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OSCILLATIONS IN A NONAUTONOMOUS DELAY LOGISTIC DIFFERENCE EQUATION

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Consider the nonautonomous delay logistic difference equation

$$\Delta y_n = p_n y_n \left(1 - \frac{y_{n-l_n}}{K}\right), \tag{E_0}$$

where $(p_n)_{n\geqq 0}$ is a sequence of nonnegative numbers, $(l_n)_{n\geqq 0}$ is a sequence of positive integers with $\lim_{n\to\infty}(n-l_n)=\infty$, and K is a positive constant. Only solutions which are positive for $n\geqq 0$ are considered. We established a sharp condition under which all solutions of (E_0) are oscillatory about the equilibrium point K. Also we obtained sufficient conditions for the existence of a solution of (E_0) which is nonoscillatory about K.

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1. Introduction

Difference equations provide an important first step in developing techniques for the analysis of dynamic phenomena in biology, both with respect to problem formulation and theoretical development. See, for example, [**1, 4, 5, 7** and **12**]. An example is the logistic difference equation, which arises in models of population dynamics as a discrete-time version of the logistic differential equation and as an approximation to a nonlinear renewal equation (see, e.g., [**3, 4, 7** and **12**]). This equation has been studied in many papers, including [**3, 8, 9, 10** and **13**]. Our aim in this paper is to study the oscillation about the positive equilibrium point of the solutions of a nonautonomous delay logistic difference equation.

The *logistic difference equation* is

$$N_{n+1} = N_n(a - bN_n),$$

where a and b are positive constants. By writing

$$y_n = \frac{b}{a} N_n,$$

the equation may be brought into canonical form

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$$y_{n+1} = ay_n(1-y_n).$$

This equation has two equilibrium points, the origin and $K=(a-1)/a$ which has physical meaning only for $a>1$. For $a>1$, the last equation takes the form

$$\Delta y_n = py_n\left(1-\frac{y_n}{K}\right),$$

where $p=a-1>0$ and the forward difference operator Δ is defined as usual, i.e. $\Delta y_n = y_{n+1}-y_n$.

In this paper, we consider the *nonautonomous delay logistic difference equation*

$$\Delta y_n = p_n y_n\left(1-\frac{y_{n-l_n}}{K}\right), \tag{E_0}$$

where $(p_n)_{n\geqq 0}$ *is a sequence of nonnegative numbers,* $(l_n)_{n\geqq 0}$ *is a sequence of positive integers with*

$$\lim_{n\to\infty}(n-l_n)=\infty,$$

and K is a positive constant.

Throughout the paper we will use the notation

$$l = -\min_{n\geqq 0}(n-l_n).$$

Clearly, l is a positive integer.

By a *solution* of (E_0) we mean a sequence $(y_n)_{n\geqq -l}$ which satisfies (E_0) for all $n\geqq 0$.

Motivated by the plausible applications of (E_0), in what follows *we consider only solutions* $(y_n)_{n\geqq -l}$ of (E_0) *with* $y_n>0$ *for every* $n\geqq 0$.

We are concerned with the oscillation of the solutions of (E_0) about the equilibrium point K. A solution $(y_n)_{n\geqq -l}$ of (E_0) is said to be *oscillatory about* K if the terms of the sequence are neither eventually greater than K nor eventually less than K. Otherwise, the solution is called *nonoscillatory about* K.

The substitution

$$x_n = \frac{y_n}{K}-1,\ n\geqq -l$$

transforms (E_0) into the equation

$$\Delta x_n + p_n(1+x_n)x_{n-l_n} = 0. \tag{E}$$

Clearly, the oscillation of $(y_n)_{n\geqq -l}$ about K is equivalent with the usual oscillation (i.e. the oscillation about the origin) of $(x_n)_{n\geqq -l}$. As usual, a solution $(x_n)_{n\geqq -l}$ of (E) is called *oscillatory* if it is neither eventually positive nor eventually negative, and otherwise the solution is called *nonoscillatory*.

Moreover, the fact that $y_n>0$ for $n\geqq 0$ is equivalent with the statement $1+x_n>0$ for $n\geqq 0$. So, in the sequel *we consider only such solutions $(x_n)_{n\geqq -l}$ of (E) which satisfy*

$$1+x_n>0 \quad \textit{for all} \quad n\geqq 0.$$

The paper is organized as follows: In Section 2 we give a sharp condition for the oscillation of all solutions of (E). Sufficient conditions for the existence of a nonoscillatory solution of (E) are presented in Section 3.

For related results concerning the oscillation of the solutions of (E) we refer to the recent paper [2]. Moreover, we note that oscillation and nonoscillation criteria for a nonautonomous delay logistic differential equation are obtained in [14].

2. A sharp condition for oscillations

The main result in this section is Theorem 1 below, which provides a sufficient condition for the oscillation of all solutions of the delay difference equation (E) where the sequence $(n-l_n)_{n\geqq 0}$ is assumed to be increasing.

Theorem 1. *Let the sequence $(n-l_n)_{n\geqq 0}$ be increasing and assume that*

$$\liminf_{n\to\infty}\left(\frac{1}{l_n}\sum_{k=n-l_n}^{n-1} p_k\right)>\limsup_{n\to\infty}\frac{l_n^{l_n}}{(l_n+1)^{l_n+1}}. \tag{i}$$

Then all solutions of (E) are oscillatory.

In order to prove Theorem 1 we make use of the following known result, in which *$(P_n)_{n\geqq 0}$ is assumed to be a sequence of nonnegative numbers.*

Theorem 0. *Let the sequence $(n-l_n)_{n\geqq 0}$ be increasing and assume that*

$$\liminf_{n\to\infty}\left(\frac{1}{l_n}\sum_{k=n-l_n}^{n-1} P_k\right)>\limsup_{n\to\infty}\frac{l_n^{l_n}}{(l_n+1)^{l_n+1}}.$$

Then:

(a) *There is no sequence of numbers $(x_n)_{n\geqq -l}$ which is eventually positive and satisfies*

$$\Delta x_n+P_n x_{n-l_n}\leqq 0 \quad \textit{for all large } n.$$

(b) *There is no sequence of numbers $(x_n)_{n\geqq -l}$ which is eventually negative and satisfies*

$$\Delta x_n + P_n x_{n-l_n} \geqq 0 \quad \textit{for all large } n.$$

It has been proved in [11] (see also [6] for the special case where $l_n = l$ for all $n \geqq 0$) that, under the assumptions of Theorem 0, all solutions of the linear delay difference equation

$$\Delta x_n + P_n x_{n-l_n} = 0$$

are oscillatory. A slight modification in the proof of this result leads to Theorem 0.

To prove Theorem 1 we also need the next lemma.

Lemma 1. *Assume that*

$$\sum_{n=0}^{\infty} p_n = \infty. \tag{ii}$$

Then every nonoscillatory solution of (E) *tends to zero as* $n \to \infty$.

Proof. Let $(x_n)_{n \geqq -l}$ be a nonoscillatory solution of (E). Then there exists an integer $n_0 \geqq -l$ such that $(x_n)_{n \geqq n_0}$ is either positive or negative.

Consider first the case where $x_n > 0$ for all $n \geqq n_0$. We choose a nonnegative integer $n_1 \geqq n_0$ such that

$$n - l_n \geqq n_0 \quad \text{for every} \quad n \geqq n_1.$$

Then from (E) it follows that $\Delta x_n \leqq 0$ for $n \geqq n_1$ and so the sequence $(x_n)_{n \geqq n_1}$ is decreasing. Hence, $\alpha \equiv \lim_{n \to \infty} x_n$ exists and is a nonnegative number. Assume, for the sake of contradiction, that $\alpha > 0$. Then from (E) we obtain

$$\Delta x_n = -p_n(1 + x_n)x_{n-l_n} \leqq -p_n x_{n-l_n} \leqq -\alpha p_n$$

for all $n \geqq n_2$, where the integer $n_2 \geqq n_1$ is chosen so that

$$n - l_n \geqq n_1 \quad \text{for every} \quad n \geqq n_2.$$

Hence, we get for $n \geqq n_2$

$$x_{n+1} - x_{n_2} = \sum_{k=n_2}^{n} \Delta x_k \leqq -\alpha \sum_{k=n_2}^{n} p_k$$

and consequently

$$\alpha - x_{n_2} \leqq -\alpha \sum_{k=n_2}^{\infty} p_k,$$

which contradicts (ii). Thus, $\alpha \equiv \lim_{n\to\infty} x_n$ must be zero.

Suppose now that $x_n<0$ for all $n \geqq n_0$ and choose the nonnegative integer $n_1 \geqq n_0$ as above. By our general hypothesis, we have $1+x_n>0$ for $n \geqq 0$. From (E) we derive $\Delta x_n \geqq 0$ for every $n \geqq n_1$, which means that $(x_n)_{n \geqq n_1}$ is an increasing sequence. This implies that $\beta \equiv \lim_{n\to\infty} x_n$ exists and is a nonpositive number. Let β be negative. Then from (E) it follows that

$$\Delta x_n = -p_n(1+x_n)x_{n-l_n} \geqq -(1+x_{n_2})\beta p_n$$

for every $n \geqq n_2$, where the integer $n_2 \geqq n_1$ is chosen as in the first case. So, we obtain for $n \geqq n_2$

$$x_{n+1} - x_{n_2} = \sum_{k=n_2}^{n} \Delta x_k \geqq -(1+x_{n_2})\beta \sum_{k=n_2}^{n} p_k$$

and hence

$$\beta - x_{n_2} \geqq -(1+x_{n_2})\beta \sum_{k=n_2}^{\infty} p_k,$$

where $1+x_{n_2}>0$. This contradicts (ii) and so $\beta \equiv \lim_{n\to\infty} x_n = 0$.

The proof of the lemma is complete.

Proof of Theorem 1. Assume, for the sake of contradiction, that there is a nonoscillatory solution $(x_n)_{n \geqq -l}$ of the difference equation (E). Then we can choose an integer $n_0 \geqq -l$ so that the sequence $(x_n)_{n \geqq n_0}$ is either positive or negative. We consider the following two cases:

Case 1: $x_n>0$ *for all* $n \geqq n_0$. Let $n_1 \geqq n_0$ be a nonnegative integer such that

$$n - l_n \geqq n_0 \quad \text{for every} \quad n \geqq n_1.$$

Then from (E) it follows that

$$\Delta x_n + p_n x_{n-l_n} \leqq 0 \quad \text{for all} \quad n \geqq n_1.$$

By Theorem 0, this is a contradiction.

Case 2: $x_n<0$ *for all* $n \geqq n_0$. In view of condition (i), we can choose a number $\varepsilon \in (0,1)$ such that

$$(1-\varepsilon)\liminf_{n\to\infty}\left(\frac{1}{l_n}\sum_{k=n-l_n}^{n-1} p_k\right) > \limsup_{n\to\infty} \frac{l_n^{l_n}}{(l_n+1)^{l_n+1}}.$$

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That is

$$\liminf_{n\to\infty}\left(\frac{1}{l_n}\sum_{k=n-l_n}^{n-1}P_k\right)>\limsup_{n\to\infty}\frac{l_n^{l_n}}{(l_n+1)^{l_n+1}},$$

where

$$P_n=(1-\varepsilon)p_n\geqq 0\quad (n=0,1,\ldots).$$

Furthermore, we observe that from condition (i) it follows that

$$\liminf_{n\to\infty}\sum_{k=n-l_n}^{n-1}p_k>0,$$

which obviously implies that condition (ii) is satisfied. Hence, in view of Lemma 1, we have

$$\lim_{n\to\infty}x_n=0.$$

Thus, if we consider a nonnegative integer $n_1^*\geqq n_0$ such that

$$n-l_n\geqq n_0\quad\text{for}\quad n\geqq n_1^*$$

and

$$x_n>-\varepsilon\quad\text{for all}\quad n\geqq n_1^*,$$

then from (E) we obtain for $n\geqq n_1^*$

$$0=\Delta x_n+p_n(1+x_n)x_{n-l_n}\leqq\Delta x_n+p_n(1-\varepsilon)x_{n-l_n}.$$

So, we have

$$\Delta x_n+P_nx_{n-l_n}\geqq 0\quad\text{for all}\quad n\geqq n_1^*.$$

In view of Theorem 0, this is impossible.

The proof of the theorem is now complete.

3. Sufficient conditions for the existence of nonoscillatory solutions

The main purpose in this section is to prove the following theorem, which establishes some conditions under which the delay difference equation (E) has a nonoscillatory solution.

Theorem 2. *Suppose that*

$$\sum_{i=0}^{L_n-1} p_{n+i}>0 \quad \text{for all} \quad n\geqq 0, \tag{iii}$$

where

$$L_n=\max\{\rho\in\{1,2,\dots\}: -l\leqq \mu-l_\mu\leqq n-1 \quad \text{for} \quad n\leqq\mu\leqq n+\rho-1\} \quad (n=0,1,\dots).$$

Moreover, assume that there exist two numbers c and γ with $0<c<\gamma<1$ such that

$$p_n<\gamma \quad \text{for every} \quad n\geqq 0 \tag{iv}$$

and

$$\frac{1}{\gamma}\prod_{k=n-l_n}^{n-1}\left(1-\frac{1}{\gamma}\tilde{p}_k\right)\geqq 1+\left(1-\frac{1}{\gamma}c\right)^{l+1} \quad \text{for all} \quad n\geqq 0, \tag{v}$$

where

$$\tilde{p}_n=\begin{cases} p_n, & \text{if} \quad n\geqq 0\\ c, & \text{if} \quad -l-1\leqq n<0.\end{cases}$$

Then there exists a positive solution $(x_n)_{n\geqq -l}$ of (E) with $\lim_{n\to\infty} x_n=0$.

We remark that in the special case where $l_n=l$ $(n=0,1,\dots)$ we also have $L_n=l$ for all $n\geqq 0$. The proof of Theorem 2 is based on the following lemma.

Lemma 2. *Suppose that* (iii) *holds, where $(L_n)_{n\geqq 0}$ is defined as in Theorem 2, and let $(z_n)_{n\geqq -l}$ be a positive solution of the delay difference inequality*

$$\Delta z_n+p_n(1+z_n)z_{n-l_n}\leqq 0. \tag{I}$$

Then there exists a positive solution $(x_n)_{n\geqq -l}$ of (E) with $\lim_{n\to\infty}x_n=0$ and such that $x_n\leqq z_n$ for $n\geqq -l$.

Note. By a *solution* of (I) we mean a sequence $(z_n)_{n\geqq -l}$ which satisfies (I) for all $n\geqq 0$.

Proof. The method of proof is similar to that of a lemma in **[11]** (see also the proof of Theorem 3 in **[6]** for the special case where $l_n=l$ for $n=0,1,\dots$).

From (I) it follows that for $\nu\geqq n\geqq 0$

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$$z_n > -(z_{\nu+1} - z_n) = -\sum_{k=n}^{\nu} \Delta z_k \geqq \sum_{k=n}^{\nu} p_k(1+z_k)z_{k-l_k}$$

and consequently

$$z_n \geqq \sum_{k=n}^{\infty} p_k(1+z_k)z_{k-l_k} \quad \text{for every} \quad n \geqq 0.$$

Next, we consider the space $\mathscr{X}$ of all sequences $(x_n)_{n \geqq -l}$ which satisfy

$$x_n = z_n \text{ for } -l \leqq n < 0, \text{ and } 0 \leqq x_n \leqq z_n \text{ for } n \geqq 0.$$

For any sequence $(x_n)_{n \geqq -l}$ in $\mathscr{X}$, we put

$$Sx_n = \begin{cases} z_n, & \text{if} \quad -l \leqq n < 0 \\ \displaystyle\sum_{k=n}^{\infty} p_k(1+x_k)x_{k-l_k}, & \text{if} \quad n \geqq 0. \end{cases}$$

We immediately see that this formula defines an operator $S: \mathscr{X} \to \mathscr{X}$. This operator is monotonic in the sense that, if $(x_n^1)_{n \geqq -l}$ and $(x_n^2)_{n \geqq -l}$ are two sequences in $\mathscr{X}$ with $x_n^1 \leqq x_n^2$ for $n \geqq -l$, then $Sx_n^1 \leqq Sx_n^2$ for all $n \geqq -l$. Now, we define

$$x_n^0 = z_n \quad \text{for} \quad n \geqq -l$$

and

$$x_n^r = Sx_n^{r-1} \text{ for } n \geqq -l \quad (r = 1, 2, \ldots)$$

and we see that $(x_n^r)_{n \geqq -l}$ belongs to $\mathscr{X}$ for every nonnegative integer r. Moreover, it is easy to see that

$$x_n^0 \geqq x_n^1 \geqq x_n^2 \geqq \cdots \quad \text{for all} \quad n \geqq -l.$$

Furthermore, we set

$$x_n = \lim_{r \to \infty} x_n^r \quad \text{for} \quad n \geqq -l$$

and we observe that $0 \leqq x_n \leqq z_n$ for every $n \geqq -l$. Moreover, we have $x_n = Sx_n$ for all $n \geqq -l$ and consequently

$$x_n=\begin{cases} z_n, & \text{if} \quad -l \leqq n < 0 \\ \sum_{k=n}^{\infty} p_k(1+x_k)x_{k-l_k}, & \text{if} \quad n \geqq 0. \end{cases}$$

This gives

$$\lim_{n\to\infty} x_n = 0.$$

Also, we get

$$\Delta x_n = -p_n(1+x_n)x_{n-l_n} \quad \text{for all} \quad n \geqq 0$$

and therefore the sequence $(x_n)_{n\geqq -l}$ is a solution of (E). Since $x_n = z_n > 0$ for $-l \leqq n < 0$, it remains to show that $x_n > 0$ for all $n \geqq 0$. Assume, for the sake of contradiction, that the sequence $(x_n)_{n\geqq 0}$ has at least one zero. Then these exists an integer $n_0 \geqq 0$ such that

$$x_n > 0 \quad \text{for} \quad -l \leqq n < n_0, \quad \text{and} \quad x_{n_0} = 0.$$

By condition (iii) and the definition of the integer L_{n_0}, we obtain

$$\begin{aligned} 0 \leqq x_{n_0+L_{n_0}} &= x_{n_0+L_{n_0}} - x_{n_0} = \sum_{\mu=n_0}^{n_0+L_{n_0}-1} \Delta x_\mu \\ &= -\sum_{\mu=n_0}^{n_0+L_{n_0}-1} p_\mu(1+x_\mu)x_{\mu-l_\mu} \\ &\leqq -\left[\min_{n_0\leqq\mu\leqq n_0+L_{n_0}-1} (1+x_\mu)x_{\mu-l_\mu}\right] \sum_{\mu=n_0}^{n_0+L_{n_0}-1} p_\mu \\ &= -\left[\min_{n_0\leqq\mu\leqq n_0+L_{n_0}-1} (1+x_\mu)x_{\mu-l_\mu}\right] \sum_{i=0}^{L_{n_0}-1} p_{n_0+i} \\ &< 0, \end{aligned}$$

which is a contradiction, and so the proof of our lemma is complete.

Proof of Theorem 2. Define (cf. [**11**, Theorem 5] or [**6**, Corollary 1])

$$z_n = \prod_{k=-l-1}^{n-1} \left(1 - \frac{1}{\gamma}\tilde{p}_k\right) \quad \text{for} \quad n \geqq -l.$$

By taking into account the fact that $0<c<\gamma$ and condition (iv), we can see that $z_n>0$ for all $n \geqq -l$. Furthermore, by using condition (v), we derive for each $n \geqq 0$

$$\begin{aligned}
\Delta z_n &= \left[\left(1-\frac{1}{\gamma}\tilde{p}_n\right)-1\right] \prod_{k=-l-1}^{n-1}\left(1-\frac{1}{\gamma}\tilde{p}_k\right) \\
&= -\frac{1}{\gamma}p_n\left[\prod_{k=-l-1}^{n-l_n-1}\left(1-\frac{1}{\gamma}\tilde{p}_k\right)\right]\left[\prod_{k=n-l_n}^{n-1}\left(1-\frac{1}{\gamma}\tilde{p}_k\right)\right] \\
&= -p_n z_{n-l_n}\left[\frac{1}{\gamma}\prod_{k=n-l_n}^{n-1}\left(1-\frac{1}{\gamma}\tilde{p}_k\right)\right] \leqq -p_n z_{n-l_n}\left[1+\left(1-\frac{1}{\gamma}c\right)^{l+1}\right] \\
&\leqq -p_n z_{n-l_n}\left[1+\prod_{k=-l-1}^{n-1}\left(1-\frac{1}{\gamma}\tilde{p}_k\right)\right] = -p_n z_{n-l_n}(1+z_n).
\end{aligned}$$

Hence, the sequence $(z_n)_{n \geqq -l}$ is a positive solution of the delay difference inequality (I). So, an application of Lemma 2 completes our proof.

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