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ALGEBRAIC IDEALS IN GROUP RINGS AND TENSOR PRODUCTS

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ABSTRACT. It is shown that if a solvable group is not locally finite, then the group algebra over a field of characterisitc 0 has no nonzero algebraic ideals.

In this paper all rings are rings with unity and all modules are unital. For the most part we are dealing with algebras over a given ground field and we are interested in existence (or nonexistence) of algebraic ideals, that is, ideals in which every element satisfies a nontrivial polynomial over the ground field. Our interest in this problem arose from the study of the Jacobson radical of tensor products [4]. In this earlier paper, we proved that if two algebras over an algebraically closed field have no nonzero algebraic ideals, then their tensor product is semiprimitive. We would like to show that either a group is locally finite, or the group algebra over a field of characteristic zero has no nonzero algebraic ideals. In this paper we prove that this is true for solvable groups. There appears to be little in the literature dealing with our problem. Amitsur [1] proved that a group algebra in characteristic zero has no nonzero nil ideals and Connell and Passman [5] gave a generalization of this to arbitrary characteristic. On the other hand, Herstein [3] noted that if a group algebra of characteristic zero is algebraic, then the group is locally finite.

Let R be an algebra over the field F. As the join of algebraic ideals is algebraic, R has a unique maximal F-algebraic ideal $A(R)$. This ideal behaves like a radical in that $A(R/A(R)) = (0)$, so we will call it the *algebraic radical* of R. This radical is invariant under all F-automorphisms and involutions on R. If $J(R)$ is the Jacobson radical of R, then $J(R) \cap A(R) = N(R)$, the (upper) nil radical of R.

Our first theorem deals with tensor products.

THEOREM 1. *Suppose that R and T are algebras over an algebraically closed field F. If $A(R) = N(T) = (0)$, then $A(R \otimes_F T) = (0)$.*

PROOF. Suppose that $A(R \otimes_F T) \neq (0)$. Let $\alpha = \Sigma_{i=1}^{k}(a_i \otimes b_i)$ be a nonzero element of $A(R \otimes_F T)$ of minimal tensor rank k. As $(a_i \otimes 1)\alpha - \alpha(a_i \otimes 1) \in A(R \otimes_F T)$ has rank less than k, we can conclude that it is zero, that is, $a_i a_j = a_j a_i$ for all i and j. Similarly $b_i b_j = b_j b_i$ for all i and j. As $A(R) = (0)$, we may choose α so that a_1 is transcendental over F. Let C be the (commutative) subalgebra of R generated by all the a_i's. As $F[a_1] - \{0\}$ is multiplicatively closed and does not contain 0, we can choose a prime ideal P of C disjoint from $F[a_1] - \{0\}$. Note that as C is finitely generated as an

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algebra over an algebraically closed field F and, as $P \cap F[a_1] = 0$, the domain C/P is not a field.

By rearrangement and modification (if necessary) of $a_2, \ldots, a_k$ and $b_1, \ldots, b_k$, we may assume that, under the natural map $\bar{}: C \longrightarrow C/P$, $\{\bar{a}_1, \ldots, \bar{a}_l\}$ is a basis for the span of $\{\bar{a}_1, \ldots, \bar{a}_k\}$ and that $\bar{a}_{l+1} = \cdots = \bar{a}_k = 0$. Finally, as $N(T) = 0$, we may choose b_1 nonnilpotent. Let D be the (commutative) subalgebra of T generated by all the b_i's and let Q be a prime ideal of D with $b_1 \notin Q$.

Under the map $\phi: C \otimes_F D \longrightarrow (C/P) \otimes_F (D/Q)$, $\phi(\alpha) \neq 0$; hence $A[(C/P) \otimes_F (D/Q)] \neq (0)$. But $(C/P) \otimes_F (D/Q)$ is an integral domain [6] which is not a field, so its algebraic radical must be zero. To see this note that as F is algebraically closed $A[(C/P) \otimes_F (D/Q)] \subseteq F$; hence, $1 \in A[(C/P) \otimes_F (D/Q)]$. However $\bar{a}_1 \otimes 1$ is transcendental over F. This contradiction completes the proof of the theorem.

It is conjectured that the Jacobson radical of a rational group algebra is zero. If this is true then the following corollary is vacuous.

COROLLARY. *Let $G_1, \ldots, G_n$ be groups with no nontrivial finite normal subgroups. If $J(\mathbb{Q}[G_1 \times \cdots \times G_n]) \neq (0)$, then precisely $n-1$ of the group algebras $\mathbb{Q}[G_i]$ are semiprimitive.*

PROOF. Let $\bar{\mathbb{Q}}$ denote the algebraic closure of the rational numbers $\mathbb{Q}$. As $J(\mathbb{Q}[G_1 \times \cdots \times G_n]) \neq (0)$, we have $J(\bar{\mathbb{Q}}[G_1 \times \cdots \times G_n]) \neq (0)$ [5, Ch.7, Theorem 2.13]. If $J(\bar{\mathbb{Q}}[G_i]) = (0)$ for all i, then $J(\bar{\mathbb{Q}}[G_1 \times \cdots \times G_n]) = (0)$ [5, Ch.7, Lemma 3.12]. On the other hand if $J(\bar{\mathbb{Q}}[G_i]) \neq (0)$ and $J(\mathbb{Q}[G_j]) \neq (0)$, where $i \neq j$, then, as both group algebras are prime with no nonzero nil ideals, $A(\bar{\mathbb{Q}}[G_i]) = A(\bar{\mathbb{Q}}[G_j]) = (0)$ (recall that in general $A(R) \cap J(R) = N(R)$). From our theorem, it follows that $A(\bar{\mathbb{Q}}[G_i \times G_j]) = (0)$ and by Theorem 1 of [4], it follows that $J(\bar{\mathbb{Q}}[G_1 \times \cdots \times G_n]) = (0)$. Using Theorem 2 of [4], we conclude that $J(\bar{\mathbb{Q}}[G_1 \times \cdots \times G_n]) = (0)$; whence, $J(\mathbb{Q}[G_1 \times \ldots \times G_n]) = (0)$, a contradiction.

Recall that a group is said to be an FC group if each element of the group has only finitely many conjugates. If p is a prime integer or zero, we say that a group is a p'-group if it has no elements of order p. In the case when $p = 0$, this means that every group is a $0'$-group.

THEOREM 2. *Let F be a field of characteristic p and let G be an* FC *p'-group. Then $A(F[G]) \neq (0)$ if and only if G is locally finite.*

PROOF. Suppose that $A(F[G]) \neq (0)$. Let H be the torsion subroup of G (so that G/H is torsion-free Abelian with a total ordering, $\leq$) [5, Lemmas 4.1.6 and 13.1.6]. If $H = G$, then the proof is completed; so assume that $H \neq G$. Let S be a complete set of coset representatives of H in G. Choose nonzero $\alpha \in A(F[G])$ and express α in the form $\alpha = x_1 s_1 + \cdots + x_n s_n$, with $x_i \in F[H]$, $s_i \in S$ and $\bar{s}_1 > \bar{s}_2 > \cdots > \bar{s}_n$ where $\bar{}: G \longrightarrow G/H$ is the natural map. We can choose α so that $\bar{s}_1 > 1$. Let N be the (finite) normal subgroup of G generated by the supports of $\{x_1, \ldots, x_n\}$. The element s_1 acts on N by conjugation and as N is finite, some power of s_1, say s_1^k, acts trivially on N. Now let

$\beta = \alpha s_1^{k-1}$. If $\lambda_t, \lambda_{t-1}, \ldots, \lambda_0 \in F$, then $\lambda_t\beta^t + \lambda_{t-1}\beta^{t-1} + \cdots \lambda_0 = \lambda_t x_1^t s_1^{tk} + \sum y_\sigma s'_\sigma$, where $y_j \in F[N]$, $s'_j \in S$ and $\bar{s}_1^{tk} > \bar{s}'_j$ for all j. If β is algebraic, then we must have $x_1^t = 0$ for some t. It follows that if $A(F[G]) \neq (0)$, then $N(F[H]) \neq (0)$, which is impossible as H is a p'-group [5, Ch.2, Theorem 3.4].

The group algebra of a locally finite group is algebraic.

THEOREM 3. *Let F be a field of characteristic p and let G be a solvable p'-group. Then $A(F[G]) \neq (0)$ if and only if G is locally finite.*

PROOF. Let $\mathcal{Z}(G)$ denote the Zalesskii subgroup of G [5, Ch.9, Theorem 1.17]. If $A(F[G]) \neq (0)$, then $A(F[G]) \cap F[\mathcal{Z}(G)] \neq (0)$; whence, $A(F[\mathcal{Z}(G)]) \neq (0)$. It follows that $\mathcal{Z}(G)$ is a torsion group (for the Zalesskii subgroup is an FC group). If $\bar{x} \in G/\mathcal{Z}(G)$ is of infinite order, then x acts on $\mathcal{Z}(G)$ through conjugation. Choose a nonzero $\alpha \in A(F[G]) \cap F[\mathcal{Z}(G)]$ and let n be a positive integer. Then $x^n\alpha \in A(F[G])$, and, as x^n is infinite cyclic, there is polynomial of degree t of which $x^n\alpha$ is a root; say $\lambda_t(x^n\alpha)^t + \cdots + \lambda_1(x^n\alpha) + \lambda_0 = 0$. Re-expressing the left side as multiples of powers of x^n we have $(\lambda_t\alpha\alpha^{x^n}\cdots\alpha^{x^{tn}})t^{nt}$+ terms in which x occurs to a power $< nt$. As this sum is zero $\alpha^{x^n}\alpha^{x^{2n}}\cdots\alpha^{x^{tn}} = 0$. Thus $F[\mathcal{Z}(G)]$ has a σ-nil ideal [2], and therefore there is a finite normal subgroup N of $\mathcal{Z}(G)$ such that $F[N]$ has a σ-nil ideal [2,Lemma 7]. As N is finite, this implies that $F[N]$ has a nil ideal which is impossible as N is a p'-group.

COROLLARY. *Let G be a non-locally finite solvable group and let H be an arbitrary group. Then $J(\mathbb{Q}[G \times H]) = (0)$.*

PROOF. By Theorem 3, $A(\mathbb{Q}[G]) = (0)$. The rational group algebra of a solvable group is semiprimitive, so by Theorem 2 of [4], $J(\mathbb{Q}[G \times H]) = (0)$.

THEOREM 4. *Let $\bar{\mathbb{Q}}$ denote the algebraic closure of the rationals and let G be a group. Then $A(\bar{\mathbb{Q}}[G])$ is the $\bar{\mathbb{Q}}$ linear span of its idempotents. If $A(\bar{\mathbb{Q}}[G])$ is locally finite, then it is von Neumann regular.*

PROOF. Let $*$ be the involution on $\bar{\mathbb{Q}}[G]$ which sends $\Sigma f_g g$ to $\Sigma \bar{f}_g g^{-1}$, where $\bar{f}_g$ denotes the complex conjugate of f_g. Amitsur proved that a nonzero symmetric element of $\bar{\mathbb{Q}}[G]$ is not nilpotent [1] and it follows in a similar manner, that a $*$-stable subalgebra of $\bar{\mathbb{Q}}[G]$ has no nonzero nil ideals. If $\alpha \in A(\bar{\mathbb{Q}}[G])$, then $\alpha = \frac{1}{2}(\alpha + \alpha^*) + \frac{1}{2}(\alpha - \alpha^*)$. Both $\mathbb{Q}[\alpha + \alpha^*]$ and $\bar{\mathbb{Q}}[\alpha - \alpha^*]$ have no nonzero nil ideals, so both are isomorphic to a finite direct sum of copies of $\bar{\mathbb{Q}}$. In particular, the ideal of $\bar{\mathbb{Q}}[\alpha+\alpha^*]$ generated by $\alpha+\alpha^*$ is a direct sum of copies of $\bar{\mathbb{Q}}$. Thus $\alpha + \alpha^* = \Sigma_i\lambda_i e_i$ with $\lambda_i \in \bar{\mathbb{Q}}$ and e_i idempotents which are multiples of $\alpha + \alpha^*$, so $e_i \in A(\bar{\mathbb{Q}}[G])$. Similarly $\alpha - \alpha^* = \Sigma_i\mu_i f_i$. It therefore follows by adding that α is a $\bar{\mathbb{Q}}$ linear sum of idempotents in the ideal $A(\bar{\mathbb{Q}}[G])$.

Finally, if $A(\bar{\mathbb{Q}}[G])$ is locally finite, then $\bar{\mathbb{Q}}[\alpha, \alpha^*]$ is a finite dimensional $*$-stable subalgebra of $\bar{\mathbb{Q}}[G]$. Thus it has no nonzero nil ideals and is von Neumann regular.

COROLLARY. *If G is a torsion-free group with the ascending chain condition on cyclic subgroups, then $A(\bar{\mathbb{Q}}[G]) = (0)$.*

PROOF. By [5, Ch.2, Corollary 3. 10], $\bar{\mathbb{Q}}[G]$ has no nontrivial idempotents.

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