PROJECTIVE-SYMMETRIC SPACES

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Introduction

Gy. Soos [1] and B. Gupta [2] have discussed the properties of Riemannian spaces V_n $(n > 2)$ in which the first covariant derivative of Weyl's projective curvature tensor is everywhere zero; such spaces they call *Projective-Symmetric spaces*. In this paper we wish to point out that all Riemannian spaces with this property are *symmetric* in the sense of Cartan [3]; that is the first covariant derivative of the Riemann curvature tensor of the space vanishes. Further sections are devoted to a discussion of projective-symmetric affine spaces A_n with symmetric affine connexion. Throughout, the geometrical quantities discussed will be as defined by Eisenhart [4] and [5].

1. Projective-symmetric Riemannian spaces

For a V_n, Weyl's projective curvature tensor $W^a{}_{bcd}$ is

$$W^a{}_{bcd} = R^a{}_{bcd} - \frac{2}{n-1}\{\delta^a{}_{[d}R_{c]b}\},$$

where $R^a{}_{bcd}$ is the curvature tensor, and $R_{bc} = R^a{}_{bca}$ the Ricci tensor, of the space. The V_n is a projective-symmetric space if and only if

$$W^a{}_{bcd;e} = 0. \tag{1.1}$$

We define the tensor $U^a{}_d$ by

$$U^a{}_d = g^{bc}\, W^a{}_{bcd} = \frac{n}{n-1}\left\{R^a{}_d - \frac{1}{n}\, R\, \delta^a{}_d\right\},$$

where $R = R^a{}_a$, and from (1.1) it follows that if the space is projective-symmetric, then

$$U^a{}_{d,e} = 0. \tag{1.2}$$

For $n > 2$, equation (1.2) and the twice-contracted Bianchi Identity $R^a{}_{b;a} = \frac{1}{2}R_{,b}$ implies

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$$R = \text{constant},$$

and thus we have $R^a{}_{b;e} = 0$. With (1.1) this gives the result

$$0 = W^a{}_{bcd;e} \Leftrightarrow R^a{}_{bcd;e} = 0,$$

from which follows:

THEOREM 1. *A Riemannian space V_n $(n > 2)$ is a projective-symmetric space if and only if it is symmetric in the sense of Cartan* [3].

For $n = 2$, $W^a{}_{bcd}$ is identically zero and (1.1) is a degenerate condition in a V_2. We remark however that a V_2 is a symmetric space if and only if it has constant scalar curvature R.

The results of Gupta [2] follow immediately since they are trivially true for symmetric spaces. The paper of Soos [2] contains theorems for projective-symmetric spaces which are generalisations of results found by Sinjukow [6] for symmetric spaces.

2. Affine spaces with symmetric connexion

For the remainder of this paper we consider the applicability of the preceding theorem in an Affine space with symmetric connexion. Such a space we will denote by A_n, its connexion by $\Gamma^a{}_{bc}$, and covariant differentiation with respect to this connexion by ";".

The curvature tensor of A_n is defined

$$B^a{}_{bcd} = 2\Gamma^a{}_{b[d,c]} + 2\Gamma^h{}_{b[d}\Gamma^a{}_{c]h}, \tag{2.1}$$

for which the identities

$$B^a{}_{b(cd)} = B^a{}_{[bcd]} = 0, \tag{2.2}$$

and Bianchi's identity

$$B^a{}_{b[cd;e]} = 0, \tag{2.3}$$

hold. The analogue of the Ricci tensor for an A_n is $B_{bc} = B^a{}_{bca}$, but in this case it is not necessarily symmetric; it follows from (2.2) that

$$S_{cd} = -2B_{[cd]}, \tag{2.4}$$

where $S_{cd} = B^a{}_{acd}$. From (2.3) we have also

$$\begin{aligned} S_{[cd;e]} &= 0, \\ B^a{}_{bcd;a} &= 2B_{b[c;d]}. \end{aligned} \tag{2.5}$$

Weyl's projective curvature tensor for an A_n is

$$W^a{}_{bcd} = B^a{}_{bcd} - \frac{1}{n+1}\,\delta^a{}_b\, S_{cd} - \frac{2}{n-1}\, B_{b[c}\delta^a{}_{d]} - \frac{2}{n^2-1}\, S_{b[c}\delta^a{}_{d]}. \tag{2.6}$$

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This tensor is invariant for projective transformations of the space and its vanishing implies that the A_n has the same paths as flat space [5]. By a *projective-symmetric affine space* we will mean an A_n $(n > 2)$ such that

$$W^a{}_{bcd;e} = 0, \tag{2.7}$$

throughout; an A_n is *symmetric* [3] if and only if

$$B^a{}_{bcd;e} = 0, \tag{2.8}$$

at all points.

Equation (2.8) implies that every symmetric A_n is a projective-symmetric A_n. Such projective-symmetric spaces we will call ***degenerate***, and from Theorem 1 we see that all projective-symmetric Riemannian spaces are degenerate in this sense. We will show that this is not true for a general A_n and will consider its validity in relation to certain sub-classes of Affine spaces.

3. A non-degenerate projective-symmetric A_n

Consider the A_n with connexion coefficients

$$\Gamma^a{}_{bc} = 2\delta^a{}_{(b}\psi_{c)}, \tag{3.1}$$

in a coordinate system $\{x^a\}$ such that

$$\frac{\partial}{\partial x^a}\psi_c = 0.$$

The latter condition is expressed covariantly as

$$\psi_{c;d}+2\psi_c\psi_d = 0. \tag{3.2}$$

The A_n is projectively related to flat space; its projective curvature tensor vanishes and therefore it is a projective-symmetric space. From (2.1) we have for this A_n

$$B^a{}_{bcd} = 2\psi_b\delta^a{}_{(c}\psi_{d)},$$

and using (3.2)

$$B^a{}_{bcd;e} = -4\psi_e B^a{}_{bcd}.$$

For $\psi_e \neq 0$, the curvature tensor of the space is non-zero and we have the result:

THEOREM 2. *There exist projective-symmetric A_n's which are non-degenerate.*

4. The decomposable A_n

If two spaces A_m and A_{n-m} are given with coordinates x^α: $(\alpha, \beta, \gamma = 1, 2, \cdots, m)$ and x^A: $(A, B, C = m+1, \cdots, n)$ and the connexions $\Gamma^\alpha_{\beta\gamma}$ and Γ^A_{BC}, then the A_n with coordinates x^a: $(a, b, c, = 1, 2, \cdots, n)$ and connexion $\Gamma^a_{bc} \equiv \{\Gamma^\alpha_{\beta\gamma}, \Gamma^A_{BC}\}$, is called the product of A_m and A_{n-m}. An A_n that is a product space is said to be *decomposable*. A geometric object in a decomposable A_n is *decomposable* if and only if its components with respect to the special coordinates are always zero when they have indices from both ranges, and the components belonging to the subspace A_n (A_{n-m}) are functions of x^α (x^A) only. In a decomposable A_n, $B^a{}_{bcd}$, B_{bc} and their covariant derivatives are decomposable; $W^a{}_{bcd}$ and $W^a{}_{bcd;e}$ are not in general decomposable.

THEOREM 3. *A projective-symmetric A_n which is decomposable is necessarily degenerate.*

We assume that $A_n \equiv \{A_m \times A_{n-m}\}$ where indices $\alpha, \beta, \gamma = 1 \cdots m$ relate to A_m, and $A, B, C = m+1, \cdots n$ relate to A_{n-m}. From the definition of the projective-curvature tensor we have for the decomposable A_n

$$W^\alpha{}_{\beta CD} = -\frac{1}{n+1}\,\delta^\alpha{}_\beta S_{CD}, \tag{4.1}$$

and

$$W^\alpha{}_{B\gamma D} = \frac{1}{n-1}\,\delta^\alpha{}_\gamma \left\{B_{BD} + \frac{1}{n+1}\,S_{BD}\right\}. \tag{4.2}$$

The assumption that A_n is a projective-symmetric space gives with (3.1)

$$S_{CD;E} = 0,$$

and therefore in (3.2)

$$B_{BD;E} = 0.$$

Similarly we have

$$B_{\beta\delta;\varepsilon} = 0,$$

and since $B_{bd;e}$ is a decomposable tensor of the A_n it follows that

$$B_{bd;e} = 0.$$

With the above, the differentiation of (2.5) gives

$$0 = W^a{}_{bcd;e} = B^a{}_{bcd;e},$$

and the decomposable A_n is a symmetric space.

5. The projective-symmetric W_n

An A_n in which there exists a symmetric two index tensor g_{ab} of rank n such that

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(5.1) $$g_{ab;c} = -2\phi_c g_{ab},$$

for some covariant vector ϕ_e is called a W_n and was first discussed by Weyl [7]. Define the contravariant tensor g^{ab} by $g^{ab}g_{bc} = \delta^a{}_c$, then from (4.1)

(5.1) $$g^{ab}{}_{;c} = 2\phi_c g^{ab}.$$

We can use g_{ab} (g^{ab}) to define a correspondence between covariant and contravariant quantities in A_n; in fact if ϕ_c is a gradient vector $\phi_{,c}$ the W_n is a Riemannian space V_n with metric tensor $\bar{g}_{ab} = e^{2\phi}g_{ab}$.

With $W_{abcd} = g_{ae}W^e{}_{bcd}$ and $B_{abcd} = g_{ae}B^e{}_{bcd}$, we define

(5.2) $$T_{ad} = g^{bc}W_{abcd},$$

and

(5.3) $$Q_{ad} = g^{bc}B_{abcd}.$$

From the Ricci identity applied to g_{ab}, and the use of (5.1) and (5.1a) we have

$$B_{(ab)cd} = -2g_{ab}\phi_{[c;d]},$$

which yields after contraction

$$Q_{ad} = B_{ad} - 4\phi_{[a;d]},$$

and

$$S_{cd} = -2n\phi_{[c;d]}.$$

We extract the symmetric and anti-symmetric parts of these equations to obtain

(5.4) $$\begin{aligned} Q_{(ad)} &= B_{(ad)}, \\ B_{[ad]} &= n\phi_{[a;d]}, \\ Q_{[ad]} &= (n-4)\phi_{[a;d]}. \end{aligned}$$

With equation (5.2), the definition of the projective curvature tensor gives

$$T_{ab} = Q_{ab} - \frac{n-2}{n^2-1}S_{ab} + \frac{1}{n-1}\{B_{ab} - Bg_{ab}\},$$

where $B = g^{bc}B_{bc} = g^{bc}Q_{bc}$. Frequent use of the relations (2.4) and (5.4) gives the decomposition

(5.5) $$\begin{aligned} T_{(ab)} &= \frac{n}{n-1}\left\{B_{(ab)} - \frac{1}{n}g_{ab}B\right\}, \\ T_{[ab]} &= \frac{n^2-4}{n(n-1)}B_{[ab]}. \end{aligned}$$

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LEMMA 1. *In a projective-symmetric W_n $(n > 2)$,*

$$T_{ab;c} = 0.$$

PROOF. $T_{ab;c} = g_{ad}g^{ef}W^d{}_{efb;c} + g_{ad;c}T^d{}_b + g^{ef}{}_{;c}W_{aefb}$. From (5.1) and (5.1a) the sum of the second and third terms on the right hand side of the above equation is zero. Hence

$$T_{ab;c} = g_{ad}g^{ef}W^d{}_{efb;c},$$

which vanishes if W_n is projective-symmetric. Q.E.D.

Lemma 1 applied to the second equation of (5.5) gives

$$B_{[ab];c} = 0, \tag{5.6}$$

and with (5.6) in the first equation of (5.5)

$$B_{ab;c} = \frac{1}{n} g_{ab} \{B_{,c} - 2B\phi_c\}. \tag{5.7}$$

LEMMA 2. *In a projective-symmetric W_n $(n > 2)$*

$$B_{a[b;c]} = 0.$$

PROOF. We have

$$0 = W^a{}_{bcd;a} = B^a{}_{bcd;a} - \frac{n-2}{n^2-1} S_{cd;b} - \frac{2}{n-1} B_{b[c;d]}.$$

From (2.3) and (5.6), $S_{cd;e} = 0$, and we see that

$$B^a{}_{bcd;a} = \frac{2}{n-1} B_{b[c;d]}.$$

However from the contracted Bianchi Identity (2.4) for the space

$$B^a{}_{bcd;a} = 2B_{b[c;d]},$$

and the result of the lemma follows.

From lemma 2 and (5.7) we have

$$g_{a[b}B_{,c]} - 2Bg_{[b}\phi_{c]} = 0,$$

and after contraction

$$(n-1)\{B_{,c} - 2\phi_c B\} = 0. \tag{5.8}$$

Referring to equation (5.7) we deduce that $B_{ab;c} = 0$, and therefore for a W_n

$$0 = W^a{}_{bcd;e} \Leftrightarrow B^a{}_{bcd;e} = 0.$$

THEOREM 4. *Every projective-symmetric W_n is degenerate.*

We also remark that if $B \neq 0$ in (5.3) then ϕ_c is necessarily a gradient:

THEOREM 5. *The "scalar curvature" B of a projective-symmetric W_n which is not a Riemannian space is necessarily zero.*

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