Fifth Meeting, 8th March 1907.

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On Area-Theory, and some applications.

By P. PINKERTON, M.A.

1. In the *Cambridge and Dublin Mathematical Journal*, vol. v., 1859, De Morgan gives the definition of the "area contained within a circuit" as the area swept out by a radius vector which has one end (the pole) fixed and the other describing the circuit (in a determinate mode), on the supposition that each element of area is positive or negative, according as the radius is revolving positively or negatively. He remarks that the definition satisfies existing notions, that it provides the necessary extension of the meaning of the word area, and proceeds to show that it gives to every circuit the same area, whatever point the pole may be. The object of this paper is to give an Area-Theory beginning with the triangle and going on to circuits bounded by straight or curved lines. The fundamental proposition is derived from Analysis, and the geometry of the applications is therefore an Analytical Geometry; indeed, one of the objects of the paper is to emphasise the advantage of keeping Analysis and Geometry in close correspondence. As evidence of the difficulty of pursuing an Area-Theory in Geometry, without the aid of Analysis, it may be noticed that Townsend in his *Modern Geometry* (1863), § 83, lays down Salmon's Theorem in this form: "If A, B, C, D be any four points on a circle taken in the order of their disposition, and P any fifth point, without, within, or upon the circle, but not at infinity, then always

$$\text{area } BCD.AP^2 - \text{area } CDA.BP^2 + \text{area } DAB.CP^2 - \text{area } ABC.DP^2 = 0,$$

regard being had only to the absolute magnitudes of the several areas which from their disposition are incapable of being compared in sign." Yet, previous to this, he uses positive and negative area of the triangle; and, later on (Chap. VII), works out at some length a formal definition of the "area of a polygon," "whether convex, reentrant, or intersecting."

2. Let (x_1, y_1) and (x_2, y_2) be the coordinates of points P_1, P_2 with reference to a rectangular Cartesian system of reference, origin O; to find an expression for the measure of $\triangle OP_1P_2$ in terms of x_1, y_1, x_2, y_2.

Let (r_1, θ_1) and (r_2, θ_2) be polar coordinates of P_1, P_2 with reference to O as pole and OX as initial line; r_1, r_2 being positive, and θ_1, θ_2 being *any* angles through which OX must turn to come into the positions OP_1, OP_2. Let $P_1\widehat{O}P_2$ be the angle through which OP_1 must turn to come into the position OP_2, under the condition that the radius vector traces out the angle O of the triangle P_1OP_2; then $P_1\widehat{O}_2P$ has sign as well as magnitude.

Then $$\theta_1 + P_1\widehat{O}P_2 = 2n\pi + \theta_2 \ (n \text{ integral or zero});$$
$$\therefore \ P_1\widehat{O}P_2 = 2n\pi + (\theta_2 - \theta_1);$$
$$\therefore \ \sin P_1OP_2 = \sin(\theta_2 - \theta_1),$$
and is positive or negative according as OP_1P_2O indicates the trigonometrically positive sense or the trigonometrically negative sense of rotation in the plane.

Now the absolute measure of $\frac{1}{2}r_1r_2\sin P_1OP_2$ is the area of triangle OP_1P_2; we introduce positive and negative area by *defining* $\frac{1}{2}r_1r_2\sin P_1OP_2$ *or* $\frac{1}{2}r_1r_2\sin(\theta_2 - \theta_1)$ *as the measure of* $\triangle OP_1P_2$, and write
$$\triangle OP_1P_2 = \tfrac{1}{2}r_1r_2\sin(\theta_2 - \theta_1) = \tfrac{1}{2}(x_1y_2 - x_2y_1),$$
$$\text{and} \quad \triangle OP_2P_1 = \tfrac{1}{2}r_2r_1\sin(\theta_1 - \theta_2) = \tfrac{1}{2}(x_2y_1 - x_1y_2).$$
The *sign* of the expression $\frac{1}{2}(x_1y_2 - x_2y_1)$ has a specific geometrical meaning, and the order of the letters OP_1P_2 has a corresponding significance.

If A, B, C are three points in a plane, we say that $\triangle ABC$ is "a positive area" or "a negative area," according as the sequence of letters ABCA indicates the positive or negative sense of circulation in the plane, as already agreed on in Trigonometry.

3. To find an expression for the measure of $\triangle P_1P_2P_3$ in terms of the coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) of three points P_1, P_2, P_3 in the plane of the axes.

$\triangle P_1P_2P_3$, that is, $\frac{1}{2}P_1P_2 . P_1P_3\sin P_2P_1P_3$, is unaltered by change of axes. Change to parallel axes through the point (x_1, y_1). Let (ξ_2, η_2), (ξ_3, η_3) be the new coordinates of P_2, P_3; then
$$\triangle P_1P_2P_3 = \tfrac{1}{2}(\xi_2\eta_3 - \xi_3\eta_2) = \tfrac{1}{2}\{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\}$$
$$= \tfrac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\}.$$

4. From § 3 comes the general Area-theorem,

$$\Delta P_1P_2P_3 = \Delta OP_1P_2 + \Delta OP_2P_3 + \Delta OP_3P_1,$$

connecting the areas (regarded as having *sign*) associated with any four coplanar points.

Cor. 1. The relation can be more systematically expressed thus: for any four coplanar points $P_1P_2P_3P_4$

$$\Delta P_2P_3P_4 - \Delta P_3P_4P_1 + \Delta P_4P_1P_2 - \Delta P_1P_2P_3 = 0.$$

Cor. 2. If A, B, C, ... K, L are collinear points, and O any other point

$$\Delta OAL = \Delta OAB + \Delta OBC + \ldots + OKL.$$

5. This theorem may be regarded as proving that if P_1, P_2, P_3 *are fixed points, and* Q *a variable point of their plane*

$$(\Delta QP_1P_2 + \Delta QP_2P_3 + \Delta QP_3P_1)$$

does not vary with Q.

The theorem in this form has the following important extension: *If* $P_1, P_2, \ldots, P_n$ *are any n given coplanar points, and* Q *a variable point of their plane,* $(\Delta QP_1P_2 + \Delta QP_2P_3 + \ldots + \Delta QP_{n-1}P_n + \Delta QP_nP_1)$ *does not vary with* Q.

Proof. If O is any base-point of the plane,

$$\Delta QP_rP_{r+1} = \Delta OQP_r + \Delta OP_rP_{r+1} + \Delta OP_{r+1}Q,$$
$$= \Delta OP_rP_{r+1} + \Delta OQP_r - \Delta OQP_{r+1}.$$

$\therefore \quad \Sigma\Delta QP_rP_{r+1} = \Sigma OP_rP_{r+1}$, for a complete cycle.

6. Now consider a simple closed plane space bounded by straight lines P_1P_2, P_2P_3 ..., $P_{n-1}P_n$, P_nP_1 in order and first suppose the boundary is *convex*. Give Q a position within the boundary. Then $(\Delta QP_1P_2 + \Delta QP_2P_3 + \ldots + \Delta QP_nP_1)$ is in absolute measure the area* of the closed space. Therefore the absolute measure of the same expression is the area* of the closed space, for *all* positions of Q.

Next suppose that the boundary is not convex. Break the area*

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* "Area" here means simply area, and is of course neither positive nor negative.

of the closed space into areas* of simple closed spaces with convex boundaries by introducing cross-lines such as P_rP_s in fig. 18. Then

$$(\triangle QP_1P_2 + \triangle QP_2P_3 + \ldots + \triangle QP_nP_1)$$

$$= \{\triangle QP_1P_2 + \ldots + \triangle QP_nP_1 + \Sigma(\triangle QP_rP_s + \triangle QP_sP_r)\}$$

= ± sums of areas* of closed spaces with convex boundaries, since each of these areas* would appear with the same sign prefixed.

Hence again

absolute measure of $(\triangle QP_1P_2 + \triangle QP_2P_3 + \ldots + \triangle QP_nP_1)$

= area* of closed space.

Hence for the most general coplanar positions of $P_1, P_2, \ldots, P_n$, we *define area* $P_1P_2 \ldots P_nP_1$ *to be*

$$(\triangle QP_1P_2 + \triangle QP_2P_3 + \ldots + \triangle QP_{n-1}P_n + \triangle QP_nP_1),$$

Q *being any coplanar point.*

7. Any one of the lines P_1P_2, P_2P_3, …, P_nP_1, supposed terminated at the extremities P_1, P_2; etc., may now cross any other. Consider fig. 19. Each of the lines P_1P_2, etc., crosses two or more of the others. Mark the crossing-points as in the figure. Then

$$\triangle QP_1P_2 = \triangle QP_1R_1 + \triangle QR_1R_2 + \triangle QR_2P_2,$$

$$\triangle QP_2P_3 = \triangle QP_2R_3 + \triangle QR_3R_4 + \triangle QR_4R_5 + \triangle QR_5P_3,$$

etc., etc.,

∴ Area $P_1P_2 \ldots P_6P_1$

$$= \text{Area } P_1R_1R_4R_5P_1 + \text{Area } P_2R_3R_2P_2 + \text{Area } P_3R_6R_5P_3$$
$$+ \text{Area } P_4R_3R_4R_7P_4 + \text{Area } P_5R_1R_2P_5 + \text{Area } P_6R_6R_7P_6.$$

In estimating Area $P_1R_1R_4R_5P_1$, etc., give Q a position within each boundary in turn, and the signs of these partial areas are seen to be, in order, +, +, −, −, −, +. This result corresponds to De Morgan's Rule for Area.

The following sections contain some applications of the above theory.

8. Note (i) that $\triangle Q_1AB$, $\triangle Q_2AB$ are of the same or of opposite sign according as Q_1, Q_2 are on the same or on opposite sides of the AB-line.

* "Area" means simply area, and of course is neither positive nor negative.

(ii) that if AB, CD are steps on the same line, $\triangle QAB$ and $\triangle QCD$ are of the same or of opposite signs according as AB, CD are steps of the same or of opposite sign.

Hence the fundamental theorem

$$\triangle QAB : \triangle QCD = AB : CD$$

is to be regarded as taking account of *sign*.

In particular, if M is the middle point of AB, $\triangle QAM = \triangle QMB$.

Euc. VI., 2 can be written out in such a way as to suit all figures. Let B_1C_1 parallel to base BC of triangle ABC meet the lines AB, AC in B_1, C_1 respectively. Then B_1, C_1 are on the same side of BC,

$$\therefore \quad \triangle BCC_1 = \triangle BCB_1,$$

$$\therefore \quad \triangle ABC + \triangle ACC_1 + \triangle AC_1B = \triangle ABC + \triangle ACB_1 + \triangle AB_1B,$$

$$\therefore \quad \triangle AC_1B = \triangle ACB_1, \text{ since } \triangle ACC_1 = 0 = \triangle AB_1B.$$

Hence $\quad AB : AB_1 = \triangle ABC : \triangle AB_1C = \triangle ABC : \triangle ABC_1$

$$= AC : AC_1.$$

Again, a direct and general proof of Ceva's Theorem can be given.

Let concurrent lines AOD, BOE, COF meet the sides BC, CA, AB of triangle ABC in D, E, F respectively.

$$\begin{aligned} BD : CD = \triangle OBD : \triangle OCD &= \triangle ABD : \triangle ACD \\ &= \triangle OAB + \triangle OBD : \triangle OAC + \triangle OCD, \\ &\qquad \text{since } \triangle ODA = 0 \\ &= -(\triangle OAB : \triangle OCA). \end{aligned}$$

Similarly $\quad CE : AE = -(\triangle OBC : \triangle OAB)$,

$$AF : BF = -(\triangle OCA : \triangle OBC)$$

$$\therefore \quad \frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = -1.$$

(iii) $\triangle Q_1AB : \triangle Q_2AB = p_1 : p_2$,

where p_1, p_2 are the *ordinates* of Q_1, Q_2 with respect to the AB-line, in other words the *perpendiculars* from Q_1, Q_2 to the AB-line, if the *perpendiculars* are regarded as *steps*.

This may be shown by taking A, B as points on the x-axis of a system of Rectangular axes and applying the formula for $\triangle P_1P_2P_3$ in terms of the coordinates of P_1, P_2, P_3.

9. If A, B, C, O are any four points in a plane and G the middle point of BC, then

$$\triangle OAB + \triangle OAC = 2\triangle OAG.$$

For $\triangle OAB + \triangle OBG + \triangle OGA = \triangle ABG = \triangle AGC$

$$= \triangle OAG + \triangle OGC + \triangle OCA$$

$\therefore\ \triangle OAB + \triangle OAC = 2\triangle OAG$, since $\triangle OBG = \triangle OGC$.

Hence, if M is the middle point of AB, P and Q two other points of the plane

$$\triangle APQ + \triangle BPQ = 2\triangle MPQ,$$

being a form of $\triangle PQA + \triangle PQB = 2\triangle PQM$.

And again, if M is half-way from A to the PQ-line,

$$\triangle APQ = 2\triangle MPQ.$$

10. If A, B, C, D are any four points of a plane; E, F, G, H the middle points of AB, BC, CD, DA respectively, then

$$\text{Area EFGH} = \tfrac{1}{2}\ \text{Area ABCD}.$$

Area EFGH $= \triangle AEF + \triangle AFG + \triangle AGH + \triangle AHE$

$\triangle AEF = \frac{1}{2}\triangle ABF = \frac{1}{4}\triangle ABC,$

$\triangle AFG = \frac{1}{2}(\triangle AFC + \triangle AFD) = \frac{1}{4}(\triangle ABC + \overline{\triangle ABD + \triangle ACD}),$

$\triangle AGH = \frac{1}{2}\triangle AGD = \frac{1}{4}\triangle ACD,$

$\triangle AHE = \frac{1}{2}\triangle ADE = \frac{1}{4}\triangle ADB,$

$\therefore$ Area EFGH $= \frac{1}{2}(\triangle ABC + \triangle ACD) = \frac{1}{2}$ Area ABCD.

11. If A, B, C, D are any four points of a plane, P and Q the middle points of AC, BD respectively, X the point of intersection of the AD- and the BC-lines, Y the point of intersection of the AB- and CD-lines, then

$$\triangle XPQ = \tfrac{1}{4}\ \text{Area ABCD},$$

$$\text{and}\quad \triangle YPQ = -\tfrac{1}{4}\ \text{Area ABCD}.$$

$2\triangle XPQ = \triangle XPD + \triangle XPB,$

$= \frac{1}{2}\triangle XCD + \frac{1}{2}\triangle XAB,$

$= \frac{1}{2}(\triangle XAB + \triangle XBC + \triangle XCD + \triangle XDA),$

since $\triangle XBC = 0 = \triangle XDA$,

$= \frac{1}{2}$ Area ABCD,

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Similarly $\triangle YPQ = -\frac{1}{4}$ Area ABCD.

Cor. Hence $\triangle XPQ + \triangle YPQ = 0$

therefore the middle point of XY is on the PQ-line, *i.e.*, the middle points of the diagonals of a complete quadrilateral are collinear.

12. If A, B, C, D are any four given points of a plane, and if a variable point P moves so that

$$m . \triangle PAB + n . \triangle PCD = \text{constant},$$

when m, n are any fixed multiples positive or negative, then the locus of P is a straight line.

Let the AB- and CD-lines meet in O. Let $OX = m . AB$ and $OY = n . CD$ in sign and magnitude, and let G be the middle point of XY.

Then $$m . \triangle PAB + n . \triangle PCD = \triangle POX + \triangle POY$$
$$= 2\triangle POG.$$

$\therefore$ locus of P is a straight line parallel to OG.

An obvious extension is that if A_1B_1, A_2B_2, ..., A_nB_n are n fixed lines in a plane, and P a variable point such that

$$a_1 . \triangle PA_1B_1 + a_2 . \triangle PA_2B_2 + \dots + a_n . \triangle PA_nB_n = \text{constant},$$

where a_1, a_2, ..., a_n are fixed multiples, positive or negative, then the locus of P is a straight line.

Cor. An equation of the first degree in areal coordinates represents a straight line.

13. The following problem illustrates the use of the theory geometrically.

Let A, B be two fixed points in a plane, C, D two variable points in the plane, such that CD is fixed in magnitude and direction and Area ABCD is fixed; to find the loci of C and D.

Draw AE parallel to CD such that $AE = DC$, in sign and magnitude.

Then $\quad$ Area ABCD $= \triangle ABE + \triangle CEB + \triangle CDAE$,

$\therefore$ Area ABCD $- \triangle ABE$

$= \triangle CEB + 2\triangle CAE$

$= \triangle CEB + \triangle CEF$, where AE is produced to F so that EF $=$ 2AE in sign and magnitude

$= 2\triangle CEG$, if G is the middle point of BF;

therefore $\triangle CEG$ is constant. Hence the locus of C is a straight line parallel to EG, and therefore the locus of D is a parallel straight line, since CD is fixed in magnitude and direction.

14. If A, B, C, O are any four coplanar points, the mean centre of the points A, B, C for multiples $\triangle OBC$, $\triangle OCA$, $\triangle OAB$, or multiples proportional to these, is the point O.

For
$$\frac{\triangle OBC}{\triangle OCA} = -\frac{\triangle BCO}{\triangle ACO} = -\frac{b}{a},$$

where b, a are the perpendiculars from B, A to OC, account being taken of sign.

$$\therefore \quad a \,.\, \triangle OBC + b \,.\, \triangle OCA = 0$$

and hence OC passes through the mean centre of ABC for multiples $\triangle OBC$, $\triangle OCA$, OAB.

Similarly OA, OB pass through the mean centre for those multiples. Therefore O is the mean centre.

Hence if A, B, C, D be any four points on a circle, and O any *fifth point in the plane*

$$OB^2 . \triangle ACD + OC^2 . \triangle ADB + OD^2 . \triangle ABC - (\triangle ABC + \triangle ACD + \triangle ADB)OA^2$$
$$= AB^2 . \triangle ACD + AC^2 . \triangle ADB + AD^2 . \triangle ABC$$
$$= \text{constant, for all positions of O.}$$

Giving O the position of the centre of the circle, and noting that

$\triangle ABC + \triangle ACD + \triangle ADB = \triangle BCD$, $\quad$ we see that

$$OA^2 . \triangle BCD - OB^2 \triangle CDA + OC^2 \triangle DAB - OD^2 . \triangle ABC = 0,$$

and $\quad AB^2 . \triangle ACD + AC^2 \triangle ADB + AD^2 \triangle ABC = 0.$

Also, if x, y, z are the areal coordinates of any point P on a circle and ABC, the triangle of reference, taking P, A, B, C in turn as mean centres of A, B, C; B, C, P; etc.; we have

$$x \, . \, PA^2 + y \, . \, PB^2 + z \, . \, PC^2 = 0,$$
$$PA^2 = yc^2 + zb^2,$$
$$PB^2 = za^2 + xc^2,$$
$$PC^2 = xb^2 + ya^2,$$

whence $$yza^2 + zxb^2 + xyc^2 = 0.$$

15. In extending the theory to areas of closed plane spaces bounded by curves or partly bounded by curves, the following Lemma is useful:

If $P_1, P_2, \ldots, P_n$ *be any n given points of a plane*, $L_1, L_2, \ldots, L_n$ *n collinear points of the plane*

$$\textit{Area } P_1P_2 \ldots P_nP_1 = \textit{Area } P_1P_2L_2L_1 + \ldots + \textit{Area } P_rP_{r+1}L_{r+1}L_r + \ldots$$
$$\ldots + \textit{Area } P_nP_1L_1L_n.$$

For

$$\text{Area } P_rP_{r+1}L_{r+1}L_r = \triangle OP_rP_{r+1} - \triangle OL_rL_{r+1} - (\triangle OP_rL_r - \triangle OP_{r+1}L_{r+1})$$

and $$\Sigma \triangle OL_rL_{r+1} = 0, \;\; \Sigma(\triangle OP_rL_r - \triangle OP_{r+1}L_{r+1}) = 0.$$

If $L_1, L_2, \ldots, L_n$ be the projections $M_1, M_2, \ldots, M_n$ of $P_1, P_2, \ldots, P_n$ on the x-axis of a rectangular Cartesian system,

$$\text{Area } P_1P_2M_2M_1 = \triangle OP_1P_2 + \triangle OP_2M_2 + \triangle OM_2M_1 + \triangle OM_1P_1$$
$$= \tfrac{1}{2}(x_1y_2 - x_2y_1 - x_2y_2 + x_1y_1)$$
$$= -\tfrac{1}{2}(x_2 - x_1)(y_1 + y_2).$$

If $L_1, L_2, \ldots, L_n$ be the projections $N_1, N_2, \ldots, N_n$ of $P_1, P_2, \ldots, P_n$ on the y-axis,

$$\text{Area } P_1P_2N_2N_1 = \tfrac{1}{2}(x_1 + x_2)(y_2 - y_1).$$

16. If $P_1P_2 \ldots P_nP_1$ specifies the boundary of a closed curve, the area of the space enclosed is defined to be

$$\underset{n=\infty}{\text{Lt}} \; \Sigma(\triangle QAP_1 + \triangle QP_1P_2 + \ldots + \triangle QP_nA),$$

where A is a fixed point on the curve and the P's are distributed on the curve according to some law such that $\underset{n=\infty}{\text{Lt}}\ P_rP_{r+1}=0$ and that a current point P moving steadily round the curve from A to A passes through $P_1, P_2, \ldots, P_n$ in succession.

Hence $\text{Area}=\frac{1}{2}\int r^2 d\theta$ from $\text{Lt}\ \Sigma r(r+\Delta r)\sin\Delta\theta$,

and $\text{Area}=\frac{1}{2}\int(xdy-ydx)$ from $\text{Lt}\ \Sigma\frac{1}{2}\{x(y+\Delta y)-(x+\Delta x)y\}$.

Again, from the expressions for Area $P_1P_2M_2M_1$ and Area $P_1P_2N_2N_1$ in § 15 it is clear that

$$\text{Area}=-\int ydx=\int xdy.$$

If HA, KB are ordinates of A, B two points on a curve represented by the equation $y=f(x)$, where $f(x)$ is a single-valued continuous function of x, then along AH and BK, $dx=0$; and along HK, $y=0$. Therefore Area $\text{AHKB}=\int_a^b ydx$, where a, b are the abscissae of A, B.

And if P is a variable point (x, y) on the curve and MP its ordinate

$$\frac{d\text{A}}{dx}=y$$

where A = Area AHMP.

For take Q a point on the curve near to P, then

$$\text{A}+\Delta\text{A}=\text{Area AHNQ}=\text{Area AHMP}+\text{Area PMNQ}$$

$$\therefore\ \Delta\text{A}=\text{Area PMNQ}=+y\Delta x$$

$$\therefore\ \frac{d\text{A}}{dx}=y.$$

If a new variable t be introduced where x, y are single-valued functions of t, and t varies always in one sense (that is, always increasing or always decreasing) from t_1 to t_2 as the current point P moves round the curve from A to A, passing through $P_1, P_2, \ldots,$ in succession, we have formulæ such as

$$\text{Area}=\frac{1}{2}\int_{t_1}^{t_1}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt.$$

It is sometimes said that t must be chosen so as to go on *increasing* when the current point P moves steadily round the boundary *leaving the area on the left.* There are two misleading

https://doi.org/10.1017/S0013091500033575 Published online by Cambridge University Press

elements in such a statement. First, t may go on *decreasing or increasing*. Secondly, in cases where the boundary crosses itself, it is not possible for the current point P to move steadily round the boundary and *always* leave the *area* on left or right.

For example, in fig. 20

$$\tfrac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt = \text{Area } AP_1P_2 \ldots P_5A$$

$$= \text{area* of space (2)} - \text{area* of space (1)},$$

space (2) being to *left* of current point, while space (1) is to *right* of current point.

In fig. 21

$$\text{Integral} = \text{Area} AP_1P_2 \ldots P_5A$$

$$= \text{twice area* of space (1)} + \text{area* of space (2)},$$

space (2) not including the shaded portion.

In fig. 22

$$\text{Integral} = \text{Area } AP_1P_2 \ldots P_{12}A$$

$$= \text{area* of shaded space} + \text{twice area* of space (4)}$$

$$- \text{sum of areas* of spaces (1), (2), (3)}.$$

It is worth noting that, using double integrals, we have

$$\iint dx\,dy = \tfrac{1}{2}\int\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt,$$

the simplest case of Stokes's Theorem.

* "Area" being here neither positive nor negative.

On Commutative Matrices.

By J. H. Maclagan-Wedderburn, M.A.