal{L}_{f_1, f_2} \neq \Phi$. In general, the question of whether $f_1, f_2 \in S$ generate a non-empty \mathcal{L}_{f_1, f_2} is the question of whether the equation $lf_1 = lf_2$ has a solution l in S . The example below illustrates that \mathcal{L}_{f_1, f_2} may be empty.

Example 3.7. Let S be the dual Baer-Levi semigroup of the type $(|X|, |X|)$ [1], that is the semigroup of all onto mappings f such that $|f^{-1}(x)| = |X|$ for each $x \in X$. Certainly S is \mathcal{G}_X -normal. Assume $X = \mathbb{N}$, so that $|X| = \aleph_0$. Fix an arbitrary $f_1 \in S$ and let

$$\mathcal{A} = \pi(f_1) = \{A_1, A_2, A_3, \dots\}$$

Partition each $A_i \in \mathcal{A}$ such that $A_i = A'_i \dot{\cup} A''_i$, $|A'_i| = |A''_i| = \aleph_0$. Let \mathcal{B} be the partition of X

given by

$$\mathcal{B} = \{A'_1, A''_1 \dot{\cup} A'_2, A''_2 \dot{\cup} A'_3, \dots\}.$$

Since \mathcal{B} is a partition of X into \aleph_0 sets, each of cardinality \aleph_0 , $\mathcal{B} \in \pi(S)$, and so there exists $f_2 \in S$ with $\pi(f_2) = \mathcal{B}$. Suppose $l \in \mathcal{L}_{f_1, f_2}$ that is $lf_1 = lf_2$ and let $l_1(A_1) = x$. Then because of the choice of \mathcal{B} we have the following chain of equalities:

$$x = lf_1(A_1) = lf_1(A''_1) = lf_2(A''_1) = lf_2(A'_2) = lf_1(A'_2) = lf_1(A_2) = \dots$$

thus

$$x = lf_1(A_1) = lf_1(A_2) = \dots,$$

that is $R(lf_1) = \{x\}$ and lf_1 is a constant in S , contradicting the construction of S , so that $\mathcal{L}_{f_1, f_2} = \Phi$. □

Recall that \mathcal{D}_{f_1, f_2} and D_{f_1, f_2} (Notation 2.6) were defined for an arbitrary \mathcal{G}_X -normal semigroup S ($f_1, f_2 \in S$). The following remark is an immediate consequence of the definition of D_{f_1, f_2} .

Remark 3.8. Let $f_1, f_2 \in S$, then $D_{f_1, f_2} \subseteq \mathcal{P}_2$. □

We proceed with two results deriving relationships between point left ideals and function left ideals.

Result 3.9. Let f_1 and f_2 be distinct elements of S , and $\mathcal{L}_{f_1, f_2} \neq \Phi$. Then

$$\mathcal{L}_{f_1, f_2} = \bigcap_{A \in D_{f_1, f_2}} \mathcal{L}_A.$$

Proof. Let $l \in \mathcal{L}_{f_1, f_2}$, that is, $lf_1 = lf_2$ and so for each $x \in \mathcal{D}_{f_1, f_2}$ we have $lf_1(x) = lf_2(x)$ (recall $f_1(x) \neq f_2(x)$) so $l \in \mathcal{L}_{(f_1(x), f_2(x))}$ and since this is true for each $x \in \mathcal{D}_{f_1, f_2}$ we conclude

$$l \in \bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{L}_{(f_1(x), f_2(x))} = \bigcap_{A \in D_{f_1, f_2}} \mathcal{L}_A, \text{ or } \mathcal{L}_{f_1, f_2} \subseteq \bigcap_{A \in D_{f_1, f_2}} \mathcal{L}_A.$$

Conversely, let $l \in \bigcap_{A \in D_{f_1, f_2}} \mathcal{L}_A$, then for each $x \in \mathcal{D}_{f_1, f_2}$, $lf_1(x) = lf_2(x)$. Now for each $y \notin \mathcal{D}_{f_1, f_2}$ we have $f_1(y) = f_2(y)$, so we deduce $lf_1 = lf_2$. That is,

$$l \in \mathcal{L}_{f_1, f_2} \text{ and } \bigcap_{A \in D_{f_1, f_2}} \mathcal{L}_A \subseteq \mathcal{L}_{f_1, f_2},$$

which proves the desired equality. □

Result 3.10 Given an $A \in \mathcal{P}_2$, there exist f_1 and f_2 in S such that

$$\mathcal{L}_A = \mathcal{L}_{f_1, f_2}.$$

Proof. On account of Result 3.9 it is sufficient to construct f_1 and f_2 such that

$$D_{f_1, f_2} = \{A\}.$$

Choose an f in \mathcal{L}_A (Lemma 3.3) and let $f(A) = z$. Let $A = \{a_1, a_2\}$. Since S is transitive (Result 1.3) there exists g in S such that $g(z) = a_1$. Let $h = (a_1, a_2)$ and

$$f_1 = gf; \quad f_2 = hf_1h^{-1}.$$

Since h moves only points in A and $f_1 \in \mathcal{L}_A$ (\mathcal{L}_A is a left ideal), we conclude $f_2 = hf_1$. For each $x \in X \setminus f_1^{-1}(A)$ we have:

$$f_1(x) = hf_1(x) = f_2(x),$$

so $\mathcal{D}_{f_1, f_2} \subseteq f_1^{-1}(A)$. Now if $x \in f_1^{-1}(A)$, that is $f_1(x) = a_i, i = 1, 2$, then

$$f_1(x) = a_i \neq h(a_i) = hf_1(x) = f_2(x),$$

hence $\mathcal{D}_{f_1, f_2} \supseteq f_1^{-1}(A)$. We conclude

$$\mathcal{D}_{f_1, f_2} = f_1^{-1}(A).$$

Thus

$$\begin{aligned} D_{f_1, f_2} &= \{\{f_1(x), f_2(x)\} : x \in \mathcal{D}_{f_1, f_2}\} \quad (\text{Notation 2.6}) \\ &= \{\{f_1(x), f_2(x)\} : x \in f_1^{-1}(A)\} \\ &= \{\{a_i, h(a_i)\} : i = 1, 2\} \\ &= \{\{a_1, a_2\}\} \\ &= \{A\}, \end{aligned}$$

as required. □

Result 3.11. Given distinct f_1 and f_2 in S , \mathcal{L}_{f_1, f_2} is a maximal function left ideal if and only if $|D_{f_1, f_2}| = 1$.

Proof. Let \mathcal{L}_{f_1, f_2} be a maximal function left ideal and suppose $A, B \in D_{f_1, f_2}, A \neq B$.

Then $A, B \in \mathcal{P}_2$ (Remark 3.8). Hence

$$\begin{aligned} \mathcal{L}_{f_1, f_2} &= \bigcap_{C \in D_{f_1, f_2}} \mathcal{L}_C \quad (\text{Result 3.9}) \\ &\subseteq \mathcal{L}_A \cap \mathcal{L}_B \\ &\subseteq \mathcal{L}_A \quad (\text{Lemma 3.4}) \\ &= \mathcal{L}_{g_1, g_2} \quad (\text{Result 3.10}), \end{aligned}$$

for some distinct $g_1, g_2 \in S$, contradicting the maximality of \mathcal{L}_{f_1, f_2} . Hence $|D_{f_1, f_2}| = 1$.

Conversely, suppose $D_{f_1, f_2} = \{A\}$, some $A \in \mathcal{P}_2$, while there exists a function left ideal \mathcal{L}_{g_1, g_2} ($g_1, g_2 \in S$) such that

$$\mathcal{L}_{g_1, g_2} \supseteq \mathcal{L}_{f_1, f_2}.$$

Since $\mathcal{L}_{g_1, g_2} = \bigcap_{B \in D_{g_1, g_2}} \mathcal{L}_B$ (Result 3.9) we have

$$\bigcap_{B \in D_{g_1, g_2}} \mathcal{L}_B = \mathcal{L}_{g_1, g_2} \supseteq \mathcal{L}_{f_1, f_2} = \mathcal{L}_A \quad (\text{Result 3.9 again}),$$

and so Lemma 3.4 ensures $D_{g_1, g_2} = \{A\}$, that is

$$\mathcal{L}_{g_1, g_2} = \mathcal{L}_A = \mathcal{L}_{f_1, f_2}. \quad \square$$

Corollary 3.12. *Given f_1 and f_2 is S , \mathcal{L}_{f_1, f_2} is a maximal left function ideal if and only if $\mathcal{L}_{f_1, f_2} = \mathcal{L}_A$, some $A \in \mathcal{P}_2$.*

Proof. Follows from Results 3.9 and 3.11. □

We show now that each automorphism ϕ of S permutes point left ideals.

Result 3.13. *Given $A \in \mathcal{P}_2$,*

$$\phi(\mathcal{L}_A) = \mathcal{L}_B,$$

for some $B \in \mathcal{P}_2$.

Proof. Choose f_1 and f_2 in S such that $\mathcal{L}_{f_1, f_2} = \mathcal{L}_A$ (Result 3.10), then

$$\begin{aligned} \phi(\mathcal{L}_A) &= \phi(\mathcal{L}_{f_1, f_2}) = \phi(\{l: lf_1 = lf_2\}) \\ &= \{\phi(l): \phi(lf_1) = \phi(lf_2)\} \\ &= \{\phi(l): \phi(l)\phi(f_1) = \phi(l)\phi(f_2)\} \\ &= \{l': l'\phi(f_1) = l'\phi(f_2)\} \\ &= \mathcal{L}_{\phi(f_1), \phi(f_2)}. \end{aligned}$$

Now Corollary 3.12 ensures \mathcal{L}_{f_1, f_2} is a maximal function left ideal, hence $\mathcal{L}_{\phi(f_1), \phi(f_2)}$ ($= \phi(\mathcal{L}_{f_1, f_2})$) is a maximal function left ideal, so there exists $B \in \mathcal{P}_2$ such that

$$\mathcal{L}_{\phi(f_1), \phi(f_2)} = \mathcal{L}_B \quad (\text{Corollary 3.12}).$$

We conclude

$$\phi(\mathcal{L}_A) = \mathcal{L}_{\phi(f_1), \phi(f_2)} = \mathcal{L}_B. \quad \square$$

Define a map

$$\mu: \{\mathcal{L}_A: A \in \mathcal{P}_2\} \rightarrow \{\mathcal{L}_A: A \in \mathcal{P}_2\}$$

via $\mu(\mathcal{L}_A) = \phi(\mathcal{L}_A)$, each $\mathcal{L}_A \subseteq S$.

Lemma 3.14. μ is a bijection.

Proof. That μ is a mapping is the content of Result 3.13. Similarly by considering the automorphism ϕ^{-1} we define a map

$$\xi: \{\mathcal{L}_A: A \in \mathcal{P}_2\} \rightarrow \{\mathcal{L}_A: A \in \mathcal{P}_2\}$$

via $\xi(\mathcal{L}_A) = \phi^{-1}(\mathcal{L}_A)$, each $\mathcal{L}_A \subseteq S$. Certainly, ξ is the inverse of μ and so μ is a bijection. \square

We now define a map

$$\lambda: \mathcal{P}_2 \rightarrow \mathcal{P}_2 \quad \text{via} \quad \lambda(A) = B, \quad \text{where} \quad \mu(\mathcal{L}_A) = \mathcal{L}_B, \quad \text{each} \quad A \in \mathcal{P}_2.$$

It is clear that

$$\lambda = \delta^{-1} \mu \delta,$$

and so Lemmas 3.5 and 3.14 ensure λ is a bijection of \mathcal{P}_2 . We call λ the *bijection of \mathcal{P}_2 associated with ϕ* .

We show that λ is *induced* by a bijection h of X , that is

$$\lambda(A) = h(A),$$

for each $A \in \mathcal{P}_2$. Note here that not every bijection of \mathcal{P}_2 is induced, as shown in Example 3.15 below.

Example 3.15. Fix A and C in \mathcal{P}_2 , $A \neq C$ and let λ be a bijection of \mathcal{P}_2 , which interchanges A and C and the identity otherwise. Choose $B \in \mathcal{P}_2$, $B = \{x, y\}$ such that $x \in A \setminus C$ and $y \in X \setminus (A \cup C)$. Note $A \cap B = \{x\}$ and $B \cap C = \emptyset$. Suppose λ is induced by $h \in \mathcal{G}_X$, then

$$h(x) = h(A \cap B) = h(A) \cap h(B) = \lambda(A) \cap \lambda(B) = C \cap B = \emptyset.$$

Thus λ is not induced. □

Observe that in the example above we had λ , a bijection of \mathcal{P}_2 , such that

$$|A \cap B| \neq |\lambda(A) \cap \lambda(B)|,$$

for some A, B in \mathcal{P}_2 . This leads us to a criterion for a bijection λ of \mathcal{P}_2 to be induced.

Result 3.16. *Let λ be a bijection of \mathcal{P}_2 . Then λ is induced if and only if $|A \cap B| = |\lambda(A) \cap \lambda(B)|$, for every $A, B \in \mathcal{P}_2$.*

Proof. If λ is induced by an $h \in \mathcal{G}_X$, then for every $A, B \in \mathcal{P}_2$, $|A \cap B| = |h(A \cap B)| = |h(A) \cap h(B)| = |\lambda(A) \cap \lambda(B)|$.

For the converse suppose that λ is a bijection of \mathcal{P}_2 such that for every $A, B \in \mathcal{P}_2$

$$|A \cap B| = |\lambda(A) \cap \lambda(B)|. \tag{*}$$

We show that λ is induced. This is done in the following three steps.

1. Given $x \in X$ there exists a unique $y \in X$ such that for every $A, B \in \mathcal{P}_2$ with $A \cap B = \{x\}$ we have $\lambda(A) \cap \lambda(B) = \{y\}$.

Take a pair A, B in \mathcal{P}_2 with $A \cap B = \{x\}$, then by the assumption (*) $\lambda(A) \cap \lambda(B) = \{y\}$, for some $y \in X$.

Take any other pair C, D in \mathcal{P}_2 with $|C \cap D| = 1$ and let $\mathcal{F} \subseteq \mathcal{P}_2$ be such that:

(α) for every distinct $F_1, F_2 \in \mathcal{F}$, $|F_1 \cap F_2| = 1$;

(β) for any $F \in \mathcal{F}$, $|A \cap F| = |B \cap F| = |C \cap F| = |D \cap F| = 1$.

We show:

$$C \cap D = \{x\} \text{ iff there exists an } \mathcal{F} \text{ (as described above) with } |\mathcal{F}| = |X|.$$

Let $A \cup B \cup C \cup D = E$, then $|E| \leq 8$ and $|X \setminus E| = |X|$.

Assume firstly that $C \cap D = \{x\}$ and let $\mathcal{F} = \{\{x, y\} : y \in X \setminus E\}$. Then \mathcal{F} satisfies (α) and (β) and $|\mathcal{F}| = |X \setminus E| = |X|$.

For the converse assume $C \cap D = \{z\}$, $z \neq x$ and $\mathcal{F} \subseteq \mathcal{P}_2$ satisfies (α) and (β) . For each $F \in \mathcal{F}$ we have $|E \cap F| > 1$. (If not, then

$$\begin{aligned} |E \cap F| &= |(A \cup B \cup C \cup D) \cap F| \\ &= |(A \cap F) \cup (B \cap F) \cup (C \cap F) \cup (D \cap F)| \leq 1. \end{aligned}$$

Using condition (β) we conclude:

$$A \cap F = B \cap F = C \cap F = D \cap F = A \cap B = \{x\},$$

or $C \cap D = \{x\}$, a contradiction).

Define a map $v: \mathcal{F} \rightarrow \mathcal{P}(E)$, where $\mathcal{P}(E)$ is the power set of E , via $v(F) = E \cap F$, each $F \in \mathcal{F}$. We show v is 1-1. Suppose $F_1, F_2 \in \mathcal{F}$ with $v(F_1) = v(F_2)$. Then

$$1 < |E \cap F_1| = |E \cap F_1 \cap F_2| \leq |F_1 \cap F_2|,$$

so that $|F_1 \cap F_2| > 1$, thus $F_1 = F_2$ (condition (α)). However $\mathcal{P}(E)$ is finite, so $|\mathcal{F}| \leq |\mathcal{P}(E)| < \aleph_0$, or $|\mathcal{F}| < |X|$. We conclude $C \cap D = \{x\}$.

Observe now that the definition of the set \mathcal{F} depends on the sets A, B, C and D . We denote this dependence by $\mathcal{F} = \mathcal{F}(A, B, C, D)$. Hence $C \cap D = \{x\}$ iff $\exists \mathcal{F}(A, B, C, D)$ with $|\mathcal{F}(A, B, C, D)| = |X|$ iff $\exists \mathcal{F}(\lambda(A), \lambda(B), \lambda(C), \lambda(D))$ with $|\mathcal{F}(\lambda(A), \lambda(B), \lambda(C), \lambda(D))| = |X|$ (assumption $(*)$)

$$\text{iff } \lambda(C) \cap \lambda(D) = \{y\}.$$

Now define a map

$$h: X \rightarrow X \text{ via } \{h(x)\} = \lambda(A) \cap \lambda(B), \text{ where } \{x\} = A \cap B, \text{ for } A, B \in \mathcal{P}_2 \text{ and each } x \in X.$$

2. h is a bijection of X .

That h is well-defined is the content of step 1. Observe that the bijection λ^{-1} of \mathcal{P}_2 is associated with the automorphism ϕ^{-1} . By considering ϕ^{-1} and λ^{-1} instead of ϕ and λ we define a map $k: X \rightarrow X$ via $\{k(x)\} = \lambda^{-1}(A) \cap \lambda^{-1}(B)$, where $\{x\} = A \cap B$, for $A, B \in \mathcal{P}_2$ and each $x \in X$. Then for each $x \in X$

$$\begin{aligned} \{kh(x)\} &= k(\lambda(A) \cap \lambda(B)), \text{ where } A \cap B = \{x\} \\ &= \lambda^{-1}\lambda(A) \cap \lambda^{-1}\lambda(B) \\ &= A \cap B \\ &= \{x\}. \end{aligned}$$

Similarly we can show $hk(x) = x$, for each $x \in X$. Thus k is the inverse of h , and so h is a bijection of X .

3. λ is induced by h .

To show λ is induced by h we must show $\lambda(A) = h(A)$ for each $A \in \mathcal{P}_2$. From the definition of h we at once have $h(A) \subseteq \lambda(A)$. Take $y \in \lambda(A)$ and let $B \in \mathcal{P}_2$ be such that $\lambda(A) \cap \lambda(B) = \{y\}$. Then $A \cap B = \{x\}$, some $x \in A$, so $h(x) = y$ and $h(A) \ni \lambda(A)$. The equality follows. \square

Remark 3.17. In view of Result 3.16 our aim now is to show that for every $A, B \in \mathcal{P}_2$

$$|A \cap B| = |\lambda(A) \cap \lambda(B)| \tag{*}$$

where λ is the bijection of \mathcal{P}_2 associated with ϕ as defined prior to Example 3.15. Observe that (*) is equivalent to the statement

$$|A \cap B| = 1 \text{ if and only if } |\lambda(A) \cap \lambda(B)| = 1, \tag{**}$$

for each $A, B \in \mathcal{P}_2$.

Indeed (*) certainly implies (**). We show the reverse implication.

Assume (**) holds. If $|A \cap B| = 2$, that is $A = B$, then $\lambda(A) = \lambda(B)$, and so $|\lambda(A) \cap \lambda(B)| = 2$. If $|A \cap B| = 1$, then by our assumption $|\lambda(A) \cap \lambda(B)| = 1$. The case $|A \cap B| = 0$ follows by elimination. \square

The next lemma illustrates the fact that the existence of a transformation f in S which is not 1-1 provides an extensive variety of elements in $\pi(S)$.

Lemma 3.18. *Given $B_1, B_2 \subseteq X$ with $B_1 \cap B_2 = \Phi$ and $|B_1| = |B_2| = 3$ there exists an $\mathcal{A} \in \pi(S)$ with $B_1 \subseteq A_1 \in \mathcal{A}, B_2 \subseteq A_2 \in \mathcal{A}$.*

Proof. Suppose that there exists a transformation f in S such that:

$$C_1, C_2 \in \pi(f) \text{ and } |C_1|, |C_2| \geq 3.$$

Choose a bijection p of X with

$$B_1 \subseteq p(C_1) \text{ and } B_2 \subseteq p(C_2).$$

Certainly $pf p^{-1} \in S$. Let

$$\mathcal{A} = \pi(pf p^{-1}) (= p(\pi(f))), A_1 = p(C_1) \text{ and } A_2 = p(C_2),$$

then $A_1, A_2 \in \mathcal{A} \in \pi(S)$ and $B_1 \subseteq A_1, B_2 \subseteq A_2$.

To construct such an f as used above we first show that there exists a g in S such that

$$g(x_1) = g(x_2) = g(x_3) = x_1, \text{ for some distinct } x_1, x_2, x_3 \in X.$$

Choose a t in S not 1-1 and let $x, x_1, x_2 \in X$ be such that

$$t(x_1) = t(x_2) = x.$$

We assume $x = x_1$ (for if $x \neq x_1$ choose $s \in S$ such that $s(x) = x_1$ (Result 1.3) and replace t by st). Let $x_4 \in R(t) \setminus \{t^{-1}(x_1)\}$ (note: $R(t) \setminus \{t^{-1}(x_1)\} \neq \Phi$, else t^2 is a constant in S) and let $x_3 \in X$ such that $t(x_3) = x_4$. Then

$$g = (x_2, x_4)t(x_2, x_4)t$$

is such that $g(x_1) = g(x_2) = g(x_3) = x_1$.

To accomplish the construction of the above f choose distinct z_1, z_2, z_3 in $R(g) \setminus \{g^{-1}(x_1)\}$ together with $y_1, y_2, y_3 \in X$ such that $g(y_i) = z_i, i = 1, 2, 3$. Let

$$k = (x_1, z_1)(x_2, z_2)(x_3, z_3) \in \mathcal{G}_X \quad \text{and} \quad f = kgk^{-1}g.$$

Let $kg(z_1) = z_4$. Then

$$f(x_1) = f(x_2) = f(x_3) = z_4$$

and

$$f(y_1) = f(y_2) = f(y_3) = z_1.$$

Now $z_1 \neq z_4$ (else $kg(z_1) = z_1$ implies $g(z_1) = x_1$ or $z_1 \in g^{-1}(x_1)$, contrary to the choice of z_1). Let $C_1 = f^{-1}(z_1), C_2 = f^{-1}(z_4)$. Then $|C_1|, |C_2| \geq 3$ and $C_1, C_2 \in \pi(f)$ as required. \square

Remark 3.19. It easily follows from Lemma 3.18 that

$$\mathcal{L}_A \cap \mathcal{L}_B \neq \Phi,$$

for every $A, B \in \mathcal{P}_2$. \square

Lemma 3.20. Let $A, B \in \mathcal{P}_2, A \neq B$. Then $|A \cap B| = 1$ iff there is a C in $\mathcal{P}_2, C \neq A$ or B , such that $\mathcal{L}_A \cap \mathcal{L}_B \subseteq \mathcal{L}_C$.

Proof. Assume $|A \cap B| = 1$ and let $C = (A \cup B) \setminus (A \cap B)$. For each l in $\mathcal{L}_A \cap \mathcal{L}_B$ (Remark 3.19):

$$l(A) = l(A \cap B) = l(B) = l(A \cup B) = l(C),$$

so that $l \in \mathcal{L}_C$ and $\mathcal{L}_A \cap \mathcal{L}_B \subseteq \mathcal{L}_C$.

For the converse suppose $A \cap B = \Phi$ and $C \in \mathcal{P}_2$ is distinct from A and B . Let $C = \{c_1, c_2\}$. Since $|A \cap C| \leq 1$ and $|B \cap C| \leq 1$ assume without loss of generality that $c_1 \in X \setminus B$ and $c_2 \in X \setminus A$. Choose

$$\mathcal{A} \in \pi(S) \quad \text{with} \quad A \cup \{c_1\} \subseteq A_1 \in \mathcal{A}, B \cup \{c_2\} \subseteq A_2 \in \mathcal{A} \quad \text{and} \quad A_1 \neq A_2 \quad (\text{Lemma 3.18}).$$

If $l \in S$ has $\pi(l) = \mathcal{A}$, then $l \in (\mathcal{L}_A \cap \mathcal{L}_B) \setminus \mathcal{L}_C$.
 This confirms that $|A \cap B| = 1$. □

Lemma 3.21. *Let A, B and C be distinct elements of \mathcal{P}_2 . Then*

$$\mathcal{L}_A \cap \mathcal{L}_B \subseteq \mathcal{L}_C \quad \text{iff} \quad \mathcal{L}_{\lambda(A)} \cap \mathcal{L}_{\lambda(B)} \subseteq \mathcal{L}_{\lambda(C)}.$$

Proof. Observe that $\mathcal{L}_A \cap \mathcal{L}_B \neq \Phi$ (Remark 3.19) and

$$\mathcal{L}_A \cap \mathcal{L}_B \subseteq \mathcal{L}_C \quad \text{iff} \quad \phi(\mathcal{L}_A \cap \mathcal{L}_B) \subseteq \phi(\mathcal{L}_C).$$

Now

$$\phi(\mathcal{L}_A \cap \mathcal{L}_B) = \phi(\mathcal{L}_A) \cap \phi(\mathcal{L}_B) = \mathcal{L}_{\lambda(A)} \cap \mathcal{L}_{\lambda(B)},$$

by the definition of λ . Also $\phi(\mathcal{L}_C) = \mathcal{L}_{\lambda(C)}$, so that

$$\phi(\mathcal{L}_A \cap \mathcal{L}_B) \subseteq \phi(\mathcal{L}_C) \quad \text{iff} \quad \mathcal{L}_{\lambda(A)} \cap \mathcal{L}_{\lambda(B)} \subseteq \mathcal{L}_{\lambda(C)},$$

and the desired equivalence is established. □

Result 3.22. *Given A and B in \mathcal{P}_2 ,*

$$|A \cap B| = 1 \quad \text{if and only if} \quad |\lambda(A) \cap \lambda(B)| = 1.$$

Proof. We have:

$$\begin{aligned} |A \cap B| = 1 & \quad \text{iff} \quad \exists C \neq A \text{ or } B \text{ such that } \mathcal{L}_A \cap \mathcal{L}_B \subseteq \mathcal{L}_C \text{ (Lemma 3.20)} \\ & \quad \text{iff} \quad \exists \lambda(C) \neq \lambda(A) \text{ or } \lambda(B) \text{ such that } \mathcal{L}_{\lambda(A)} \cap \mathcal{L}_{\lambda(B)} \subseteq \mathcal{L}_{\lambda(C)} \\ & \quad \quad (\lambda \text{ is a bijection and Lemma 3.21)} \end{aligned}$$

$$\text{iff} \quad |\lambda(A) \cap \lambda(B)| = 1 \quad \text{(Lemma 3.20 again).} \quad \square$$

From Results 3.16, 3.22 and Remark 3.17 we readily deduce

Result 3.23. *λ is induced by a bijection of X .* □

Now we are ready to show that a constant-free \mathcal{G}_X -normal semigroup containing a transformation which is not 1-1 (that is a semigroup of Type 3), possesses only inner automorphisms.

Result 3.24. *Let S be a constant-free \mathcal{G}_X -normal semigroup containing a transformation which is not 1-1. Then each automorphism ϕ of S is inner, that is for some $h \in \mathcal{G}_X$*

$$\phi(f) = hfh^{-1}, \quad \text{for each } f \in S.$$

Proof. Let h be the bijection which induces λ (Result 3.23). In what follows we use

the fact that for any distinct $x_1, x_2 \in X$

$$\phi(\mathcal{L}_{\{x_1, x_2\}}) = \mathcal{L}_{\lambda(\{x_1, x_2\})} = \mathcal{L}_{\{h(x_1), h(x_2)\}}.$$

Take an arbitrary $f \in S$, $x \in X$ and let $y \in X$ with $f(x) \neq f(y)$ (that is $f \notin \mathcal{L}_{\{x, y\}}$). Then

$$\phi(\mathcal{L}_{\{f(x), f(y)\}}) = \mathcal{L}_{\{hf(x), hf(y)\}}.$$

Let $\phi(g) \in \phi(\mathcal{L}_{\{f(x), f(y)\}})$. Then $g \in \mathcal{L}_{\{f(x), f(y)\}}$ or $gf(x) = gf(y)$. It follows that $\phi(gf) \in \mathcal{L}_{\{h(x), h(y)\}}$, hence

$$\phi(g)\phi(f)h(x) = \phi(g)\phi(f)h(y).$$

Note that $f \notin \mathcal{L}_{\{x, y\}}$ implies $\phi(f) \notin \phi(\mathcal{L}_{\{x, y\}})$ or $\phi(f) \notin \mathcal{L}_{\{h(x), h(y)\}}$, that is

$$\phi(f)h(x) \neq \phi(f)h(y).$$

Thus $\phi(g) \in \mathcal{L}_{\{\phi(f)h(x), \phi(f)h(y)\}}$ and we conclude

$$\phi(\mathcal{L}_{\{f(x), f(y)\}}) \subseteq \mathcal{L}_{\{\phi(f)h(x), \phi(f)h(y)\}}.$$

This in turn implies

$$\mathcal{L}_{\{hf(x), hf(y)\}} \subseteq \mathcal{L}_{\{\phi(f)h(x), \phi(f)h(y)\}}.$$

Hence $\{hf(x), hf(y)\} = \{\phi(f)h(x), \phi(f)h(y)\}$ (Lemma 3.4).

Since the choice of y is independent of x (providing $y \neq x$) we conclude

$$\phi(f)h(x) = hf(x), \text{ for each } x \in X,$$

so that

$$\phi(f) = hf h^{-1}, \text{ as required.} \quad \square$$

Conclusion

We return to

Theorem 1.1. *Every automorphism of a \mathcal{G}_X -normal semigroup S is inner.*

Proof. If S is a semigroup of Type 1, that is, contains a constant transformation, then we appeal to Sullivan [12, Theorem 1].

If S is a semigroup of Type 2, that is, a semigroup of 1–1 transformations, the result is given in 2.14 and 1.4.

If S is a semigroup of Type 3, that is, a semigroup containing a transformation which is not 1–1, then the result is given in 3.24.

This completes the proof of Theorem 1.1. □

Remark. If X is a finite set and S is a semigroup of transformations of X which is not contained in \mathcal{G}_X , then S is \mathcal{G}_X -normal if and only if all automorphisms of S are inner [13].

However, this is not the case for an infinite set X . While, as we showed, every \mathcal{G}_X -normal semigroup S has only inner automorphisms, there are examples [5] of semigroups which are neither subsemigroups of \mathcal{G}_X , nor \mathcal{G}_X -normal, yet have only inner automorphisms.

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