# $B^{\#}$ MODULAR ANNIHILATOR ALGEBRAS†

*by* J. DUNCAN

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## 1. Introduction

The structure of various classes of annihilator algebras has been known for some time. Bonsall and Goldie (**1**) considered semi-simple Banach algebras $\mathfrak{A}$ with the properties

(i) $r(L) \equiv \{x\colon x \in \mathfrak{A},\ yx = 0(y \in L)\} \neq (0)$ for each proper closed left ideal $L$ of $\mathfrak{A}$,

(ii) $l(K) \equiv \{x\colon x \in \mathfrak{A},\ xy = 0(y \in K)\} \neq (0)$ for each proper closed right ideal $K$ of $\mathfrak{A}$.

Bonsall obtained more detailed results in (**2**) by imposing an extra topological condition on $\mathfrak{A}$. Smiley (**7**) generalised the work of Bonsall and Goldie by asking that only condition (i) should hold. In fact he also proved the main structure theorem for topological rings instead of Banach algebras. It is known that conditions (i) and (ii) are not equivalent.

Recently Yood (**8**) has considered semi-prime topological rings $\mathfrak{R}$ with the property

(i)′ $r(L) \neq (0)$ for each closed maximal modular left ideal $L$ of $\mathfrak{R}$.

He shows that condition (i)′ is in fact equivalent to

(ii)′ $l(K) \neq (0)$ for each closed maximal modular right ideal $K$ of $\mathfrak{R}$.

In other words, his " left " annihilator class coincides with his " right " annihilator class.

In this paper we shall consider an annihilator condition which is slightly weaker than the condition considered by Yood. We say that a Banach algebra $\mathfrak{A}$ is a *modular annihilator algebra* if it has a family of maximal modular left ideals $\{L_\lambda\colon \lambda \in \Lambda\}$ such that

(1) $r(L_\lambda) \neq (0)(\lambda \in \Lambda)$,

(2) $\cap\{L_\lambda\colon \lambda \in \Lambda\} = (0)$.

It is immediate that a modular annihilator algebra is semi-simple. For convenience we shall confine our attention to complex Banach algebras. In § **2** we shall obtain structure theorems for modular annihilator algebras. In

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particular, Theorem 1 will show that there is no need to distinguish between " left " and " right " modular annihilator algebras.

We say that any Banach algebra $\mathfrak{A}$ is a $B^{\#}$ *algebra* if

(3) given $a \in \mathfrak{A}$ with $\| a \| = 1$, and given $\varepsilon$ with $0<\varepsilon<1$, there exists $a^{\#} \in \mathfrak{A}$ such that $\| a^{\#} \| = 1$ and

$$\|(aa^{\#})^n \| \geqq (1-\varepsilon)^n \quad (n = 1, 2, \ldots).$$

In § **3** we specialise to $B^{\#}$ modular annihilator algebras with dense socle and obtain more detailed structural results. (We recall that for a semi-simple algebra the socle is the sum of the minimal left (or right) ideals.)

Bonsall and Goldie conjectured that their annihilator algebras were *dual* (at least in the topologically simple case) in the sense that

(iii) $L = l(r(L))$ for each closed left ideal $L$ of $\mathfrak{A}$,
(iv) $K = r(l(K))$ for each closed modular right ideal $K$ of $\mathfrak{A}$.

It seems natural to say that a modular annihilator algebra is *modular dual* if it satisfies

(iii)′ $L = l(r(L))$ for each closed modular left ideal $L$ of $\mathfrak{A}$,
(iv)′ $K = r(l(K))$ for each closed modular right ideal $K$ of $\mathfrak{A}$.

We shall show that a $B^{\#}$ modular annihilator algebra with dense socle is modular dual.

Finally, in § **4** we shall show how the $B^{\#}$ condition is related to a problem concerning Banach spaces in normed duality.

Unless otherwise stated, all definitions are to be taken as in Rickart (**6**).

## 2. The structure of modular annihilator algebras

**Theorem 1.** *Let $\mathfrak{A}$ be a modular annihilator algebra. Then there is a family of maximal modular right ideals $\{K_\mu\colon \mu \in M\}$ such that*

(i) $l(K_\mu) \neq (0)$ $(\mu \in M)$,
(ii) $\cap\{K_\mu\colon \mu \in M\} = (0)$.

**Proof.** Let $\{e_\mu\colon \mu \in M\}$ be the family of all primitive idempotents of $\mathfrak{A}$. This family is non-empty by the analogue for left ideals of (**8**), Lemma 3.2. Further, each $L_\lambda$ is of the form $\mathfrak{A}(1-e_\mu)$ for some $\mu \in M$. Let $K_\mu = (1-e_\mu)\mathfrak{A}$, so that $K_\mu$ is a maximal modular right ideal of $\mathfrak{A}$ for each $\mu \in M$. By (**6**), Lemma 2.8.2 we have $l(K_\mu) = \mathfrak{A}e_\mu$ so that condition (i) is satisfied.

Let $K = \cap\{K_\mu\colon \mu \in M\}$. Suppose that $K \neq (0)$. By the argument in (**6**), Lemma 2.1.11, for each $\mu \in M$, $u \in \mathfrak{A}$ either $ue_\mu\mathfrak{A}$ is a minimal right ideal or $ue_\mu\mathfrak{A} = (0)$. If $K$ contains no minimal right ideal, then we have $Ke_\mu\mathfrak{A} = (0)$ $(\mu \in M)$, and so $K \subset l(e_\mu\mathfrak{A})$ $(\mu \in M)$. Thus, by (**6**), Lemma 2.8.2, $K \subset \mathfrak{A}(1-e_\mu)$ $(\mu \in M)$. It follows that $K \subset \cap\{L_\lambda\colon \lambda \in \Lambda\} = (0)$. This contradiction implies that $K$ contains a minimal right ideal and so a primitive idempotent $e$. We now have $e\mathfrak{A} \subset K \subset (1-e)\mathfrak{A}$. This is again a contradiction and therefore condition (ii) is satisfied.

**Theorem 2.** *Let $\mathfrak{A}$ be a modular annihilator algebra and let $\mathfrak{B}$ be the closure of the socle of $\mathfrak{A}$. Then $r(\mathfrak{B}) = (0)$, and $\mathfrak{B}$ is a semi-simple Banach algebra such that*

(i) $r(L) \neq (0)$ *for each maximal modular left ideal $L$ of $\mathfrak{B}$,*

(ii) $l(K) \neq (0)$ *for each maximal modular right ideal $K$ of $\mathfrak{B}$,*

(iii) $r(I) \neq (0)$ *for each proper closed two-sided ideal $I$ of $\mathfrak{B}$ .*

**Proof.** If $r(\mathfrak{B}) \neq (0)$, then $r(\mathfrak{B})$ is a non-zero right idea lof $\mathfrak{A}$. It follows as in the proof of Theorem 1 that $r(\mathfrak{B})$ contains a primitive idempotent. But $\mathfrak{A}$ is semi-simple and $\mathfrak{B}$ is two-sided and so $\mathfrak{B} \cap r(\mathfrak{B}) = (0)$. This is a contradiction since $\mathfrak{B}$ contains all the primitive idempotents of $\mathfrak{A}$. Hence $r(\mathfrak{B}) = (0)$. We now observe that if $e$ is a primitive idempotent for $\mathfrak{A}$ then it is also a primitive idempotent for $\mathfrak{B}$. In fact $\mathfrak{B}e \subset \mathfrak{A}e$, and $\mathfrak{A}e \subset \mathfrak{B}$ so that $\mathfrak{A}e = \mathfrak{A}e^2 \subset \mathfrak{B}e$. Thus $\mathfrak{A}e = \mathfrak{B}e$ and so $e\mathfrak{A}e = e\mathfrak{B}e$. $\mathfrak{B}$ is semi-simple since it is a two-sided ideal of $\mathfrak{A}$. Thus $\mathfrak{B}$ is a semi-simple Banach algebra with dense socle. Statements (i) and (ii) now follow immediately from (**5**), Lemma 3.1. Finally, let $I$ be a proper closed two-sided ideal of $\mathfrak{B}$. Then there is a primitive idempotent $e$ of $\mathfrak{B}$ such that $e \notin I$. Let $I_e = (\mathfrak{B}e\mathfrak{B})^-$. By the argument of (**6**), Lemma 2.8.8, $I_e$ is a minimal-closed two-sided ideal. Thus $I \cap I_e = (0)$ or $I \cap I_e = I_e$. Since $e \notin I$ we must have $I \cap I_e = (0)$. We now have $II_e \subset I \cap I_e = (0)$, so that $r(I) \neq (0)$ as required.

**Corollary 1.** *$\mathfrak{A}$ has a unique norm topology.*

**Proof.** This is immediate from (**6**), Corollary 2.5.8.

**Corollary 2.** *The following statements are equivalent.*

(i) $r(L) \neq (0)$ *for each maximal modular left ideal $L$ of $\mathfrak{A}$.*

(ii) $\mathfrak{A}/\mathfrak{B}$ *is a radical algebra.*

**Proof.** This is immediate from (**8**), Theorem 3.4.

**Theorem 3.** *Let $I$ be a minimal-closed two-sided ideal of a modular annihilator algebra $\mathfrak{A}$. Then $I$ is a semi-simple topologically simple Banach algebra such that $r(L) \neq (0)$ for each maximal modular left ideal $L$ of $I$.*

**Proof.** As above $\mathfrak{B}$ denotes the closure of the socle of $\mathfrak{A}$. We observe that there is a primitive idempotent $e$ of $\mathfrak{A}$ such that $I = (\mathfrak{B}e\mathfrak{B})^-$. We now regard $I$ as an ideal of $\mathfrak{B}$ and the result follows simply from Theorem 3.7 and Lemma 3.10 of (**8**).

**Corollary.** *$\mathfrak{B}$ is the direct topological sum of topologically simple modular annihilator algebras.*

We recall that two Banach spaces $X$ and $Y$ are said to be in *normed duality* with respect to $\langle\, ,\, \rangle$, if $\langle\, ,\, \rangle$ is a non-degenerate bilinear form on $X \times Y$ such that there is some $M > 0$ with

$$|\langle x, y\rangle| \leqq M \| x \| \| y \| \quad (x \in X, y \in Y).$$

For such a pair of spaces $(X, Y, \langle\, ,\rangle)$ we shall use the concept of the *adjoint* of a bounded operator as described in (3). We shall denote by $\mathfrak{B}(X, Y, \langle\, ,\rangle)$ the algebra of all bounded operators on $X$ which have an adjoint on $Y$ with respect to $\langle\, ,\rangle$. It is shown in (3) that $\mathfrak{B}(X, Y, \langle\, ,\rangle)$ is a Banach algebra with respect to the norm defined by

$$\| T \| = \max \{ | T |, | T^* | \}$$

where $| T |, | T^* |$ are the operator norms in $\mathfrak{B}(X), \mathfrak{B}(Y)$ respectively. We shall denote by $F(X, Y, \langle\, ,\rangle)$ the algebra of all bounded operators on $X$ of finite rank which have an adjoint on $Y$ with respect to $\langle\, ,\rangle$. We shall denote by $\mathfrak{F}(X, Y, \langle\, ,\rangle)$ the closure of $F(X, Y, \langle\, ,\rangle)$ in $\mathfrak{B}(X, Y, \langle\, ,\rangle)$. We shall use the concept of a *dual representation* of a Banach algebra as defined in (3).

**Theorem 4.** *Let $\mathfrak{A}$ be a topologically simple modular annihilator algebra. Then there is a faithful dual representation $a \to T_a$ of $\mathfrak{A}$ on some pair $(X, Y, \langle\, ,\rangle)$ such that*

(i) $\| T_a \| \leqq \| a \| \; (a \in \mathfrak{A})$,

(ii) *if $A$ is the image of $\mathfrak{A}$ under $a \to T_a$ then*

$$F(X, Y, \langle\, ,\rangle) \subset A \subset \mathfrak{F}(X, Y, \langle\, ,\rangle).$$

**Proof.** $\mathfrak{A}$ is certainly a left primitive complex Banach algebra with minimal one-sided ideals. By (3) Theorem 14 there exists a faithful dual representation $a \to T_a^1$ of $\mathfrak{A}$ on some pair $(X_1, Y_1, \langle\, ,\rangle_1)$ whose image $A$ satisfies

$$F(X_1, Y_1, \langle\, ,\rangle_1) \subset A.$$

Since $\mathfrak{A}$ is topologically simple we also have $A \subset \mathfrak{F}(X_1, Y_1, \langle\, ,\rangle_1)$. By (3), Theorem 3, Corollary 1 there exists $f \in \mathfrak{A}'$ such that the dual representation $a \to T_a^f$ of $\mathfrak{A}$ on $(X_f, Y_f, \langle\, ,\rangle_f)$ is equivalent to the dual representation $a \to T_a^1$ of $\mathfrak{A}$ on $(X_1, Y_1, \langle\, ,\rangle_1)$. Since we have

$$| T_a^f | \leqq \| a \|, \quad |(T_a^f)^* | \leqq \| a \| (a \in \mathfrak{A}),$$

the proof is now complete.

**Remarks.** (1) We point out that the underlying pair is essentially unique in the sense that if $a \to T_a^2$ is any other faithful dual representation of $\mathfrak{A}$ on a pair $(X_2, Y_2, \langle\, ,\rangle_2)$ with property (ii) above, then $a \to T_a^2$ is equivalent to the dual representation $a \to T_a$ of the theorem. The proof of this remark follows by a simple extension of (6), Theorem 2.5.19.

(2) We shall have occasion to choose our underlying pair $(X, Y, \langle\, ,\rangle)$ as follows. Let $L$ be a minimal left ideal of $\mathfrak{A}$ and let $K$ be a minimal right ideal of $\mathfrak{A}$. By (6), Lemma 2.4.13 there exist bicontinuous isomorphisms $U, V$ of $L, K$ with $X, Y$ respectively. Let

$$\langle l, k \rangle = \langle Ul, Vk \rangle \quad (l \in L, k \in K).$$

Then $(L, K, \langle\, ,\rangle)$ are Banach spaces in normed duality. Let $a \to T_a$ be the equivalent dual representation of $\mathfrak{A}$ on $(L, K, \langle\, ,\rangle)$ implemented by the mappings

$U$, $V$. Then $a \rightarrow T_a$ is simply the left regular representation of $\mathfrak{A}$ on $L$ and $a \rightarrow T_a^*$ the right regular anti-representation of $\mathfrak{A}$ on $K$. Hence condition (i) is satisfied for this representation.

We now give some illustrative examples of modular annihilator algebras.

1. Let $\mathfrak{A}$ be any semi-simple Banach algebra which contains minimal one-sided ideals. If $\mathfrak{B}$ is the closure of the socle of $\mathfrak{A}$ then $\mathfrak{B}$ is a modular annihilator algebra. Let $\{e_\lambda: \lambda \in \Lambda\}$ be the family of all primitive idempotents of $\mathfrak{A}$. We may easily show that $r(\mathfrak{A}(1-e_\lambda)) \neq (0)(\lambda \in \Lambda)$ and $\cap\{\mathfrak{A}(1-e_\lambda): \lambda \in \Lambda\} = l(\mathfrak{B})$. In particular, if $l(\mathfrak{B}) = (0)$, then $\mathfrak{A}$ is a modular annihilator algebra.

2. Any primitive complex Banach algebra with minimal one-sided ideals is a modular annihilator algebra. This fact follows easily from (**6**), Theorem 2.4.12.

3. Given any Banach space $X$, $\mathfrak{B}(X)$, the Banach algebra of all bounded linear operators on $X$, is a modular annihilator algebra. (In this case we may even take $X$ to be a real Banach space and so $\mathfrak{B}(X)$ a real Banach algebra.) The closure of the socle of $\mathfrak{B}(X)$ is of course $\mathfrak{F}(X, X', (\ ,\ ))$ where $X'$ is the Banach space of all continuous linear functionals on $X$ and $(\ ,\ )$ is the natural bilinear form on $X \times X'$.

4. Let $\mathfrak{I}(X)$ denote the subalgebra of $\mathfrak{B}(X)$ consisting of all inessential operators on $X$ (see (**4**)). Since $\mathfrak{I}(X)/\mathfrak{F}(X, X', (\ ,\ ))$ is a radical algebra, $\mathfrak{I}(X)$ is a modular annihilator algebra with $r(L) \neq (0)$ for each maximal modular left ideal $L$ of $\mathfrak{I}(X)$. Further $\mathfrak{I}(X)$ is the largest subalgebra of $\mathfrak{B}(X)$ with this property.

## 3. $B^{\#}$ modular annihilator algebras with dense socle

The first result of this section is stated in the notation of (**2**). The proof is a simple extension of the methods of Theorems 3, 4 and 6 of (**2**).

**Theorem 5.** *Let $\mathfrak{A}$ be a $B^{\#}$ modular annihilator algebra with dense socle. Then $\mathfrak{A}$ has minimal norm and is isometrically isomorphic with the $B(\infty)$ sum of its minimal-closed two-sided ideals, each of which satisfies the $B^{\#}$ condition.*

We recall from (**3**) that if $(X, Y, \langle\ ,\ \rangle)$ are Banach spaces in normed duality, then there are two natural mappings $x \rightarrow \hat{x}$, $y \rightarrow \hat{y}$ from $X$ into $Y'$, $Y$ into $X'$ respectively given by

$$\hat{x}(y) = \langle x, y \rangle \quad (y \in Y),$$

$$\hat{y}(x) = \langle x, y \rangle \quad (x \in X).$$

**Theorem 6.** *Let $\mathfrak{A}$ be a topologically simple $B^{\#}$ modular annihilator algebra. Then there is a pair $(X, Y, \langle\ ,\ \rangle)$ of Banach spaces in normed duality such that*

(i) *$x \rightarrow \hat{x}$, $y \rightarrow \hat{y}$ are isometries,*

(ii) *$\mathfrak{A}$ is isometrically isomorphic with $\mathfrak{F}(X, Y, \langle\ ,\ \rangle)$.*

**Proof.** Let $a \to T_a$ be the dual representation of $\mathfrak{A}$ on $(X, Y, \langle\ ,\ \rangle)$ given by Theorem 4. Since $\mathfrak{A}$ has minimal norm it follows from Theorem 4 (i) that

$$|T_a| = |T_a^*| = \|a\| \quad (a \in \mathfrak{A}).$$

Thus the image of $\mathfrak{A}$ under $a \to T_a$ is closed in $\mathfrak{B}(X, Y, \langle\ ,\ \rangle)$ and so must be $\mathfrak{F}(X, Y, \langle\ ,\ \rangle)$ by Theorem 4 (ii). Further, for each $x \in X$, $y \in Y$ we have $|x \otimes y| = |y \otimes x|$ so that

$$\|x\| \|\hat{y}\| = \|y\| \|\hat{x}\| \quad (x \in X, y \in Y).$$

Hence there exists $k > 0$ such that

$$\|\hat{x}\| = k\|x\| \quad (x \in X),$$

$$\|\hat{y}\| = k\|y\| \quad (y \in Y).$$

Now consider the pair $(X, \hat{Y}, (\ ,\ ))$ of Banach spaces in normed duality. We have for each $x \in X$

$$\begin{aligned} \sup\{|(x, \hat{y})| : \|\hat{y}\| \leqq 1\} &= k^{-1} \sup\{|\langle x, ky\rangle| : \|ky\| \leqq 1\} \\ &= k^{-1}\|\hat{x}\| \\ &= \|x\|. \end{aligned}$$

Condition (i) is now satisfied for the new pair $(X, \hat{Y}, (\ ,\ ))$. Since also

$$\mathfrak{F}(X, Y, \langle\ ,\ \rangle) = \mathfrak{F}(X, \hat{Y}, (\ ,\ )),$$

the proof is now complete.

**Remark.** In particular we may again take our pair of the form $(L, K, \langle\ ,\ \rangle)$ with $L$, $K$ minimal left, right ideals respectively and $l \to \hat{l}$, $k \to \hat{k}$ isometries. To see this, choose $x_0 \in X$, $y_0 \in Y$ with $\|x_0\| = \|y_0\| = 1$. Let $L$ be the minimal left ideal of $\mathfrak{A}$ isometrically isomorphic with $\{x \otimes y_0 : x \in X\}$, and let $K$ be the minimal right ideal of $\mathfrak{A}$ isometrically isomorphic with $\{x_0 \otimes y : y \in Y\}$. Since

$$|x \otimes y_0| = \|x\| \|\hat{y}_0\| = \|x\| \quad (x \in X),$$

$L$ is isometrically isomorphic with $X$. Similarly $K$ is isometrically isomorphic with $Y$. If $\langle\ ,\ \rangle$ is the induced bilinear form on $L \times K$, then we have $l \to \hat{l}$, $k \to \hat{k}$ isometries for the pair $(L, K, \langle\ ,\ \rangle)$.

Let $\{E_\lambda : \lambda \in \Lambda\}$ be any family of Banach spaces. For $1 \leqq p < \infty$ we denote by $\Sigma^{(p)}\{E_\lambda : \lambda \in \Lambda\}$ the set of all functions $f$ on $\Lambda$ such that $f_\lambda \in E_\lambda (\lambda \in \Lambda)$ and

$$\|f\| = (\Sigma\{\|f_\lambda\|^p : \lambda \in \Lambda\})^{1/p} < \infty.$$

We denote by $\Sigma^{(\infty)}\{E_\lambda : \lambda \in \Lambda\}$ the set of all functions $f$ on $\Lambda$ such that $f_\lambda \in E_\lambda (\lambda \in \Lambda)$ and

$$\|f\| = \sup\{\|f_\lambda\| : \lambda \in \Lambda\} < \infty.$$

It is routine to verify that for $1 \leqq p \leqq \infty$, $\Sigma^{(p)}\{E_\lambda : \lambda \in \Lambda\}$ is a Banach space (with addition and scalar multiplication defined pointwise).

**Proposition 1.** *For each $\lambda \in \Lambda$ let $(X_\lambda, Y_\lambda, \langle\,,\rangle_\lambda)$ be a pair of Banach spaces in normed duality. Suppose that the bounds $M_\lambda$ satisfy $M = \sup\{M_\lambda: \lambda \in \Lambda\} < \infty$. Let $X = \Sigma^{(1)}\{X_\lambda: \lambda \in \Lambda\}$, $Y = \Sigma^{(\infty)}\{Y_\lambda: \lambda \in \Lambda\}$ and*

$$\langle x, y\rangle = \Sigma\{\langle x_\lambda, y_\lambda\rangle_\lambda: \lambda \in \Lambda\} \quad (x \in X, y \in Y).$$

*Then $(X, Y, \langle\,,\rangle)$ are Banach spaces in normed duality. If, further, for each $\lambda \in \Lambda$, $x_\lambda \to \hat{x}_\lambda$, $y_\lambda \to \hat{y}_\lambda$ are isometries, then $x \to \hat{x}$, $y \to \hat{y}$ are isometries.*

**Proof.** $\langle\,,\rangle$ is non-degenerate and bilinear since each $\langle\,,\rangle_\lambda$ has these properties. Further, for each $x \in X$, $y \in Y$,

$$\begin{aligned} |\langle x, y\rangle| &\leqq \Sigma\{M_\lambda \| x_\lambda \| \| y_\lambda \|: \lambda \in \Lambda\} \\ &\leqq M \| y \| \Sigma\{\| x_\lambda \|: \lambda \in \Lambda\} \\ &= M \| x \| \| y \|. \end{aligned}$$

Therefore $(X, Y, \langle\,,\rangle)$ are Banach spaces in normed duality.

Suppose now that $x_\lambda \to \hat{x}_\lambda$, $y_\lambda \to \hat{y}_\lambda$ are isometries for each $\lambda \in \Lambda$, so that

$$|\langle x_\lambda, y_\lambda\rangle_\lambda| \leqq \| x_\lambda \| \| y_\lambda \| \quad (x_\lambda \in X_\lambda, y_\lambda \in Y_\lambda).$$

Then for each $y \in Y$

$$\begin{aligned} \| \hat{y} \| &= \sup\{|\langle x, y\rangle|: \| x \| \leqq 1\} \\ &\leqq \sup\{\Sigma|\langle x_\lambda, y_\lambda\rangle_\lambda|: \| x \| \leqq 1\} \\ &\leqq \sup\{\Sigma \| x_\lambda \| \| y_\lambda \|: \| x \| \leqq 1\} \\ &\leqq \| y \|. \end{aligned}$$

Given $\varepsilon > 0$, there exists $\lambda_0 \in \Lambda$ such that $\| y \| - \frac{1}{2}\varepsilon \leqq \| y_{\lambda_0} \|$. Further there exists $x_{\lambda_0} \in X_{\lambda_0}$ such that $\| x_{\lambda_0} \| = 1$ and $\| y_{\lambda_0} \| - \frac{1}{2}\varepsilon \leqq |\langle x_{\lambda_0}, y_{\lambda_0}\rangle|$. Define $x$ on $\Lambda$ by

$$x_\lambda = \begin{cases} x_{\lambda_0} & (\lambda = \lambda_0) \\ 0 & (\lambda \neq \lambda_0). \end{cases}$$

Then $x \in X$, $\| x \| = 1$ and $\| y \| - \varepsilon \leqq |\langle x, y\rangle|$. It follows that $y \to \hat{y}$ is an isometry, and similarly $x \to \hat{x}$ is an isometry.

**Theorem 7.** *Let $\mathfrak{A}$ be a modular annihilator algebra with dense socle. Then there is a faithful dual representation $a \to T_a$ of $\mathfrak{A}$ on some pair $(X, Y, \langle\,,\rangle)$ such that*

(i) $\| T_a \| \leqq \| a \|$ $(a \in \mathfrak{A})$,

(ii) *if $A$ is the image of $\mathfrak{A}$ under $a \to T_a$ then $A \subset \mathfrak{F}(X, Y, \langle\,,\rangle)$.*

*If, further, $\mathfrak{A}$ is a $B^{\#}$ algebra then we may choose $(X, Y, \langle\,,\rangle)$ such that $x \to \hat{x}$, $y \to \hat{y}$ are isometries and $a \to T_a$, $a \to T_a^*$ are isometries.*

**Proof.** Let $\mathfrak{A}_0$ be the least subalgebra of $\mathfrak{A}$ containing the family $\{I_\lambda: \lambda \in \Lambda\}$ of minimal-closed two-sided ideals of $\mathfrak{A}$, so that we have $\mathfrak{A}_0^- = \mathfrak{A}$. Let $i_\lambda \to T_{i_\lambda}$ be the dual representation of $I_\lambda$ on $(X_\lambda, Y_\lambda, \langle\,,\rangle_\lambda)$ as given in Remark (2) after Theorem 4. Note that we may suppose that each $\langle\,,\rangle_\lambda$ has bound 1, for if $\langle\,,\rangle_\lambda$ has bound $M_\lambda$ we simply replace $\langle\,,\rangle_\lambda$ by the non-degenerate bilinear form $M_\lambda^{-1}\langle\,,\rangle_\lambda$. Let $(X,\ Y,\ \langle\,,\rangle)$ be the pair of Banach spaces in normed duality defined in Proposition 1.

Let $a\in\mathfrak{A}_0$ so that $a = a_1+a_2+\ldots+a_n$ where $a_i\in I_{\lambda_i}$ $(1\leqq i\leqq n)$. Note that $a_ia_j = 0$ $(i\neq j)$. Define $T_a$ on $X$ by

$$(T_ax)_\lambda = \begin{cases} T_{a_i}x_{\lambda_i} & (\lambda=\lambda_i,\ 1\leqq i\leqq n)\\ 0 & (\text{otherwise}).\end{cases}$$

Then $T_a$ is linear on $X$ and we have

$$\begin{aligned}\| T_ax \| &= \sum_{i=1}^n \| T_{a_i}x_{\lambda_i} \| \\ &\leqq \sum_{i=1}^n | T_{a_i} | \, \| x_{\lambda_i} \| \\ &\leqq \sum_{i=1}^n \| x_{\lambda_i} \| \sup\{\| a_ib_i \| : b_i\in I_{\lambda_i},\quad \| b_i \| \leqq 1\} \\ &= \sum_{i=1}^n \| x_{\lambda_i} \| \sup\{\| ab_i \| : b_i\in I_{\lambda_i},\quad \| b_i \| \leqq 1\} \\ &\leqq \sum_{i=1}^n \| x_{\lambda_i} \| \, \| a \| \\ &\leqq \| a \| \, \| x \|.\end{aligned}$$

It follows that $T_a\in\mathfrak{B}(X)$ and $| T_a | \leqq \| a \|$ $(a\in\mathfrak{A}_0)$. Define $S_a$ on $Y$ by

$$(S_ay)_\lambda = \begin{cases} T^*_{a_i}y_{\lambda_i} & (\lambda=\lambda_i,\ 1\leqq i\leqq n)\\ 0 & (\text{otherwise}).\end{cases}$$

Arguing as above we obtain $S_a\in\mathfrak{B}(Y)$ and $| S_a | \leqq \| a \|$ $(a\in\mathfrak{A}_0)$. For each $x\in X$, $y\in Y$ we have

$$\begin{aligned}\langle T_ax,\ y\rangle = \sum_{i=1}^n \langle T_{a_i}x_{\lambda_i},\ y_{\lambda_i}\rangle_{\lambda_i} &= \sum_{i=1}^n \langle x_{\lambda_i},\ T^*_{a_i}y_{\lambda_i}\rangle_{\lambda_i} \\ &= \langle x,\ S_ay\rangle,\end{aligned}$$

so that $S_a = T_a^*$ $(a\in\mathfrak{A}_0)$.

Now extend $a\to T_a$, $a\to S_a$ by continuity from $\mathfrak{A}_0$ to $\mathfrak{A}$. This gives a dual representation $a\to T_a$ of $\mathfrak{A}$ on $(X,\ Y,\ \langle\,,\rangle)$ satisfying condition (i). Further $a\to T_a$ is faithful. Otherwise, the kernel would contain a primitive idempotent $e_\lambda$ belonging to some $X_\lambda$ and this is clearly impossible since $| T_{e_\lambda} | = \| e_\lambda \|$. Suppose that $i\in I_\lambda$ corresponds to an operator on $X_\lambda$ of the form $x_\lambda\otimes y_\lambda$. Then clearly $T_i\in F(X,\ Y,\ \langle\,,\rangle)$. Since $a\to T_a$ is a continuous isomorphism we now have $T_i\in\mathfrak{F}(X,\ Y,\ \langle\,,\rangle)$ $(i\in I_\lambda)$, from which we obtain $T_a\in\mathfrak{F}(X,\ Y,\ \langle\,,\rangle)$ $(a\in\mathfrak{A})$ so that condition (ii) holds.

Suppose now that $\mathfrak{A}$ is a $B^\#$ algebra. By Theorem 6 we may choose each $(X_\lambda,\ Y_\lambda,\ \langle\,,\rangle_\lambda)$ so that $x_\lambda\to\hat{x}_\lambda$, $y_\lambda\to\hat{y}_\lambda$ are isometries. By Proposition 1 $x\to\hat{x}$, $y\to\hat{y}$ are isometries. Finally it follows from (i) and Theorem 5 that $a\to T_a$ $a\to T_a^*$ are isometries.

We remark that the inclusion $F(X, Y, \langle\,,\rangle) \subset A$ holds if and only if $\mathfrak{A}$ is topologically simple. We also point out that there are many possible choices of underlying pairs in duality. For example we might take $X = \Sigma^{(p)}\{X_\lambda\colon \lambda \in \Lambda\}$, $Y = \Sigma^{(q)}\{Y_\lambda\colon \lambda \in \Lambda\}$, $\langle x, y\rangle = \Sigma\{\langle x_\lambda, y_\lambda\rangle_\lambda\colon \lambda \in \Lambda\}$, where $p > 1$ and

$$p^{-1}+q^{-1} = 1.$$

In the case when $\mathfrak{A}$ is not topologically simple there remains the problem of determining the " best possible " underlying pair in duality.

The remainder of this section is concerned with the description of the closed modular one-sided ideals of $B^{\#}$ modular annihilator algebras with dense socle. For this we need some notation and a preliminary result.

Let $(X, Y, \langle\,,\rangle)$ be a pair of Banach spaces in normed duality. Given $E \subset X$, $F \subset Y$ we define

$$E^0 = \{y\colon \langle x, y\rangle = 0 \quad (x \in E)\},$$
$${}^0F = \{x\colon \langle x, y\rangle = 0 \quad (y \in F)\}.$$

Given $T \in \mathfrak{B}(X)$ we define

$$R(T) = \{Tx\colon x \in X\},$$
$$N(T) = \{x\colon Tx = 0\}.$$

**Proposition 2.** *Let $(X, Y, \langle\,,\rangle)$ be Banach spaces in normed duality such that $x \to \hat{x}$, $y \to \hat{y}$ are isometries. Let $E \in \mathfrak{B}(X)$ be such that $E$, $E^*$ are compact and 1 is an eigenvalue for $E$. Then*

$$R(E^*-I^*) = N(E-I)^0.$$

**Proof.** We shall identify $Y$ with $\hat{Y}$ and thus take the bilinear form as the natural form $(\,,)$. Since $E$ is compact with eigenvalue 1 we have

$$\dim (N(E-I)) = n$$

where $0 < n < \infty$. Since $E^*$ is compact we have that $R(E^*-I^*)$ is norm closed and codim $(R(E^*-I^*))$ is finite, say $m$. Moreover we have $m \geqq n$, since it is trivial that

$$R(E^*-I^*) \subset N(E-I)^0.$$

Suppose that $m = n+p$ with $p \geqq 1$. Let $(E^*-I^*)'$ denote the usual adjoint of $E^*-I^*$ on $Y'$, and let us agree to identify $X$ with $\hat{X}$. It is now trivial to verify that $(E^*-I^*)'$ is an extension of $E-I$ and so

$$E^{*\prime}X \subset X.$$

Moreover we have

$$\dim (N((E^*-I^*)')) = \operatorname{codim} (R(E^*-I^*)) = n+p.$$

It follows that there exist linearly independent vectors $u_1, u_2, \ldots, u_p$ in $Y' \smallsetminus X$ such that

$$E^{*\prime}u_j = u_j \quad (j = 1, 2, \ldots, p).$$

Now let $F$ be the restriction of $E^{*\prime}$ to the subspace of $Y'$ generated by $X$ and $\{u_j\colon j = 1, 2, \ldots, p\}$, and let $P$ be the spectral projection for $F$ at 1. We then have that $F$ has an adjoint on $Y$ ($E^*$ in fact) and

$$F = FP + F(I-P)$$

where $(I-P)X \subset X$. It follows that

$$F = P + E(I-P)$$

and thereafter that $F - E = Q$ where $Q$ is of the form

$$Q = \sum_{j=1}^{p} u_j \otimes \phi_j$$

for non-zero $\phi_j \in Y$. Since $FX \subset X$ we must have $QX \subset X$. But

$$Qx = \sum_{j=1}^{p} (x, \phi_j)u_j$$

and so $QX \subset X$ if and only if $\phi_j = 0$ ($j = 1, 2, \ldots, p$). This contradiction shows that codim $(R(E^* - I^*)) = n$ and we therefore conclude that

$$R(E^* - I^*) = N(E-I)^0.$$

**Remark.** The above result is well known for the case $(X, X', (\,,\,))$. It may even be true for an arbitrary pair of Banach spaces in normed duality.

**Theorem 8.** *Let $\mathfrak{A}$ be a topologically simple $B^\#$ modular annihilator algebra. Given a proper closed modular left ideal $L$ of $\mathfrak{A}$ we have*

(i) *$L$ is the intersection of a finite number of maximal modular left ideals,*

(ii) *there exists $p \in \mathfrak{A}$ such that $p^2 = p$ and $L = \mathfrak{A}(1-p)$,*

(iii) $L = l(r(L))$.

*A similar result holds for the proper closed modular right ideals.*

**Proof.** It is clearly sufficient to make the proof for left ideals. In view of Theorem 6 we shall identify $\mathfrak{A}$ with an algebra of the form $\mathfrak{F}(X, Y, \langle\,,\,\rangle)$ where $(X, Y, \langle\,,\,\rangle)$ are Banach spaces in normed duality such that $x \to \hat{x}$ and $y \to \hat{y}$ are isometries. Let $L$ be a proper closed modular left ideal of $\mathfrak{A}$ with modular identity $E$. Let $M$ be the closure of the subspace of $Y$ generated by

$$\{T^*y \colon y \in Y, T \in L\}.$$

By (6) Corollary (2.4.19) we have that $L$ contains every operator of finite rank whose adjoint has range contained in $M$.

We show first that ${}^0M \neq (0)$. Indeed we have $r(L) \neq (0)$ so that $r(L)$ contains operators of rank one. Therefore there exists non-zero $x_0 \in X$ such that $Tx_0 = 0$ ($T \in L$). It follows that

$$\langle x_0, T^*y\rangle = 0 \quad (y \in Y, T \in L)$$

and so $x_0 \in {}^0M$. Since $E$ is a modular identity for $L$ we have that

$$x \otimes y - x \otimes yE \in L \quad (x \in X, y \in Y).$$

Choose $x \neq 0$ and we obtain $R(E^* - I^*) \subset M$. Since $E$ is compact $N(E-I)$ is finite dimensional. Let $x_1, x_2, \ldots, x_n$ be a basis for $N(E-I)$. By (6), Lemma

(2.4.9) we may choose $y_1, y_2, \ldots, y_n \in Y$ such that

$$\langle x_i, y_j \rangle = \delta^i_j.$$

Now let $P = \sum_{j=1}^{n} x_j \otimes y_j$ so that $P$ is a projection with range $N(E-I)$. We shall show that $P$ is a modular identity for $L$. Since the operators of rank one generate $\mathfrak{A}$ it is sufficient to show that

$$x \otimes y - x \otimes yP \in L \quad (x \in X, y \in Y).$$

For this it is sufficient to show that $R(P^* - I^*) \subset M$. Applying Proposition 2 twice we obtain

$$R(P^* - I^*) = N(P-I)^0 = N(E-I)^0 = R(E^* - I^*) \subset M.$$

We now have

$$\begin{aligned}\mathfrak{A}(1-P) \subset L \subset \{T\colon T^*Y \subset M\} &= \{T\colon Tx_j = 0 (j = 1, 2, \ldots, n)\} \\ &= l(P\mathfrak{A}) = \mathfrak{A}(1-P)\end{aligned}$$

since $P^2 = P$. It follows that

$$L = \mathfrak{A}(1-P) = \{T\colon Tx_j = 0 (j = 1, 2, \ldots, n)\}$$

and the result is clear.

**Remark.** It is in fact clear that the above result holds for any topologically simple modular annihilator algebra whose underlying pair $(X, Y, \langle\, , \rangle)$ satisfies the condition that $x \to \hat{x}$, $y \to \hat{y}$ are bicontinuous. We conjecture that the result holds for all topologically simple modular annihilator algebras.

**Proposition 3.** *Let $\mathfrak{A}$ be a $B^{\#}$ modular annihilator algebra with dense socle, let $I_1, I_2, \ldots, I_n$ be minimal-closed two-sided ideals of $\mathfrak{A}$, and let $I = I_1 \oplus I_2 \oplus \ldots \oplus I_n$. Then $\mathfrak{A} = I \oplus r(I)$.*

**Proof.** Let $\mathfrak{A}_0 = I \oplus r(I)$. Then $r(\mathfrak{A}_0) = r(I) \cap r(r(I)) = (0)$ and so $\mathfrak{A}_0^- = \mathfrak{A}$ by Theorem 2. We have that $\mathfrak{A}_0/r(I)$ is isomorphic with $I$ and is a normed algebra under the usual infimum norm. Transfer this infimum norm to $I$ and we obtain $\|[i]\| \leqq \| i \| (i \in I)$. Since $\mathfrak{A}$ is the $B(\infty)$ sum of its minimal-closed two-sided ideals we have that $I$ is closed. It is proved in (7) that $I$ is a $B^{\#}$ algebra and thus since $I$ is a modular annihilator algebra with dense socle, $I$ has minimal norm. We thus have $\|[i]\| = \| i \| (i \in I)$. Thus $\mathfrak{A}_0/r(I)$ is a Banach algebra under the infimum norm. We conclude that $\mathfrak{A}_0$ is closed in $\mathfrak{A}$ and so $\mathfrak{A}_0 = \mathfrak{A}$.

**Theorem 9.** *Let $\mathfrak{A}$ be a $B^{\#}$ modular annihilator algebra with dense socle. Then $\mathfrak{A}$ is modular dual.*

**Proof.** As in Theorem 8 we shall simply consider the left ideals of $\mathfrak{A}$. Let $\{I_\lambda\colon \lambda \in \Lambda\}$ be the family of all the minimal-closed two-sided ideals of $\mathfrak{A}$. Let $L$ be any closed modular left ideal of $\mathfrak{A}$. For each $\lambda \in \Lambda$ let $L_\lambda = L \cap I_\lambda$. Then $L_\lambda$ is a closed left ideal of $I_\lambda$. Since $L$ is modular, there exists $e \in \mathfrak{A}$ such that $\mathfrak{A}(1-e) \subset L$. By Proposition 3 we may write $e = i_\lambda + j_\lambda$ where $i_\lambda \in I_\lambda$ and

$j_\lambda \in r(I_\lambda)$. Thus

$$I_\lambda(1-i_\lambda) \subset I_\lambda(1-e)+I_\lambda j_\lambda = I_\lambda(1-e) \subset L \cap I_\lambda = L_\lambda,$$

so that $L_\lambda$ is modular. If $L_\lambda$ is proper, we have by Theorem 8 that $L_\lambda = I_\lambda(1-p_\lambda)$ where $p_\lambda \in I_\lambda$ and $p_\lambda^2 = p_\lambda$. If $L_\lambda = I_\lambda$, then the same is trivially true with $p_\lambda = 0$.

We show next that $L = (\Sigma\{L_\lambda : \lambda \in \Lambda\})^-$. It is clear that $(\Sigma\{L_\lambda : \lambda \in \Lambda\})^- \subset L$. Now let $l \in L$. Given any minimal-closed two-sided ideals $I_1, I_2, \ldots, I_n$ we have by Proposition 3 that

$$l = i_1+i_2+\ldots+i_n+x$$

where $x \in r(I)$, $I = I_1 \oplus I_2 \oplus \ldots \oplus I_n$. Since $\mathfrak{A}$ is isometrically isomorphic with the $B(\infty)$ sum of $\{I_\lambda : \lambda \in \Lambda\}$, given $\varepsilon > 0$ we may choose the above elements $i_r$ such that $\| x \| < \varepsilon$. We show below that each $i_r \in L \cap I_r$ and it will then follow that $l \in (\Sigma\{L_\lambda : \lambda \in \Lambda\})^-$. We have

$$I_r i_r = I_r l \subset L \cap I_r = I_r(1-p_r)$$

and so $I_r i_r p_r I_r = (0)$. This implies $i_r p_r I_r = (0)$ (since $I_r$ is semi-simple) and so $i_r \in I_r(1-p_r) = L \cap I_r$.

We shall denote left and right annihilators in $I_\lambda$ by $l^\lambda$ and $r^\lambda$. We observe that $r(L) \cap I_\lambda = r^\lambda(L_\lambda)$. For, if $x \in I_\lambda$ and $Lx = (0)$, then $L_\lambda x = (0)$ so that $x \in r(L_\lambda) \cap I_\lambda$ and thus $x \in r^\lambda(L_\lambda)$. Conversely, if $x \in r^\lambda(L_\lambda)$ then $r(I_\lambda)x = (0)$ and $L_\lambda x = (0)$. Since $L \subset L_\lambda \oplus r(I_\lambda)$, we have $Lx = (0)$ and so $x \in r(L) \cap I_\lambda$. By arguing as above we further obtain $r(L) = r^\lambda(L_\lambda) \oplus (r(L) \cap r(I_\lambda))$. Thus

$$l(r(L)) = l(r^\lambda(L_\lambda)) \cap l(r(L) \cap r(I_\lambda)).$$

Since $l(r^\lambda(L_\lambda)) \cap I_\lambda = l^\lambda(r^\lambda(L_\lambda))$ and $l(r(L) \cap r(I_\lambda)) \supset I_\lambda$ it follows that

$$l(r(L)) \cap I_\lambda = l^\lambda(r^\lambda(L_\lambda)).$$

If $L_\lambda = I_\lambda$, then clearly $l^\lambda(r^\lambda(L_\lambda)) = L_\lambda$. If $L_\lambda$ is proper then $l^\lambda(r^\lambda(L_\lambda)) = L_\lambda$ by Theorem 8. Finally, since $l(r(L))$ is also a closed modular left ideal,

$$\begin{aligned} l(r(L)) &= (\Sigma\{l(r(L)) \cap I_\lambda : \lambda \in \Lambda\})^- \\ &= (\Sigma\{l^\lambda(r^\lambda(L_\lambda)) : \lambda \in \Lambda\})^- \\ &= (\Sigma\{L_\lambda : \lambda \in \Lambda\})^- \\ &= L. \end{aligned}$$

The proof is now complete.

## 4. The $B^\#$ condition and Banach spaces in normed duality

Let $(X, Y, \langle\, ,\, \rangle)$ be Banach spaces in normed duality and let $x \to \hat{x}$, $y \to \hat{y}$ be the associated natural mappings. It is remarked in (3) that we may have both the mappings $x \to \hat{x}$, $y \to \hat{y}$ bicontinuous, or only one bicontinuous, or neither bicontinuous. We show here how the condition that both mappings be bicontinuous is related to the $B^\#$ condition on the associated algebra $\mathfrak{F}(X, Y, \langle\, ,\, \rangle)$.

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**Theorem 10.** *Let $(X, Y, \langle\, , \rangle)$ be Banach spaces in normed duality. If $x \to \hat{x}$ and $y \to \hat{y}$ are both isometries, then $\mathfrak{F}(X, Y, \langle\, , \rangle)$ is a $B^{\#}$ algebra. Conversely, if $\mathfrak{F}(X, Y, \langle\, , \rangle)$ is a $B^{\#}$ algebra, then there are equivalent norms on $X$ and $Y$ for which $x \to \hat{x}$, $y \to \hat{y}$ are isometries.*

**Proof.** Suppose that $x \to \hat{x}$, $y \to \hat{y}$ are isometries. Then $| T^* | \leqq | T |$ and so $\| T \| = | T | (T \in \mathfrak{F}(X, Y, \langle\, , \rangle))$. Let $T \in \mathfrak{F}(X, Y, \langle\, , \rangle)$ with $| T | = 1$ and suppose $0 < \varepsilon < 1$. Then there exists $x \in X$ such that $\| x \| = 1$ and

$$1 - \tfrac{1}{2}\varepsilon \leqq \| Tx \| \leqq 1.$$

Since $\| Tx \| = \| \widehat{Tx} \|$, there exists $y \in Y$ such that $\| y \| = 1$ and

$$1 - \varepsilon \leqq \langle Tx, y \rangle \leqq 1.$$

Let $T^{\#} = x \otimes y$ so that $| T^{\#} | = \| x \| \| \hat{y} \| = \| x \| \| y \| = 1$. Further, $|(TT^{\#})^n | = \langle Tx, y \rangle^{n-1} Tx \otimes y$ and hence

$$|(TT^{\#})^n | \geqq (1 - \varepsilon)^n (n = 1, 2, \ldots),$$

so that $\mathfrak{F}(X, Y, \langle\, , \rangle)$ is a $B^{\#}$ algebra.

The converse follows simply from the argument used in Theorem 6.

We point out that it is not difficult to extend the above theorem to the case in which $x \to \hat{x}$, $y \to \hat{y}$ are only bicontinuous. For this we need a weakened $B^{\#}$ condition. We say that $\mathfrak{A}$ is a $B^{\#}_{\alpha}$ *algebra* if there is $\alpha$ with $0 < \alpha \leqq 1$ such that given $a \in \mathfrak{A}$ with $\| a \| = 1$ and given $\varepsilon$ with $0 < \varepsilon < \alpha$ there exists $a^{\#} \in \mathfrak{A}$ such that $\| a^{\#} \| = 1$ and

$$\|(aa^{\#})^n \| \geqq (\alpha - \varepsilon)^n (n = 1, 2, \ldots).$$

If $x \to \hat{x}$, $y \to \hat{y}$ are bicontinuous, then for some $\alpha$, $\mathfrak{F}(X, Y, \langle\, , \rangle)$ is a $B^{\#}_{\alpha}$ algebra. Conversely, if $\mathfrak{F}(X, Y, \langle\, , \rangle)$ is a $B^{\#}_{\alpha}$ algebra, then $x \to \hat{x}$, $y \to \hat{y}$ are bicontinuous. The proof of these statements is similar to the proof of Theorem 10.

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King's College, Aberdeen