Canad. Math. Bull. Vol. **18** (5), 1975

A BANACH SPACE WHOSE ELEMENTS ARE CLASSES OF SETS OF CONSTANT WIDTH

BY
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Let K be a compact subset of the real Euclidean space E^n. We say that K has *constant width* if the distance between each pair of distinct parallel hyperplanes which support K is constant. The collection of all compact convex subsets of E^n which have constant width is denoted $\mathscr{K}^n$.

The metric for E^n induces the Hausdorff metric h on $\mathscr{K}^n$, and the linear structure of E^n induces a corresponding algebraic structure for $\mathscr{K}^n$(see §1). The algebraic structure does not make $\mathscr{K}^n$ a vector space. However, Ewald and Shephard [4] have shown that by considering equivalence classes rather than individual elements of $\mathscr{K}^n$, a normed linear space $(K^n, |\cdot|)$ can be obtained.

In [4] an example was given showing that $(K^n, |\cdot|)$ is incomplete, whereas the opposite is true for the metric space $(\mathscr{K}^n, h)$. One may therefore ask if there is a norm under which K^n becomes a complete space. We shall show that such a renorming of K^n is possible; in fact, there is a norm, $\|\cdot\|$, such that $(K^n, \|\cdot\|)$ is a conjugate Banach space.

It also turns out that the extremal structure of the closed unit ball, V, of $(K^n, \|\cdot\|)$ is closely related to certain geometric properties of the elements of $\mathscr{K}^n$. Namely, there is a correspondence between the scalar multiples of the extreme points of V and the indecomposable elements of $\mathscr{K}^n$. (A subset K is *indecomposable* in $\mathscr{K}^n$ if $K=K_1+K_2$ with K_1, K_2 in $\mathscr{K}^n$ implies that at least one of the subsets K_1 or K_2 is equal to $\lambda K+x$ for some $\lambda>0$ and some $x \in E^n$.) By using this correspondence it is possible to obtain results concerning the approximation properties of the indecomposable sets of constant width, similar to those obtained by Shephard [5, Chapter 15], [7], and Berg [1].

1. **The Ewald-Shephard Embedding.** The family $\mathscr{K}^n$ is closed under the operations of scalar multiplication ($\lambda K=\{\lambda x : x \in K\}$), and Minkowski addition ($K_1+K_2=\{x_1+x_2 : x_1 \in K_1, x_2 \in K_2\}$). Also, since all the members of $\mathscr{K}^n$ are convex (in fact, strictly convex), the cancellation law holds for Minkowski addition (that is, $K_1+K=K_2+K$ implies that $K_1=K_2$). Despite these properties, $\mathscr{K}^n$ is not a vector space, for any member of $\mathscr{K}^n$, other than a singleton, does not possess an additive inverse.

In [4], Ewald and Shephard showed how a linear structure may be introduced

Received by the editors November 17, 1971 and, in revised form, January 5, 1973.

https://doi.org/10.4153/CMB-1975-119-6 Published online by Cambridge University Press

by considering equivalence classes of $\mathscr{K}^n$: Two members K_1 and K_2 of $\mathscr{K}^n$ are defined to be equivalent, $K_1 \approx K_2$, if there are closed balls B_1 and B_2 such that

$$K_1+B_1 = K_2+B_2. \tag{1}$$

The equivalence class containing K is denoted $[K]$, and the family of equivalence classes of $\mathscr{K}^n$ is denoted K^n. (In [4], this family of equivalence classes is denoted $\mathscr{P}^n_w$). A linear structure is introduced on K^n by defining the following operations:

$$\begin{aligned} \lambda[K] &= [\lambda K] \\ [K_1]+[K_2] &= [K_1+K_2]. \end{aligned} \tag{2}$$

These definitions are consistent, and it follows that K^n is a vector space whose null element is the class of all closed balls of E^n (including those with zero radius).

As mentioned previously, in [4] a norm $|\cdot|$ was defined on K^n. This norm may be described by using the fact that each element K of $\mathscr{K}^n$ has a unique insphere and a unique circumsphere. These spheres have a common center, and if $r(K)$ and $R(K)$ are their respective radii, then $r(K)+R(K)$ is the width of K [3, p. 125]. In view of this, the number $R(K)-r(K)$ is the same for each $K \in [K]$, and $|[K]|$ is defined to be this number.

It was proved in [4] that if h denotes the Hausdorff metric for $\mathscr{K}^n$ ($h(K_1, K_2)=\inf\{\lambda>0: K_1 \subset K_2+\lambda B; K_2 \subset K_1+\lambda B\}$, where B is the closed unit ball of E^n), then the natural mapping

$$i:(\mathscr{K}^n, h) \rightarrow (K^n, |\cdot|)$$

is continuous.

As well as proving that $(K^n, |\cdot|)$ is not complete, an example was provided to show that the closed unit ball of $(K^n, |\cdot|)$ was not strictly convex [4, p. 8]. The same example serves to illustrate the fact that, with the norm $|\cdot|$, it is possible that K_1, K_2, and K_3 be indecomposable in $\mathscr{K}^n$ while

$$[K_1] = [K_2]+[K_3]$$

and,

$$\tfrac{1}{2}\,|[K_1]| = |[K_2]| = |[K_3]| = 1.$$

This shows that in general the indecomposable elements of $\mathscr{K}^n$ do not correspond to the extreme points of the closed unit ball of $(K^n, |\cdot|)$.

We would mention at this point that as well as considering $\mathscr{K}^n$, it was shown in [4] and [8] how more general families of convex sets can be embedded in normed linear spaces. In each case, properties of the vector spaces corresponded to geometrical properties of convex sets. In particular, it was shown that the study of complementary subspaces of K^n is equivalent to certain decomposition theorems for the members of $\mathscr{K}^n$.

2. **A complete norm for K^n.** To obtain a complete norm for K^n, we first choose as a representative of each class $[K]$ that member of $[K]$ whose Steiner point (see

below) is the origin and which has minimal width. The norm of $[K]$ will then be defined as the width of this representative.

The *Steiner point* of a convex set K is defined to be [7]:

$$s(K) = \int_{\Omega} u\, H(K, u)\, d\omega \Big/ \int_{\Omega} \langle w, u\rangle^2\, d\omega. \tag{3}$$

where $u \in \Omega = \{u \in E^n : \|u\| = 1\}$, and $H(K, u)$ is the support functional for K:

$$H(K, u) = \sup\{\langle x, u\rangle : x \in K\}, \tag{4}$$

and where w is an arbitrary, but fixed unit vector.

The mapping $K \to s(K)$ from $(\mathscr{K}^n, h)$ into E^n has the following properties which may be readily deduced from (3) and (4) (see [7]).

(5) s continuous and $s(K) \in K$

$$s(\lambda K) = \lambda s(K)$$

$$s(K_1 + K_2) = s(K_1) + s(K_2). \tag{6}$$

By using the cancellation law for addition of convex sets, it is readily seen that

2.1. *$K_1 \approx K_2$ if and only if either $K_1 = K_2 + B_2$ or $K_2 = K_1 + B_1$, where B_i is a closed ball (perhaps of zero radius).*

In view of 2.1, given an equivalence class $[K]$ there is clearly a unique $K_0 \in [K]$ with the following properties:

(7) The Steiner point $s(K_0)$ of K_0 is the origin of E^n.

(8) For any other member K_1 of $[K]$, there is some closed ball B_1 such that

$$K_1 = K_0 + B_1.$$

The element K_0 of $[K]$ defined by (7) and (8) above will be denoted $a[K]$, and is called the *apex* of $[K]$. Note that if $[K]$ contains an indecomposable member K_1 of $\mathscr{K}^n$, then K_1 is a translate of $a[K]$.

The linear space K^n may be given a norm by

$$\|[K]\| = \operatorname{diam} a\,[K] \tag{9}$$

The proof that (9) actually defines a norm for K^n is straightforward and has therefore been omitted.

The unit ball of $(K^n, \|\cdot\|)$ will be denoted V. Most of our results about $(K^n, \|\cdot\|)$ are summed up by the following two theorems:

2.2. *The space $(K^n, \|\cdot\|)$ is a conjugate space. If $\mathscr{T}$ denotes the topology induced on V by the norm $|\cdot|$, then $\mathscr{T}$ coincides with the restriction of some w^*-topology to V.*

2.3. *Let $K \in \mathscr{K}^n$ with $a[K] \neq \{\bar{0}\}$. Then $a[K]$ is an indecomposable element of $\mathscr{K}^n$ if and only if $[K]/\|[K]\|$ is an extreme point of V.*

https://doi.org/10.4153/CMB-1975-119-6 Published online by Cambridge University Press

In order to prove 2.2, first recall that the Mackey-Arens Theorem [9, p. 248] states that if A and B are linear spaces in duality with respect to the bilinear form $\langle , \rangle$, and if $\mathscr{T}$ is any locally convex linear topology for A, then the family of $\mathscr{T}$-continuous linear functionals on A is identical with B if, and only if

$$w(A, B) \subset \mathscr{T} \subset m(A, B). \tag{10}$$

Here, $w(A, B)$ is the linear topology with a local base consisting of all subsets of the form $G^0 = \{f \in A : \langle f, x\rangle \leq 1$, for each $x \in G\}$, where G is a finite subset of B. The linear topology $m(A, B)$ is the one with a local base consisting of all subsets W^0, where W is a $w(B, A)$-compact, convex, circled (i.e., $\lambda W \subset W$ for all λ with $|\lambda| \leq 1$) subset of B.

The following simple and probably well-known, consequence of the Mackey-Arens Theorem will be used.

2.4. *Let E be a normed linear space with closed unit ball V. Suppose that $\mathscr{T}$ is a locally convex Hausdorff linear topology for E such that V is $\mathscr{T}$-compact. Let F be the space of all $\mathscr{T}$-continuous linear functionals on E. Then F can be given a norm $\|\cdot\|$ such that $F^* = E$, and the weak* topology, $w(E, F)$, coincides with $\mathscr{T}$ on V.*

Proof. By (10), $w(E, F) \subset \mathscr{T}$, and so V is a $w(E, F)$-compact circled convex subset of E. Similarly, for each $\lambda \neq 0$, λV is a $w(E, F)$-compact circled convex subset of E. In fact, the family $\mathscr{B} = \{\lambda V : \lambda \neq 0\}$ is an admissable family for the pairing $\langle E, F\rangle$ [6, p. 167]. Therefore, the family $\{B^0 : B \in \mathscr{B}\}$ is a local base for a linear topology $\mathscr{U}$ on F with the property that $w(F, E) \subset \mathscr{U}$ [6, pp. 167, 168]. It is also clear from the manner in which $\mathscr{U}$ has been defined that $\mathscr{U} \subset m(F, E)$ and thus the Mackey-Arens Theorem assures that the set of continuous linear functionals on $(F, \mathscr{U})$ is exactly E. Clearly, since $\mathscr{U}$ is Hausdorff, $\mathscr{U}$ is a norm topology whose closed unit ball is V^0, showing that E is a conjugate space.

The identity map $(V, \mathscr{T}) \to (V, w(E, F))$ is continuous (because $w(E, F) \subset \mathscr{T}$), and since $(V, w(E, F))$ is Hausdorff and $(V, \mathscr{T})$ is compact it follows that $(V, \mathscr{T})$ and $(V, w(E, F))$ are homeomorphic. $\square$

2.5. *Let $\mathscr{T}$ denote the $|\cdot|$ topology of K^n, and let V denote the closed unit ball of $(K^n, \|\cdot\|)$. Then V is $\mathscr{T}$-compact.*

Proof. In $(\mathscr{K}^n, h)$ let V' denote the subset $\{K : \operatorname{diam} K = 1$ and $s(K) = \bar{0}\}$. The fact that s is continuous together with Blaschke's Selection Theorem [3, p. 64] implies that V' is compact. It is readily verified that the natural map $i : (\mathscr{K}^n, h) \to (K^n, \mathscr{T})$ maps V' onto V, and since i is continuous, V must be compact (in the topology $\mathscr{T}$), being the continuous image of a compact space. $\square$

The results 2.4 and 2.5 together yield 2.2. We also remark that the normed space F constructed in 2.4 may not be complete. If $\tilde{F}$ denotes its completion, then of course

https://doi.org/10.4153/CMB-1975-119-6 Published online by Cambridge University Press

$\tilde{F}^*=E$. By the Banach-Alaoglu Theorem, $(V, w(E, \tilde{F}))$ is compact and it readily follows that $(V, \mathscr{T})$ and $(V, w(E, F))$ are homeomorphic (see also Theorem 3.4 of this paper).

A further remark seems to be in order. If E is a conjugate space, there may be Banach spaces F_1 and F_2 such that $F_1^*=F_2^*=E$, while F_1 and F_2 are not linearly isometric. In this case, there is no guarantee that $(V, w(E, F_1))$ and $(V, w(E, F_2))$ are homeomorphic. For example, one may take $F_1=c_0$ and $F_2=c$, with $F_1^*=F_2^*=l_1$.

Incidentally, note that Theorem 2.2 implies that $(K^n, \|\cdot\|)$ is the conjugate of some separable Banach space, due to the fact that some weak* topology for V is metrizable [2, p. 426].

To prove 2.3 we will need

2.6. *Let K_1 and K_2 be elements of K^n and suppose that $a[K_1+K_2]=K_1+K_2$. Then, for every pair (λ_1, λ_2) of non-negative scalars, $a[\lambda_1K_1+\lambda_2K_2]=\lambda_1K_1+\lambda_2K_2$.*

Proof. If $\lambda_1=\lambda_2=0$, the proposition is clearly true. Therefore we may assume that $\lambda_1\geq\lambda_2$ and $\lambda_1>0$. Now suppose that $a[\lambda_1K_1+\lambda_2K_2]=K\neq\lambda_1K_1+\lambda_2K_2$. By (6) $s(\lambda_1K_1+\lambda_2K_2)=\bar{0}=s(K)$, and so the definition of $a[\lambda_1K_1+\lambda_2K_2]$ implies that

$$\lambda_1K_1+\lambda_2K_2 = K+B,$$

where B is a closed ball of positive radius. Therefore,

$$K_1+K_2 = \lambda_1^{-1}(\lambda_1-\lambda_2)K_2+\lambda_1^{-1}K+\lambda_1^{-1}B.$$

Since $\lambda_1^{-1}B$ is a closed ball of positive radius, this shows that

$$a[K_1+K_2] \neq K_1+K_2,$$

which contradicts the hypothesis of the proposition. $\square$

Proof of 2.3. Let $K_0=a[K_0]$. Since $K_0\not\equiv\{\bar{0}\}$, $\|[K_0]\|\neq 0$, and so K_0 is indecomposable in $\mathscr{K}^n$ if and only if $K_0/\|[K_0]\|$ is an indecomposable element of $\mathscr{K}^n$. Therefore, there is no loss of generality in assuming that $\|[K_0]\|=1$.

Suppose that $[K_0]$ is not an extreme point of the unit ball of K^n, i.e. $[K_0]=\lambda[K_1]+(1-\lambda)K_2$ for some two points $[K_1]$ and $[K_2]$ of norm 1 and for some λ between 0 and 1. Since K_0 is the apex of $[K_0]$, we have $K_0+B=\lambda K_1+(1-\lambda)K_2$ where K_1, K_2 are apices and B is a closed ball. Since the widths of K_0, K_1, and K_2 are all equal to unity, it follows that B is the singleton $\{\bar{0}\}$ and so $K_0=\lambda K_1+(1-\lambda)K_2$, that is, K_0 is decomposable in $\mathscr{K}^n$.

Suppose now that K_0 is decomposable in $\mathscr{K}^n$. Then, there are elements K_1 and K_2 of $\mathscr{K}^n$ for which

$$K_1+K_2 = K_0,$$

with $K_i\neq\lambda K_0+x$ for any $\lambda>0$, $x\in R^n$. Since $s(K_0)=\bar{0}$, we have $s(K_1)=-s(K_2)$, and so we may assume that $s(K_1)=s(K_2)=\bar{0}$. Since $a[K_0]=K_0$, by 2.6 it follows that $a[K_1]=K_1$ and $a[K_2]=K_2$. Since $\|[K_i]\|=\text{diam } K_i$ for $i=0, 1, 2$, we have

$$\|[K_0]\| = \|[K_1]\|+\|[K_2]\| = 1$$
$$\Rightarrow \qquad [K_0] = \|[K_1]\|\,[K_1/\|[K_1]\|]+\|[K_2]\|\,[K_2/\|[K_2]\|],$$

which completes the proposition. □

3. **Comparison of K^n and $\mathscr{K}^n$.** This section is primarily concerned with the properties of the two mappings $a:K^n\to\mathscr{K}^n$ and $i:\mathscr{K}^n\to K^n$, where a takes the equivalence class $[K]$ to its apex $a[K]$, and where i is the natural map taking each set K to its equivalence class $[K]$.

It is always true that $ia[K]=[K]$, but in general $a[i(K)]$ will be different than K.

As already mentioned, it has been shown in [4] that the mapping $i:(\mathscr{K}^n, h)\to(K^n, |\cdot|)$ is continuous. Our first result shows that this is not true if the norm $|\cdot|$ is replaced by the stronger norm $\|\cdot\|$.

3.1. *The natural map $i:(\mathscr{K}^n, h)\to(K^n, \|\cdot\|)$ is not continuous.*

Proof. Let A_i denote a regular Reuleaux m-gon $m=2i+1$, $i=1, 2, \ldots$, whose centroid is the origin of E^2 and whose diameter is 1.

We shall use the following facts:

(11) $s(A_i) = \bar{0}$, $i = 1, 2, \ldots.$

(12) For $i \neq j$, $A_i - A_j$ always has a rough point in its boundary.

(13) $\lim_{i\to\infty} h(A_i, B/2) = 0$, where B is the unit ball of E^2.

The preceding statements show that $i:(\mathscr{K}^2, h)\to(K^2, \|\cdot\|)$ is not continuous. To see this, from (11) and (12) we conclude that $a[A_i-A_j]=A_i-A_j$, and hence $\|[A_i-A_j]\|=2$ for $i\neq j$. This shows that $\{[A_i]:i=1, 2, \ldots\}$ is certainly not a Cauchy sequence in $(K^2, \|\cdot\|)$. On the other hand, (13) shows that $\{A_i:i=1, 2, \ldots\}$ is a sequence in $(\mathscr{K}^2, h)$ converging to $B/2$. Thus, $i:(\mathscr{K}^2, h)\to(K^2, \|\cdot\|)$ is not continuous.

To prove (11), note that there are unitary transformations T whose only fixed point is the origin and for which $T(A_i)=A_i$. It follows from the definition of s that $s(T(A_i))=T(s(A_i))$ for any linear transformation, T, and hence $s(A_i)=\bar{0}$.

To prove (12), let U_i be the subset of the unit circle with the property that $u \in U_i$ attains its supremum on A_i at a rough point of A_i. Then, U_i consists of $2i+1$ closed arcs, equal in length, equally spaced around the unit circle, with each arc in U_i subtending an angle of $\pi/(2i+1)$ at the origin. If $i\neq j$, it is clear that $U_i \cap (-U_j)$ must contain an open interval U' in the boundary of the unit sphere. Then, each $u \in U'$ supports both A_i and $-A_j$ at rough points say, x_i and $-x_j$, and it follows that $u \in U'$ supports $A_i+(-A_j)$ at x_i-x_j. Hence x_i-x_j is a rough point of A_i-A_j.

The proof of (13) is left to the reader.

In order to show that the natural map $(\mathscr{K}^n, h)\to(K^n, \|\cdot\|)$ is not continuous for

$n>2$, first modify the sets A_i in E so that they all have a vertex on the axis through $\bar{0}$ and a point p. Then each A_i is symmetric about this axis. Let A_i' be the set of constant width in E^n which is obtained by revolving A_i about the axis through $\bar{0}$ and p. Then using arguments similar to the ones above, it can be shown that (11), (12), and (13) hold with A_i' replacing A_i. □

The continuity of the mapping $a:(K^n, \|\cdot\|)\to(\mathscr{K}^n, h)$ may be deduced from the following lemma.

3.2. *Let* $[K_1]$ *and* $[K_2]$ *be elements of* $(K^n, \|\cdot\|)$. *The following two statements are equivalent.*

(i) $\|[K_1]-[K_2]\|\leq\varepsilon$

(ii) *There exists a member* $K\in\mathscr{K}^n$ *with* diam $K\leq\varepsilon/2$ *such that*

$$K_1-K = K_1+K,$$

whenever $K_i\in[K_i]$, $i=1, 2$, *is chosen so that* $s(K_i)=\bar{0}$ *and* diam $K_1=$diam K_2.

Proof. To show that (i) implies (ii), suppose that $\|[K_1]-[K_2]\|\leq\varepsilon$. Letting $[K_3]=[K_1-K_2]/2$ it is clear that

$$[K_1]-[K_3] = [K_2]+[K_3].$$

If K_1 and K_2 have the same diameter, for each $K_3\in[K_3]$ we must have, by definition of the equivalence classes,

$$K_1-K_3 = K_2+K_3.$$

Choosing K_3 to be the apex of $[K_3]$, we have diam $K_3=\|[K_3]\|\leq\varepsilon/2$, which shows that (i) implies (ii).

Now note that if $K_1-K=K_1+K$ then there is some closed ball B centered at the origin with

$$(K_1-K)-(K_2+K) = B$$

$$K_1-K_2+2K-2K = 2K+B$$

$$K_1-K_2+B' = 2K+B,$$

where B' is the closed ball $2K-2K$. Therefore,

$$a(K_1-K_2] = a[K_1-K_2+B'] = a[2K+B],$$

and since $\|[K_1-K_2]\|=$diam $a[K_1-K_2]$, we have

$$\|[K_1-K_2]\| \leq 2(\text{diam } K) = \varepsilon,$$

which completes the lemma. □

3.3. *The mapping* $a:(K^n, \|\cdot\|)\to(\mathscr{K}^n, h)$ *is one-one and uniformly continuous.*

Proof. Clearly a is one-one. To see that it is uniformly continuous, suppose that

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$\|[K_1]-[K_2]\|\leq\varepsilon$. Then, $|\|[K_1]\|-\|[K_2]\||\leq\|[K_1]-[K_2]\|\leq\varepsilon$. Assuming that $\|[K_1]\|\leq\|[K_2]\|$, this implies that there is a positive number $r_1\leq\varepsilon/2$ such that

$$\operatorname{diam}(a[K_1]+r_1B) = \operatorname{diam} a[K_2],$$

where B is the closed unit ball of E^n.

By 3.2, there is a set $K\in\mathscr{K}^n$ with diam $K=r_2\leq\varepsilon/2$ for which

$$a[K_1]+r_1B-K = a[K_2]+K.$$

By first adding, and then subtracting K from this equation, we find

$$a[K_1]+(r_1+r_2)B = a[K_2]+2K \supset a[K_2]$$

and

$$a[K_1] \subset [aK_1]+r_1B-2K = a[K_2]+r_2B$$

showing that $h(a[K_1], a[K_2])\leq r_1+r_2\leq\varepsilon$. □

For the remainder of this section, F will denote the normed linear space constructed in 2.4; that is, F consists of all those linear functionals continuous on $(K^n, |\cdot|)$, and the norm of F is the one whose closed unit ball is the polar V^0 of the closed unit ball V of $(K^n, \|\cdot\|)$. In this case, F is incomplete, and its completion will be denoted by $\tilde{F}$. (To check that F is incomplete, let $W=\{[K]\in K^n: |[K]|\leq 1\}$. It is readily verified that W is a $w(K^n, F)$-bounded subset that is not $\|\cdot\|$-bounded and this situation could not occur if F were complete [6, p. 170].)

In the remarks following 2.5, it was pointed out that the three topologies, $\mathscr{T}$, $w(K^n, F)$, and $w(K^n, \tilde{F})$, coincide on V, where $\mathscr{T}$ denotes the topology induced by the norm $|\cdot|$. As would be expected, this result can be generalized.

3.4. *Let X be a convex subset of K^n. If X is compact in any one of the topologies $\mathscr{T}$, $w(K^n, F)$ or $w(K^n, \tilde{F})$, then X is compact in the remaining two topologies. Consequently the three topologies coincide on X.*

Proof. Even without convexity, it is clear that $\mathscr{T}$-compactness of X implies $w(K^n, F)$-compactness and $w(K^n, \tilde{F})$-compactness implies $w(K^n, F)$-compactness since $w(K^n, F)$ is weaker than both $\mathscr{T}$ and $w(K^n, \tilde{F})$. It therefore suffices to show that $w(K^n, F)$-compactness of X implies $\mathscr{T}$-compactness and $w(K^n, \tilde{F})$-compactness of X. Here, the convexity of X plays a crucial role, for together with the $w(K^n,F)$-compactness of X it implies that X is a strongly bounded subset of $K^n=F^*$ [6, p. 170], that is, there is some positive number m with $\|[K]\|\leq m$ for all $[K]\in X$.

Since $w(K^n, F)\subset\mathscr{T}$, this shows that X is a $\mathscr{T}$-closed subset of the $\mathscr{T}$-compact set $mV=\{[K]\in K^n: \|[K]\|\leq m\}$. Thus, X is $\mathscr{T}$-compact, and the same reasoning shows that X is $w(K^n, \tilde{F})$-compact.

It follows readily that the three topologies coincide on X. The identity mapping from $(X, \mathscr{U})$ onto $(X, w(E, F))$, where $\mathscr{U}$ is either $\mathscr{T}$ or $w(E, \tilde{F})$, shows that $(X, w(E, F))$ is a Hausdorff space which is the continuous one-one image of the compact space $(X, \mathscr{U})$, and so $(X, \mathscr{U})$ and $(X, w(E, F))$ are homeomorphic. □

A corollary to 3.4 is

3.5. *Let $\mathscr{U}$ be any of the topologies $\mathscr{T}$, $w(K^n, F)$, or $w(K^n, \tilde{F})$. The natural map $i:(\mathscr{K}^n, h)\rightarrow(K^n, \mathscr{U})$ is continuous.*

Proof. Let $\{K_i:i=1, 2, \ldots\}$ be a sequence in $(\mathscr{K}^n, h)$ converging to K_0. The set $\{\text{diam } K_i:i=0, 1, 2, \ldots\}$ is then bounded above, and so $\{[K_i]:i=0, 1, 2, \ldots\}$ is a strongly bounded subset of $(K^n, \|\cdot\|)$. By [6, pp. 171–172], the $w(K^n, F)$-closed convex hull C of $\{[K_i]:i=0, 1, 2, \ldots\}$ is $w(K^n, F)$-compact. Since the three topologies agree on C, and since $i:(\mathscr{K}^n, h)\rightarrow(K^n, \mathscr{T})$ is continuous [4], it follows that $\{[K_i]\}$ converges, in the topology $\mathscr{U}$, to $[K_0]$. □

We remark that the mapping $a:(K^n, \mathscr{U})\rightarrow(\mathscr{K}^n, h)$ is not continuous, where $\mathscr{U}$ is any one of the three topologies being discussed. Indeed, let $\{A_i' \in \mathscr{K}^n:i=1, 2, \ldots\}$ be the sequence of sets constructed in the proof of 3.1. Then, by 3.5, $\{[A_i']\}$ $\mathscr{U}$-converges to the null element, $[B]$, of K^n. However, $a[A_i']=A_i'$ and $a[B]=\{\bar{0}\}$, showing that $a:(K^n, \mathscr{U})\rightarrow(\mathscr{K}^n, h)$ is not continuous.

On the other hand, if X is a bounded subset of $(K^n, \|\cdot\|)$ it is always possible to define a mapping from $(X, \mathscr{U})$ into $(\mathscr{K}^n, h)$ which is continuous: For each $[K] \in X$ let $u_X[K]$ denote that unique member of $[K]$ for which $s(u_X[K])=\bar{0}$ and $\text{diam } u_X[K]=\sup\{\|[C]\|:[C] \in X\}$. Then, u_X is well defined and has the following properties:

(14) $$iu_X[K] = [K],$$

(15) u_X is continuous from $(X, \mathscr{U})$ into $(\mathscr{K}^n, h)$, where $\mathscr{U}$ is any of the three topologies discussed above.

Assertion (14) is clearly true, and a proof that (15) is valid follows readily from property (14) together with the fact that $u_X(X)$ is contained in a copact subset of $(\mathscr{K}^n, h)$ (by Blaschke's Theorem).

4. **Approximation properties of the indecomposable elements of $\mathscr{K}^n$.** Let $\mathscr{C}$ be a collection of closed convex subsets of E^n. The convex subset C is said to be *approximable* by $\mathscr{C}$ [7] if there is a sequence $\{C_m:m=1, 2, \ldots\}$ in $(\mathscr{K}^n, h)$ which converges to C where each C_m can be expressed

$$C_m = \sum_{i=1}^{q_m} \lambda_i C_{m,i}, \qquad \lambda_i > 0, \qquad C_{m,i} \in \mathscr{C}.$$

Shephard [5, Chapter 15] and [7], has shown that if P is a polytope which is indecomposable in the class $\mathscr{P}$ of all polytopes in E^n, and if $\mathscr{C}$ is a subclass of $\mathscr{P}$ which is invariant under positive homotheties ($\lambda C+x \in \mathscr{C}$ whenever $\lambda>0$ and $C \in \mathscr{C}$), then P is approximable by $\mathscr{C}$ if and only if P is in the closure of $\mathscr{C}$.

A consequence of 2.2 and 2.3 is

4.1. *Every element of $(\mathscr{K}^n, h)$ is approximable by the class of indecomposable elements of $\mathscr{K}^n$.*

If $\mathscr{C} \subset \mathscr{K}^n$ is invariant under positive homotheties, and if K is indecomposable in $\mathscr{K}^n$, then K is approximable by $\mathscr{C}$ only if K is in the closure of $\mathscr{C}$.

Proof. As usual, V denotes the closed unit ball of $(K^n, \|\cdot\|)$, and $\mathscr{U}$ denotes any of the three topologies of Theorem 3.4.

In proving that each K in $\mathscr{K}^n$ is approximable by the family of indecomposable elements of $\mathscr{K}^n$, it is clear that we may restrict attention to the case where diam $K=1$ and $s(K)=\bar{0}$. For this case, $[K] \in V$. Since V is $\mathscr{U}$-compact, it follows from the Krein-Milman Theorem [6, p. 131] that there is a sequence $\{[K_m], m=1, 2, \ldots\}$ which $\mathscr{U}$-converges to $[K]$, where each $[K_m]$ is a convex combination of the extreme points of V, that is;

$$[K_m] = \sum_{i=1}^{q_m} \lambda_i [K_{m,i}],$$

where $\lambda_i > 0$, $\Sigma \lambda_i = 1$, and $[K_{m,i}]$ is extreme.

From the definition of the mapping u_V (described at the end of §3) it is easily seen that

$$u_V[K_m] = \sum_{i=1}^{q_m} \lambda_i u_V[K_{m,i}],$$

and since $[K_{m,i}]$ is extreme, $u_V[K_{m,i}] = a[K_{m,i}]$, showing that $u_V[K_{m,i}]$ is indecomposable. The continuity of u_V together with the fact that $u_V[K]=K$ now shows that K is approximable by the family of indecomposable elements of $\mathscr{K}^n$.

Regarding the second assertion, we may assume that K is indecomposable in $\mathscr{K}^n$ with diam $K=1$ and $s(K)=\bar{0}$. If K were approximable by $\mathscr{C}$, then, K would be approximable by $\mathscr{C}_1 = \{[C] \in \mathscr{C} : \text{diam } C=1, s(C)=\bar{0}\}$. In fact, it is easily checked that K would be the limit of some sequence of convex combinations of $\mathscr{C}_1$. It follows that $[K]$ would be in the $\mathscr{U}$-closed convex hull D of $i(\mathscr{C}_1) = \{[C] : C \in \mathscr{C}_1\}$. Clearly, $[K]$ must be an extreme point of D, because $D \subset V$ and $[K]$ is an extreme point of V. Since the $\mathscr{U}$-closure of $i(\mathscr{C}_1)$ is $\mathscr{U}$-compact (due to the fact that the closure of $\mathscr{C}_1$ is compact in $(\mathscr{K}^n, h)$), the Milman Theorem [6, p. 132] shows that $[K]$ must be in the closure of $i(\mathscr{C}_1)$. The continuity of the mapping u_V now shows that K is in the closure of $\mathscr{C}_1$, completing the proof of 4.1. □

REMARKS. The fact that the Milman Theorem and the Krein-Milman Theorem are both closely related to the approximation properties of indecomposable sets is known—for example, see the paper of Berg [1]. We would also mention that the results of [1] are stated for the family of all compact convex sets of E^n, but remain valid for any closed subfamily $\mathscr{C}$ with the following property: $\mathscr{C}$ is invariant under translations, scalar multiplication and Minkowski addition. (Indecomposability would then be defined with respect to the subfamily $\mathscr{C}$.) Theorem 4.1 may therefore be obtained as a consequence of the more general results of [1].

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https://doi.org/10.4153/CMB-1975-119-6 Published online by Cambridge University Press