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Abstract

We study the action of the inertia operator on the motivic Hall algebra and prove that it is diagonalizable. This leads to a filtration of the Hall algebra, whose associated graded algebra is commutative. In particular, the degree 1 subspace forms a Lie algebra, which we call the Lie algebra of *virtually indecomposable* elements, following Joyce. We prove that the integral of virtually indecomposable elements admits an Euler characteristic specialization. In order to take advantage of the fact that our inertia groups are unit groups in algebras, we introduce the notion of *algebroid*.

Contents

Introduction

For simplicity, let us work over a field k. (Later, k will be replaced by a noetherian ring R.)

Let $\mathfrak{M}$ be an abelian k-linear algebraic stack. Roughly, this means that $\mathfrak{M}$ is at the same time a k-linear abelian category with finite-dimensional hom-spaces, and an algebraic stack, locally of

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finite type over k. (The precise definition of linear algebraic stack is Definition 1.9. In the body of the paper we work with exact, instead of abelian, categories; see the beginning of § 3.)

Examples we are interested in include:

(i) $\mathfrak{M} = \mathfrak{Coh}_Y$, the moduli stack of coherent $\mathscr{O}_Y$-modules, for a projective k-variety Y;

(ii) $\mathfrak{M} = \mathfrak{Rep}_Q$, the moduli stack of representations of a quiver Q on finite-dimensional k-vector spaces;

(iii) (the case $Y = Q = \operatorname{Spec} k$) $\mathfrak{M} = \mathfrak{Vect}$, the stack of finite-dimensional k-vector spaces. In this case, $\mathfrak{Vect}(S)$, for a k-scheme S, is the exact $\mathscr{O}(S)$-linear category of vector bundles over S, and $\mathfrak{Vect}(k)$ is the abelian k-linear category of finite-dimensional k-vector spaces. As an algebraic stack, $\mathfrak{Vect}$ is

$$\mathfrak{Vect} = \operatorname{Spec} k \amalg B\,\mathrm{GL}_1 \amalg B\,\mathrm{GL}_2 \amalg \dots .$$

Algebroids. There is a canonical sheaf of algebras $\mathfrak{A} \to \mathfrak{M}$ over $\mathfrak{M}$. The set of sections of $\mathfrak{A}$ over the S-valued point x of $\mathfrak{M}$ is the $\mathscr{O}(S)$-algebra $\mathfrak{A}_x = \operatorname{End}(x)$. For $\mathfrak{M} = \mathfrak{Vect}$, the point x is a vector bundle over S, and $\operatorname{End}(x)$ is the $\mathscr{O}(S)$-module of endomorphisms of x.

There is also a canonical isomorphism of group sheaves $\mathfrak{A}^\times \to I_{\mathfrak{M}}$ over $\mathfrak{M}$, where $I_{\mathfrak{M}}$ is the *inertia stack* of $\mathfrak{M}$. (Recall that the sections of $I_{\mathfrak{M}}$ over the S-valued point x of $\mathfrak{M}$ are the automorphisms of x, in other words, the units in the algebra of endomorphisms.)

We call a triple (X, A_X, ι) an *algebroid* (see Definition 1.38 and Remark 1.44) if X is an algebraic stack, $A_X \to X$ is a representable sheaf of finite $\mathscr{O}$-algebras over X (or *finite type algebras*, as we call them; see Definition 1.18), and $\iota : A_X^\times \to I_X$ is an open immersion of relative group schemes over X, making the canonical diagram

$$\begin{array}{ccc} A_X^\times & \xrightarrow{\ \iota\ } & I_X \\ & \searrow & \downarrow \\ & & \underline{\mathrm{Aut}}_X(A_X) \end{array}$$

commute.

So, $\mathfrak{M}$ with its canonical sheaf of algebras $\mathfrak{A}$ is an example of an algebroid. In this case, ι is an isomorphism, yielding what we call a *strict* algebroid.

Algebroids are generalizations of linear algebraic stacks (they are linear over their coarse moduli spaces, if they are strict). They are slightly more flexible. For example, schemes can be considered as algebroids in a canonical way. If the algebraic stack X is the base of an algebroid, then the connected component I_X° of its inertia stack I_X is the group of units in an algebra. This is the main significance of algebroids for us.

Just like algebraic stacks, algebroids form a 2-category, in which 2-fibred products exist. Whenever (X, A_X) is an algebroid, and $Y \to X$ is an *inert* morphism of algebraic stacks, i.e., $I_Y^\circ = I_X^\circ|_Y$ (see Definition 1.47), the stack Y is endowed with a natural structure of an algebroid via $A_X|_Y$. Examples of inert morphisms include monomorphisms and projections $Z \times X \to X$, for schemes Z. A *locally closed immersion* of algebroids $(Y, A_Y) \to (X, A_X)$ is a morphism where $Y \to X$ is a locally closed immersion of algebraic stacks, such that $A_Y = A_X|_Y$. Every scheme Z is an algebroid via the definition $A_Z = 0_Z$.

A key observation is that if (X, A) is an algebroid, then (I_X, I_A) is another algebroid. In fact, I_A, the inertia stack of the stack A (the total 'space' of the sheaf of algebras A), is equal to the subalgebra of $A|_{I_X}$ fixed under its tautological automorphism. We call (I_X, I_A) the *inertia algebroid* of (X, A). It comes with a canonical morphism to (X, A). There is also a semi-simple connected version of the algebroid inertia, denoted by $I^{\circ,\mathrm{ss}}$.

The motivic Hall algebra. We define *stack functions* to be representable morphisms of algebroids $(X, A_X) \to (\mathfrak{M}, \mathfrak{A})$, where X is of finite type. The *Hall algebra* $K(\mathfrak{M})$ of $\mathfrak{M}$ is the $\mathbb{Q}$-vector space on the isomorphism classes of stack functions modulo the *scissor relations relative* $\mathfrak{M}$:

$$[X \to \mathfrak{M}] = [Z \to X \to \mathfrak{M}] + [X \backslash Z \to X \to \mathfrak{M}]$$

and the *inert bundle relations relative* $\mathfrak{M}$:

$$[Y \to X \to \mathfrak{M}] = [F \times X \to X \to \mathfrak{M}].$$

Here, $[X \to \mathfrak{M}]$ denotes the Hall algebra element defined by a stack function with base X. Also, $Z \to X$ is a closed immersion of algebroids, with open complement $X \backslash Z$, and $Y \to X$ is an *inert* fibre bundle (Definition 1.53) with special structure group and fibre F, all endowed with their canonical algebroid structure. (Examples of inert fibre bundles are étale locally trivial ones.)

We have the following structures on the Hall algebra.

(i) *Module structure.* Let $K(\mathrm{Var})$ denote the Grothendieck ring of varieties over k. We denote the motivic weight of the affine line by $q = [\mathbb{A}^1] \in K(\mathrm{Var})$. By $[Z] \cdot [X \to \mathfrak{M}] = [Z \times X \to \mathfrak{M}]$ we define a $K(\mathrm{Var})$-module structure on $K(\mathfrak{M})$.

(ii) *Multiplication.* By $[X \to \mathfrak{M}] \cdot [Y \to \mathfrak{M}] = [X \times Y \to \mathfrak{M} \times \mathfrak{M} \xrightarrow{\oplus} \mathfrak{M}]$ we define a commutative multiplication on $K(\mathfrak{M})$, and $K(\mathfrak{M})$ is a $K(\mathrm{Var})$-algebra with this multiplication.

(iii) *Hall product.* Using the stack of short exact sequences in $\mathfrak{M}$, we can define a Hall algebra product $[X \to \mathfrak{M}] * [Y \to \mathfrak{M}]$ on $K(\mathfrak{M})$. For details, see § 3. The module $K(\mathfrak{M})$ is a $K(\mathrm{Var})$-algebra also with respect to the Hall product.

(iv) *Unit.* The unit with respect to both products is represented by $1 = [\operatorname{Spec} k \xrightarrow{0} \mathfrak{M}]$.

(v) *Inertia endomorphism.* The algebroid inertia defines an operator $I : K(\mathfrak{M}) \to K(\mathfrak{M})$ via $I[X \to \mathfrak{M}] = [I_X \to X \to \mathfrak{M}]$. This inertia operator is linear over $K(\mathrm{Var})$ and multiplicative, $I(x \cdot y) = I(x) \cdot I(y)$, with respect to the commutative multiplication. The same facts hold for the connected semi-simple inertia operator $I^{\circ,\mathrm{ss}} : K(\mathfrak{M}) \to K(\mathfrak{M})$.

There is also a 'non-representable' version of the Hall algebra, where we drop the representability requirement for stack functions and simply define a stack function to be a morphism of algebroids $X \to \mathfrak{M}$, with X of finite type. The representable Hall algebra is a subalgebra (with respect to both products) of the non-representable one. Our results on the diagonalizability of the various operators I, $I^{\circ,\mathrm{ss}}$, E_n hold true also in the non-representable Hall algebra, but the algebraic results on the structure of the Hall algebra need representability. For simplicity, we restrict ourselves therefore to the representable case from the beginning.

Usually, when defining the Hall algebra of $\mathfrak{M}$, one requires the bundle relations also for non-inert morphisms. The connected inertia operator does not respect such relations, and we therefore do not include them.

Example. A stack function $X \to \mathfrak{Vect}$ is the same thing as an algebroid (X, A) together with a faithful representation, i.e., a vector bundle V over X together with a monomorphism of algebras $A \to \operatorname{End}(V)$, making the canonical diagram

$$\begin{array}{ccc} A_X^\times & \xrightarrow{\iota} & I_X \\ & \searrow & \downarrow \\ & & \mathrm{GL}(V) \end{array}$$

commute.

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Examples of stack functions with values in $\mathfrak{Vect}$ include subalgebras $A \subset M_{n\times n}$. (The induced morphism of algebroids is $(BA^\times, A^\times \backslash A) \to (B\,\mathrm{GL}_n, \mathrm{GL}_n \backslash M_{n\times n})$.) The elements of $K(\mathfrak{Vect})$ defined by $A \subset M_{n\times n}$ and $B \subset M_{n\times n}$ are equal if and only if A and B are conjugate in $M_{n\times n}$.

The subalgebra U of $K(\mathfrak{Vect})$ with respect to the Hall product, generated by the $[n] = [B\,\mathrm{GL}_n \to \mathfrak{Vect}]$, is free on these elements $[n]$, for $n > 0$, as a unitary $\mathbb{Q}$-algebra. In the literature, U is known as the Hopf algebra of *non-commutative symmetric functions*; see [Car07, Example 4.1(F)].

(If we add the (non-inert) vector bundle relations relative to $\mathfrak{Vect}$, see, e.g., [Bri12], we get

$$[\lambda_1] * \ldots * [\lambda_r] = \frac{[\mathrm{GL}_n]}{[P(\lambda)]}[n] = \binom{n}{\lambda_1 \ldots \lambda_n}_q [n].$$

Here $n = \sum \lambda_i$, and $\binom{n}{\lambda_1 \ldots \lambda_n}_q$ denotes the q-deformed multinomial coefficient, which gives the motivic weight of the flag variety of type λ. We have also denoted the parabolic subgroup of GL_n of type λ by $P(\lambda)$. Hence, the $\mathbb{Q}$-algebra obtained by dividing U by the vector bundle relations is the commutative polynomial algebra over $\mathbb{Q}$, on the symbols $[1], [2], [3], \ldots$. This is the Hopf algebra of symmetric functions.)

The spectrum of semi-simple inertia. The main point of this work is to study the spectral theory of the semi-simple inertia operator $I^{\circ,\mathrm{ss}}$ on $K(\mathfrak{M})$.

Before announcing our results, let us do a few sample calculations. They contain some of the central ideas of this paper. Only strict algebroids will occur, so we write I^{ss} instead of $I^{\circ,\mathrm{ss}}$.

We consider $\mathfrak{M} = \mathfrak{Vect}$. The linear stack of line bundles defines the stack function $[B\,\mathrm{GL}_1 \to \mathfrak{Vect}] \in K(\mathfrak{Vect})$. We have

$$\begin{aligned} I^{\mathrm{ss}}[B\,\mathrm{GL}_1 \to \mathfrak{Vect}] &= [\mathrm{GL}_1^{\mathrm{ss}} \times B\,\mathrm{GL}_1 \to \mathfrak{Vect}] \\ &= [\mathrm{GL}_1 \times B\,\mathrm{GL}_1 \to \mathfrak{Vect}] \\ &= (q-1)[B\,\mathrm{GL}_1 \to \mathfrak{Vect}]. \end{aligned}$$

This proves that $[B\,\mathrm{GL}_1 \to \mathfrak{Vect}]$ is an eigenvector of I^{ss}, with corresponding eigenvalue $(q-1) \in K(\mathrm{Var})$.

Because I^{ss} is an algebra morphism with respect to the commutative product, it immediately follows that every $(q-1)^r$ for $r \geqslant 0$ is an eigenvalue of I^{ss}, with corresponding eigenvector $[B\,\mathrm{GL}_1^r \to B\,\mathrm{GL}_n \to \mathfrak{Vect}] \in K(\mathfrak{Vect})$.

These are not the only eigenvalues of I^{ss}. In fact, let us consider the stack function of all rank 2 vector bundles $[B\,\mathrm{GL}_2 \to \mathfrak{Vect}]$. Recall that the inertia stack of $B\,\mathrm{GL}_2$ is the quotient stack $\mathrm{GL}_2 /_{\mathrm{ad}} \mathrm{GL}_2$, where GL_2 acts on itself by the adjoint action. The semi-simple part of GL_2 decomposes as $\mathrm{GL}_2^{\mathrm{eq}} \sqcup \mathrm{GL}_2^{\mathrm{neq}}$, according to whether the two eigenvalues of an element of GL_2 are equal or not equal. By the scissor relations, we have

$$\begin{aligned} I^{\mathrm{ss}}[B\,\mathrm{GL}_2] &= [\mathrm{GL}_2^{\mathrm{eq}} /_{\mathrm{ad}} \mathrm{GL}_2] + [\mathrm{GL}_2^{\mathrm{neq}} /_{\mathrm{ad}} \mathrm{GL}_2] \\ &= [\Delta \times B\,\mathrm{GL}_2] + [T^*/_{\mathrm{ad}} N] \\ &= (q-1)[B\,\mathrm{GL}_2] + x. \end{aligned}$$

Here, Δ is the one-parameter subgroup of scalar matrices, and T is the maximal torus of diagonal matrices in GL_2. Further notation: $T^* = T \backslash \Delta$, N is the normalizer of T in GL_2, and $x = [T^*/_{\mathrm{ad}} N]$.

Next, we calculate $I^{\mathrm{ss}} x$. In fact, we have $I^{\mathrm{ss}}_{T^*/N} = I_{T^*/N} = (T^* \times T)/N$, by the 'stabilizer formula' for the inertia stack of a quotient stack

$$I_{Y/G} = \{(y, g) \in Y \times G \mid yg = y\}/G.$$

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We note that $N = T \rtimes \mathbb{Z}_2$ acts on $T^* \times T$ diagonally, via its quotient $\mathbb{Z}_2$, by swapping the entries of T. We embed T into $\mathbb{A}^2$ equivariantly with respect to $\mathbb{Z}_2$ and then decompose $\mathbb{A}^2$ as $T \sqcup (\mathrm{GL}_1 \times 0) \sqcup (0 \times \mathrm{GL}_1) \sqcup (0,0)$. This gives

$$[(T^* \times \mathbb{A}^2)/N] = [(T^* \times T)/N] + [T^* \times (\mathrm{GL}_1 \times 0 \sqcup 0 \times \mathrm{GL}_1)/N] + [T^* \times (0,0)/N]. \qquad (1)$$

We have a pullback diagram of algebroids.

$$\begin{array}{ccc} (T^* \times \mathbb{A}^2)/N & \longrightarrow & T^*/N \\ \downarrow & & \downarrow \\ (T^* \times \mathbb{A}^2)/\mathbb{Z}_2 & \longrightarrow & T^*/\mathbb{Z}_2 \end{array}$$

It shows that the vector bundle $(T^* \times \mathbb{A}^2)/N \to T^*/N$ is a pullback of the vector bundle $(T^* \times \mathbb{A}^2)/\mathbb{Z}_2 \to T^*/\mathbb{Z}_2$. The latter is a vector bundle over a scheme, and is therefore Zariski-locally trivial by Hilbert's Theorem 90. The same is then true for any pullback bundle. Hence, we conclude that

$$[(T^* \times \mathbb{A}^2)/N] = q^2[T^*/N],$$

using the scissor relations, or the inert bundle relations. So, from (1), we conclude that

$$q^2 x = I^{\mathrm{ss}} x + (q-1)[T^*/T] + x,$$

which we solve for $I^{\mathrm{ss}} x$ to get

$$I^{\mathrm{ss}} x = (q^2-1)x - (q-1)^2(q-2)[BT].$$

We already know that $I^{\mathrm{ss}}[BT] = (q-1)^2[BT]$, and so we conclude that $[B\,\mathrm{GL}_2]$, x, and $[BT]$ generate an I^{ss}-invariant subspace of $K(\mathfrak{Vect})$, and the matrix of I^{ss} on this subspace is

$$\begin{pmatrix} q-1 & 0 & 0 \\ 1 & q^2-1 & 0 \\ 0 & -(q-1)^2(q-2) & (q-1)^2 \end{pmatrix}.$$

This matrix is lower triangular, with distinct scalars on the diagonal, and is therefore diagonalizable over the field $\mathbb{Q}(q)$. So, on this subspace, I^{ss} is diagonalizable, with eigenvalues $(q-1)$, (q^2-1), and $(q-1)^2$. If we decompose $[B\,\mathrm{GL}_2]$ as a sum of eigenvectors, we get the eigenvectors

$$\begin{aligned} &[B\,\mathrm{GL}_2] - \frac{1}{q(q-1)}x - \frac{1}{q}[BT] && \text{with eigenvalue } (q-1), \\ &\frac{1}{q(q-1)}x - \frac{q-2}{2q}[BT] && \text{with eigenvalue } (q^2-1), \\ &\tfrac{1}{2}[BT] && \text{with eigenvalue } (q-1)^2. \end{aligned}$$

A very important observation is that when we add together the eigencomponents whose eigenvalues have the same order of vanishing at $q = 1$, we get coefficients in $\mathbb{Q}$, instead of $\mathbb{Q}(q)$. In the above example, we add together the components of $[B\,\mathrm{GL}_2]$ with eigenvalues $(q-1)$ and (q^2-1) to obtain $[B\,\mathrm{GL}_2] - \frac{1}{2}[BT]$.

Another important observation is that diagonalizing I^{ss} does not, despite appearances, require us to invert $(q-1)$. In fact, the algebroid x appearing in the above argument is divisible

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by $(q-1)$, although the quotient is not a strict algebroid any longer. To see this, consider the action of $\mathbb{G}_m$ on T^* by left multiplication. It commutes with the adjoint action by N. Let us denote the quotient $\mathbb{G}_m\backslash T^*$ by $\widetilde{T}^*$. We can endow the stack $\widetilde{T}^*/N$ with the structure of an algebroid, making the quotient map $T^*/N \to \widetilde{T}^*/N$ into an inert morphism of algebroids with fibre $\mathbb{G}_m$. (For the details, see Proposition 1.61.) The inert bundle relations then imply that $x = [T^*/N]$ is indeed divisible by $(q-1)$.

The matrices in $\widetilde{T}^*$ of trace zero have the full group N as stabilizer. Decomposing $\widetilde{T}^*$ into matrices of trace zero, and matrices of non-zero trace, we see that

$$\widetilde{T}^*/N = BN \sqcup (\mathbb{G}_m\backslash\{1\}) \times BT.$$

Hence, $\widetilde{T}^*/N$ is not a strict algebroid, as N is not connected, and is not the group of units in any algebra. This is, in fact, the reason for considering non-strict algebroids at all. (For more on this, see Example 2.18.)

(In the above calculations, we have suppressed the algebra part A of the various algebroids (X, A). We leave it to the reader to supply the natural algebra for each algebroid mentioned.)

Results. We now summarize the main results of this paper.

THEOREM 1 (Diagonalizability of $I^{\circ,\mathrm{ss}}$). *The operator $I^{\circ,\mathrm{ss}}$ on*

$$K(\mathfrak{M})(q) = K(\mathfrak{M}) \otimes_{\mathbb{Q}[q]} \mathbb{Q}(q)$$

is diagonalizable, the eigenvalues are indexed by partitions λ, and the eigenvalue corresponding to the partition λ is the cyclotomic polynomial

$$\mathcal{Q}(\lambda) = \prod (q^{\lambda_i} - 1).$$

In other words, we have a direct sum decomposition

$$K(\mathfrak{M})(q) = \bigoplus_{\lambda} K^{\lambda}(\mathfrak{M})$$

into subspaces invariant under $I^{\circ,\mathrm{ss}}$, and $I^{\circ,\mathrm{ss}}|_{K^{\lambda}(\mathfrak{M})}$ is multiplication by $\mathcal{Q}(\lambda)$.

The same theorem holds for the operator I^{ss} in the context of strict algebroids. We also prove a stronger version avoiding denominators divisible by $(q-1)$, but this version only works for algebroids.

The proof of this theorem is a generalization of the above sample calculation for the stack of rank 2 vector bundles. One goal of § 1 is to set up the necessary notation.

THEOREM 2 (Graded structure of $K(\mathfrak{M})$). *There is a direct sum decomposition*

$$K(\mathfrak{M}) = \bigoplus_{r \geqslant 0} K^{r}(\mathfrak{M}) \tag{2}$$

such that

$$K^{r}(\mathfrak{M})(q) = \bigoplus_{\mathrm{ord}_{q=1} \mathcal{Q}(\lambda) = r} K^{\lambda}(\mathfrak{M}).$$

Moreover, the commutative product is graded with respect to (2).

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Again, the same theorem holds in the context of strict algebroids.

The fact about the gradedness of the commutative product is expected from the fact that the semi-simple inertia respects the commutative product (it follows from this fact over $\mathbb{Q}(q)$, but is true over $\mathbb{Q}$).

Geometrically, the descending filtration $K^{\geqslant r}(\mathfrak{M})$ induced by the grading (2) can be described as follows: $K^{\geqslant r}(\mathfrak{M})$ is the $\mathbb{Q}$-span of all stack functions $[X \to \mathfrak{M}]$, for which the algebra of global sections $\Gamma(X, A_X)$ admits at least r orthogonal non-zero central idempotents, where A_X is the algebra of the algebroid (X, A_X).

The direct summands $K^r(\mathfrak{M})$ are the common eigenspaces of the family of commuting operators $(E_n)_{n\geqslant 0}$, where $E_n(X)$ is the stack of decompositions of $1 \in A_X$ into a sum of n orthogonal labelled idempotents. The eigenvalues of the operators E_n are integers, and the whole family of operators (E_n) is diagonalizable over $\mathbb{Q}$. The proof of this fact proceeds by proving that the (E_n) preserve the descending filtration described geometrically above and have distinct integer diagonal entries.

It turns out that the ascending filtration $K^{\leqslant n}(\mathfrak{M})$ associated with the grading in the above theorem can be described as

$$K^{\leqslant n}(\mathfrak{M}) = \ker E_{n+1}.$$

Let us also point out that

$$K^0(\mathfrak{M}) = K(\mathrm{DM}),$$

where $K(\mathrm{DM})$ is the Grothendieck $\mathbb{Q}$-algebra of Deligne–Mumford stacks (see § 2). For this reason, $K(\mathrm{DM})$ is a more natural ring of scalars than $K(\mathrm{Var})$, and we will use $K(\mathrm{DM})$ throughout. (On the other hand, in the context of strict algebroids, we have $K^0(\mathfrak{M}) = K(\mathrm{Var})$.)

If we denote by $\pi_r : K(\mathfrak{M}) \to K(\mathfrak{M})$ the projection operator onto the summand $K^r(\mathfrak{M})$ and form the generating series $\pi_t = \sum_{r\geqslant 0} \pi_r t^r$, then we have

$$\pi_t = \sum_{n\geqslant 0} \binom{t}{n} E_n. \tag{3}$$

All the above results could be proved for pairs (X, A) of algebraic stacks X endowed with finite type algebras A, instead of algebroids or strict algebroids. One simply replaces $I^{\circ,\mathrm{ss}}$ by $A^{\times,\mathrm{ss}}$.

Theorem 3 (Filtered nature of the Hall algebra). *The Hall product is filtered with respect to the filtration $K^{\leqslant r}(\mathfrak{M})$ induced by the grading (2). Moreover, for the associated graded algebra, we have*

$$\mathrm{gr}(K(\mathfrak{M}), *) = (K(\mathfrak{M}), \cdot).$$

In other words, if $x \in K^{\leqslant r}(\mathfrak{M})$ and $y \in K^{\leqslant s}(\mathfrak{M})$, then $x * y \in K^{\leqslant r+s}(\mathfrak{M})$, and

$$x * y \equiv x \cdot y \mod K^{\leqslant r+s-1}(\mathfrak{M}).$$

The proof of this theorem uses not much more than some simple combinatorics involving relabelling of direct sum decompositions, and compatibilities between direct sum decompositions of short exact sequences and splittings of short exact sequences.

The theorem implies that the one parameter family of algebras $(\mathscr{K}(\mathfrak{M}), *)$ given by the Rees construction

$$\mathscr{K}(\mathfrak{M}) = \bigoplus_{n\geqslant 0} t^n K^{\leqslant n}(\mathfrak{M})$$

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is a deformation quantization of (i.e., a one-parameter flat family of algebras with special fibre) the commutative algebra $(K(\mathfrak{M}),\cdot)$. Hence, the graded algebra $(K(\mathfrak{M}),\cdot)$ inherits a Poisson bracket $\{,\}$ of degree -1. In particular, $K^1(\mathfrak{M})$ is a Lie algebra, and it turns out that the Lie bracket on $K^1(\mathfrak{M})$ is equal to the commutator bracket associated to $*$.

The filtered algebra $(K(\mathfrak{M}),*)$ is a *filtered quantization* of the Poisson algebra $(K(\mathfrak{M}),\cdot,\{,\})$, as defined in [Sch16]. Of course, we have more structure here, as our filtration comes from a natural grading, but it is not clear to us what this additional structure on our filtered quantization is useful for.

Following Joyce [Joy07a], we call $K^1(\mathfrak{M})$ the Lie algebra of *virtually indecomposable* elements of $K(\mathfrak{M})$, with the notation $K^{\mathrm{vir}}(\mathfrak{M})=K^1(\mathfrak{M})$.

We denote the projection onto $K^{\mathrm{vir}}(\mathfrak{M})$ by π^{vir}. With this notation, we have, as a special case of (3),

$$\pi^{\mathrm{vir}}=\sum_{n>0}\frac{(-1)^{n+1}}{n}E_n.$$

In terms of eigenspaces of semi-simple inertia, we have

$$K^{\mathrm{vir}}(\mathfrak{M})(q)=K^{(q-1)}(\mathfrak{M})\oplus K^{(q^2-1)}(\mathfrak{M})\oplus K^{(q^3-1)}(\mathfrak{M})\oplus\dots.$$

THEOREM 4 (Hall algebra logarithms). *Let $\mathfrak{N}\subset\mathfrak{M}$ be a 'small enough' substack, closed under extensions and direct summands, and not intersecting* $\operatorname{Spec}k\xrightarrow{0}\mathfrak{M}$. *Then*

$$\varepsilon_t[\mathfrak{N}]=\sum_{n\geqslant 0}\binom{t}{n}[\mathfrak{N}]^{*n}\in\hat{\mathscr{K}}(\mathfrak{M})_+.$$

In particular, the $*$-logarithm

$$\varepsilon[\mathfrak{N}]=\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n}[\mathfrak{N}]^{*n}\in\hat{K}^{\mathrm{vir}}(\mathfrak{M})_+$$

is virtually indecomposable.

For the precise definition of 'small enough', see § 3.3. For example, if $\mathfrak{M}=\mathfrak{Coh}_Y$ for a curve Y, we could take $\mathfrak{N}$ to consist of all non-zero semi-stable vector bundles of a fixed slope. Since $\mathfrak{N}$ is typically not of finite type, to make sense of $[\mathfrak{N}]$ we have to pass to a certain completion $\hat{K}(\mathfrak{M})_+$ of $K(\mathfrak{M})$. See § 3.3 for details.

THEOREM 5 (No poles). *Let $K(\mathrm{St})$ be the Grothendieck $\mathbb{Q}$-algebra of algebraic stacks, modulo all bundle relations with special structure group (inert or not). Consider the map*

$$\int:K(\mathfrak{M})\longrightarrow K(\mathrm{St})$$
$$[(X,A)\to(\mathfrak{M},\mathfrak{A})]\longmapsto[X],$$

which forgets the structure map to $(\mathfrak{M},\mathfrak{A})$, and the algebroid structure over the stack X. If $x\in K^{\leqslant r}(\mathfrak{M})$, then $(q-1)^r\int x$ is a regular element of $K(\mathrm{St})$; i.e., under the identification

$$K(\mathrm{St})=K(\mathrm{Var})\left[\frac{1}{q},\frac{1}{q-1},\frac{1}{q^2-1},\dots\right],$$

it can be written with a denominator that does not vanish at $q=1$.

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Moreover, suppose we have a grading monoid Γ for $\mathfrak{M}$:

$$\mathfrak{M} = \coprod_{\gamma\in\Gamma} \mathfrak{M}_\gamma.$$

We say that x has 'degree' γ if $x \in \mathfrak{M}_\gamma$. We need Γ to be endowed with a $\mathbb{Z}$-valued bilinear form χ such that for every object x in $\mathfrak{M}_\gamma$ and y in $\mathfrak{M}_\beta$:

(i) every extension of y by x is in $\mathfrak{M}_{\beta+\gamma}$;

(ii) the stack of extensions of y by x is the quotient of a vector space E_1 by another vector space E_0, acting trivially, such that $\dim E_0 - \dim E_1 = \chi(\beta,\gamma)$.

For the precise assumptions, see § 4.2. They are satisfied if $\mathfrak{M} = \mathfrak{Rep}_Q$, or if $\mathfrak{M} = \mathfrak{Coh}_Y$ and Y is a smooth curve, or, more generally, if $\mathfrak{M}$ is *hereditary*.

Then we have a commutative diagram as follows.

$$\begin{array}{ccc}
(\mathscr{K}(\mathfrak{M}),*) & \xrightarrow{t\longmapsto 0} & (K(\mathfrak{M}),\cdot\,) \\
{\scriptstyle\int}\Big\downarrow{\scriptstyle t\mapsto(q-1)} & & {\scriptstyle\int}\Big\downarrow{\scriptstyle q\mapsto 1} \\
K(\mathrm{St})_{\mathrm{reg}}[\Gamma] & \xrightarrow{q\longmapsto 1} & K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma]
\end{array}$$

Here we use the Γ-graded integral, which is essentially a generating function, indexed by Γ, of the integrals of the components of degree $\gamma\in\Gamma$ of a given stack function.

The upper horizontal arrow in this diagram is the specialization map, which exists because of Theorem 3. The left vertical arrow exists by the 'no poles' theorem (Theorem 5). It is a standard fact that this map is an algebra morphism, i.e., it respects the $*$-product, if the target $K(\mathrm{St})_{\mathrm{reg}}[\Gamma]$ is endowed with its q-deformed product twisted by χ. It is a formal consequence of the commutativity of this diagram that the right vertical map is a morphism of Poisson algebras if the target $K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma]$ is endowed with its bracket induced by χ. We call the right vertical map the *semi-classical motivic integral.*

In particular, we deduce that

$$\begin{aligned}\int_{q=1} : K^{\mathrm{vir}}(\mathfrak{M}) &\longrightarrow K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma]\\
x_\gamma &\longmapsto \left((q-1)\int x_\gamma\right)\bigg|_{q=1} u^\gamma = \mathrm{res}_{q=1}\left(\int x_\gamma\right)u^\gamma \end{aligned} \tag{4}$$

is a morphism of Lie algebras. The target $K(\mathrm{St})_{\mathrm{reg}}/(q-1)$ of the semi-classical motivic integral is equal to $K(\mathrm{Var})/((q-1)+\mathrm{Ann}(q-1))$.

The proof of the no poles theorem combines the above results about the diagonalizability of $I^{\circ,\mathrm{ss}}$, especially in its form avoiding denominators divisible by $(q-1)$, with the result that for an algebroid (X,A), the stack $I_X^{\circ,\mathrm{ss}}$ has regular motivic weight, i.e.,

$$[I_X^{\circ,\mathrm{ss}}] \in K(\mathrm{St})_{\mathrm{reg}}.$$

We think of this as a motivic version of Burnside's lemma. The more natural-looking conjecture that for an algebraic stack X, the motivic weight of I_X is contained in $K(\mathrm{Var}) \subset K(\mathrm{St})$ is most likely false.

Discussion. To produce counting invariants for $\mathfrak{M}$, we need to look for subcategories $\mathfrak{N}\subset\mathfrak{M}$, to which we can apply Theorem 4, giving us virtually indecomposable elements $\varepsilon[\mathfrak{N}]$, to which we can apply the integral (4), yielding generating functions with coefficients in $K(\mathrm{St})_{\mathrm{reg}}/(q-1)$. (See Remark 4.12 for details.) We can apply the Euler characteristic to these elements of

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$K(\mathrm{St})_{\mathrm{reg}}/(q-1)$ to obtain rational numbers. In the hereditary case, the fact that (4) is a morphism of Lie algebras gives the relations among generating functions one is interested in. This leads to wall-crossing formulas, and other results. For details we refer to the works of Joyce, Joyce–Song, and many others.

To deal with the Calabi–Yau 3 case, one needs to insert the correct motivic vanishing cycle weights, to define the integral. This is done by Joyce and Song in [JS12]. We leave it to future work to supply the details in our context.

The work of Joyce on configurations in abelian categories contains results which correspond to ours, but his definitions are more ad hoc. In fact, one reason for writing the present article was to give a more conceptual treatment of Joyce's results. We do not prove that our notion of 'virtual indecomposable object' coincides with Joyce's (except for in the case of $\mathfrak{M} = \mathfrak{Vect}$, see the appendix), but instead prove that our notion has the same properties as Joyce's and is just as useful. (Of course, the counting invariants we obtain are the same as the ones obtained by Joyce: the formula $\int_{q=1} \log_*(1 + [\mathfrak{M}])$, see (39), does not involve explicitly the definition of virtually indecomposable object.)

We think of the Lie algebra $K^{\mathrm{vir}}(\mathfrak{M})$ as an analogue of the Lie algebra of primitive elements in a cocommutative Hopf algebra. In fact, one may ask whether $(K(\mathfrak{M}), *)$ is equal to the universal enveloping algebra of the Lie algebra $K^{\mathrm{vir}}(\mathfrak{M})$. To deduce such a statement from structure theorems for Hopf algebras, one would need to enhance $K(\mathfrak{M})$ to a cocommutative Hopf algebra. We have not been able to construct the necessary coproduct. We view the family of operators (E_n) as something of a replacement. It lets us prove at least some of the results expected of a cocommutative Hopf algebra, in particular Theorems 3 and 4.

1. Linear algebraic stacks and algebroids

1.1 Algebraic stacks

Let us briefly summarize our conventions about algebraic stacks.

We choose a noetherian base ring R (commutative and with unit), and we fix our base category $\mathscr{S}$ to be the category of R-schemes, endowed with the étale topology. Over $\mathscr{S}$ we have a canonical sheaf of R-algebras $\mathscr{O}_{\mathscr{S}}$; it is represented by $\mathbb{A}^1 = \mathbb{A}^1_{\operatorname{Spec} R}$ and called the *structure sheaf*.

We will assume our algebraic stacks to be locally of finite type. Thus, an *algebraic stack* is a stack over the site $\mathscr{S}$ that admits a presentation by a groupoid $X_1 \rightrightarrows X_0$, where X_0 and X_1 are algebraic spaces, *locally of finite type* over R, the source and target morphisms $s, t : X_1 \to X_0$ are smooth, and the diagonal $X_1 \to X_0 \times X_0$ is of finite type. In fact, all algebraic stacks we encounter will have affine diagonal.

By a *stratification* of an algebraic stack X, we mean a morphism of algebraic stacks $X' \to X$ that is a surjective monomorphism and that admits a finite decomposition $X' = \coprod_i X_i$ such that every $X_i \to X$ is a locally closed embedding of algebraic stacks.

If G is an algebraic group acting on the algebraic space X, we will denote the quotient stack by X/G, because we fear the more common notation $[X/G]$ would lead to confusion with the notation for elements of various K-groups of schemes and stacks.

Suppose $G \to X$ is a relative group scheme over the stack X. The *connected component* of G, denoted by G°, is the subsheaf of G defined by requiring a section $g \in G(S)$ to factor through $G^\circ(S)$ if and only if for all points (equivalently geometric points) s of S, we have $g(s) \in G^\circ_s$. If $G \to X$ is smooth, the connected component $G^\circ \subset G$ is represented by an open substack of

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G, which is a smooth group scheme with geometrically connected fibres over X. (See [SGA3, Exposé VI$_{\rm B}$, Théoreme 3.10].)

If the inertia stack I_X of an algebraic stack X is smooth over X, the connected component I_X° is an algebraic stack. We can apply the *rigidification construction* (see, for example, [AOV08, Appendix]) to $I_X^\circ \subset I_X$ and obtain a (uniquely determined) Deligne–Mumford stack $\overline{X}$, together with a morphism $X \to \overline{X}$, making X a *connected gerbe* over $\overline{X}$ (which means that the relative inertia of X over $\overline{X}$ has connected fibres). The structure morphism $X \to \overline{X}$ is smooth.

A gerbe $X \to \overline{X}$ is an *isotrivial gerbe* if it admits a section over a finite étale $\overline{X}$-stack. If $X \to \overline{X}$ is a smooth gerbe over a Deligne–Mumford stack, there exists a stratification $\overline{X}' \to \overline{X}$ such that the restriction of the gerbe X to $\overline{X}'$ is isotrivial. (This follows from the fact that a quasi-finite morphism of Deligne–Mumford stacks is generically finite. This, in turn, follows from Zariski's main theorem [LMB00, § 16].)

Let us also remark that every Deligne–Mumford stack admits a stratification by *integral normal* Deligne–Mumford stacks, although we do not use this fact.

Sheaves on algebraic stacks. We need to clarify the notions of vector bundle, coherent sheaf, and representable sheaf of $\mathscr{O}_X$-modules, and how they relate to each other.

An algebraic stack X is a fibred category $X \to \mathscr{S}$. The category X inherits a topology from $\mathscr{S}$, called the étale topology, and X endowed with this topology is the *big étale site* of X. Sheaves over X are by definition sheaves on this big étale site. For example, $\mathscr{O}_{\mathscr{S}}$ induces a sheaf of R-algebras on X, which is denoted by $\mathscr{O}_X$ and called the *structure sheaf* of X. It is represented by $\mathbb{A}^1_X$.

A sheaf $\mathscr{F}$ over X induces, for every object x of X lying over the object U of $\mathscr{S}$, a sheaf on the usual (small) étale site $U_{\text{ét}}$ of the scheme U, denoted $\mathscr{F}_U$. Moreover, for every morphism $\alpha: y \to x$ lying over $f: V \to U$, we obtain a morphism of sheaves $\alpha^*: f^{-1}(\mathscr{F}_U) \to \mathscr{F}_V$. (The α^* satisfy an obvious cocycle condition and the condition that they are isomorphisms if f is étale.) For example, the structure sheaf $\mathscr{O}_X$ induces the structure sheaf on $U_{\text{ét}}$, for every such x/U. The data of the small étale sheaves $\mathscr{F}_U$, together with the compatibility morphisms α^*, satisfying the two parenthetical conditions, is equivalent to the data defining $\mathscr{F}$ (see [SGA4, Exp. IV, 4.10]). The functor $\mathscr{F} \mapsto \mathscr{F}_U$ is the sheaf pullback morphism of a morphism of sites $U_{\text{ét}} \to X$, from the small étale site of U to the big étale site of X. In particular, $\mathscr{F} \mapsto \mathscr{F}_U$ is exact. Both $U_{\text{ét}}$ and X are ringed sites, and $\mathscr{F} \mapsto \mathscr{F}_U$ is also the sheaf of modules pullback of the morphism of ringed sites $U_{\text{ét}} \to X$. Therefore, the functor $\mathscr{F} \to \mathscr{F}_U$ is also exact when considered as a functor from the category of big sheaves of $\mathscr{O}_X$-modules to the category of small sheaves of $\mathscr{O}_{U_{\text{ét}}}$-modules.

If $\mathscr{F}$ and $\mathscr{G}$ are sheaves of $\mathscr{O}_X$-modules, then $\mathscr{H}om(\mathscr{F},\mathscr{G})$ is again a sheaf of $\mathscr{O}_X$-modules. In particular, for a sheaf of $\mathscr{O}_X$-modules, we have the dual $\mathscr{F}^\vee = \mathscr{H}om(\mathscr{F},\mathscr{O}_X)$.

Note that, in general, the natural homomorphism $\mathscr{H}om(\mathscr{F},\mathscr{G})_U \to \mathscr{H}om(\mathscr{F}_U,\mathscr{G}_U)$ is not an isomorphism; see Example 1.6.

Coherent sheaves. A sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is *locally coherent* if for every x/U, the sheaf $\mathscr{F}_U$ is a coherent sheaf of $\mathscr{O}_{U_{\text{ét}}}$-modules. (This terminology is inspired by [Sta15, Tag 06WJ].) It is *cartesian* if all compatibility morphisms $\alpha^*: f^*\mathscr{F}_U \to \mathscr{F}_V$ are isomorphisms of sheaves of $\mathscr{O}_{V_{\text{ét}}}$-modules. A sheaf that is both locally coherent and cartesian is *coherent*. The coherent sheaf $\mathscr{F}$ is said to be *locally free coherent* if every $\mathscr{F}_U$ is locally free. (This is equivalent to $\mathscr{F}$ being locally free.)

For example, a groupoid presentation $X_1 \rightrightarrows X_0$ of X and a coherent sheaf $\mathscr{F}_0$ on X_0, together with an isomorphism $s^*\mathscr{F}_0 \to t^*\mathscr{F}_0$, satisfying the usual cocycle condition on $X_2 = X_1 \times_{X_0} X_1$, give rise to a coherent sheaf $\mathscr{F}$ on X. If $\mathscr{F}_0$ is locally free, then $\mathscr{F}$ is locally free.

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A *vector bundle* over X is a representable morphism $E \to X$ endowed with an addition operation and a compatible $\mathbb{G}_m$-action such that the pullback of $E \to X$ to any scheme $U \to X$ is a vector bundle on U.

The sheaf of sections of a vector bundle over X is a coherent sheaf. In fact, the notions of vector bundle and locally free coherent sheaf are equivalent, and we will use them interchangeably. (The inverse functor is given by $\mathscr{E} \mapsto \operatorname{Spec}_X \operatorname{Sym}_{\mathscr{O}_X} \mathscr{E}^\vee$.) The cokernel of a homomorphism of vector bundles is coherent. In fact, every cokernel of a homomorphism of coherent sheaves is coherent.

If the cokernel of a homomorphism of vector bundles is locally free, we call the homomorphism a *strict* homomorphism of vector bundles. For a strict homomorphism of vector bundles, the image and the kernel, as well as the cokernel, are locally free.

A *strict monomorphism* of vector bundles is a strict homomorphism whose kernel is zero. A homomorphism of vector bundles is a strict monomorphism/an epimorphism if and only if over every geometric point of X, the induced linear map is injective/surjective. A homomorphism of vector bundles, which is a monomorphism of sheaves, is a strict monomorphism of bundles.

Let $\varphi : E \to F$ be a homomorphism of vector bundles over the algebraic stack X. The flattening stratification $X' \to X$ of $\operatorname{cok}\varphi$ serves also as a *strictening stratification* for φ. This means that an object of $X(S)$ lifts to $X'(S)$ if and only if φ_S is strict.

In general, the kernel (in the category of big sheaves of $\mathscr{O}_X$-modules) of a homomorphism of vector bundles is locally coherent, but not coherent.

By [Sta15, Tag 06WK], a sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ is coherent if and only if there exists a smooth covering $X_i \to X$ of X by finite type affine schemes X_i such that for every i, the restriction $\mathscr{F}_i$ of $\mathscr{F}$ to the big étale site of X_i is isomorphic to the cokernel of a homomorphism of vector bundles.

PROPOSITION 1.1. *Suppose that $\mathscr{F}$ is a coherent sheaf on the algebraic stack X. Then:*

(i) *for every x/U, we have $(\mathscr{F}^\vee)_U = (\mathscr{F}_U)^\vee$;*
(ii) *$\mathscr{F}^\vee$ is locally coherent;*
(iii) *$\mathscr{F}^\vee$ is represented by a an algebraic stack, which is of finite type and affine over X, namely, $\operatorname{Spec}_X \operatorname{Sym}_{\mathscr{O}_X} \mathscr{F}$;*
(iv) *the canonical homomorphism $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism of sheaves of $\mathscr{O}_X$-modules;*
(v) *if $\mathscr{F}$ is locally free coherent, then $\mathscr{F}^\vee$ is a vector bundle.*

Moreover, the functor $\mathscr{F} \mapsto \mathscr{F}^\vee$ is a fully faithful functor from the category of coherent sheaves to the category of locally coherent sheaves of $\mathscr{O}_X$-modules. It maps right exact sequences of coherent sheaves to left exact sequences of locally coherent sheaves.

Proof. The canonical homomorphism $\mathscr{H}om(\mathscr{F},\mathscr{G})_U \to \mathscr{H}om(\mathscr{F}_U,\mathscr{G}_U)$ is an isomorphism for all U in $\mathscr{S}$, if $\mathscr{F}$ is cartesian. This implies the first claim.

The second claim follows from the first (see also [Sta15, Tag 06WM]).

For the third claim, see [LMB00, (14.2.6)].

For the fourth claim, notice that both $\mathscr{F}^\vee$ and $\mathscr{O}_X$ are representable and affine over X, namely, $\mathscr{F}^\vee = \operatorname{Spec}_X \operatorname{Sym}_{\mathscr{O}_X} \mathscr{F}$ and $\mathscr{O}_X = \mathbb{A}^1_X$. They are also both endowed with $\mathbb{G}_m$-actions, via scalar multiplication. We consider the big sheaf of $\mathbb{G}_m$-equivariant X-morphisms from $\operatorname{Spec}_X \operatorname{Sym}_{\mathscr{O}_X} \mathscr{F}$ to $\mathbb{A}^1_X$, denoted by $\underline{\operatorname{Hom}}_{\mathbb{G}_m}(\mathscr{F}^\vee, \mathbb{A}^1)$. The sections of $\underline{\operatorname{Hom}}_{\mathbb{G}_m}(\mathscr{F}^\vee, \mathbb{A}^1)$ over $S \to X$ are the $\mathbb{G}_m$-equivariant S-morphisms $\operatorname{Spec}_S \operatorname{Sym}_{\mathscr{O}_S} \mathscr{F}_S \to \mathbb{A}^1_S$ or, equivalently, the morphisms of sheaves

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of $\mathscr{O}_S$-algebras $\mathscr{O}_S[t] \to \operatorname{Sym}_{\mathscr{O}_S} \mathscr{F}_S$ of degree 1. These are the same as the sections of $\mathscr{F}_S$ over S. This proves that $\underline{\operatorname{Hom}}_{\mathbb{G}_m}(\mathscr{F}^\vee, \mathbb{A}^1) = \mathscr{F}$.

On the other hand, the sections of $\mathscr{H}om(\mathscr{F}^\vee, \mathscr{O})$ over $S \to X$ are the $\mathscr{O}$-linear homomorphisms $\operatorname{Spec}_S \operatorname{Sym}_{\mathscr{O}_S} \mathscr{F}_S \to \mathbb{A}^1_S$, which are, in particular, $\mathbb{G}_m$-equivariant. Thus we have a natural map $\mathscr{H}om(F^\vee, \mathscr{O}) \to \underline{\operatorname{Hom}}_{\mathbb{G}_m}(\mathscr{F}^\vee, \mathbb{A}^1)$. In total, we have a canonical map

$$\mathscr{F}^{\vee\vee} = \mathscr{H}om(\mathscr{F}^\vee, \mathscr{O}) \longrightarrow \underline{\operatorname{Hom}}_{\mathbb{G}_m}(\mathscr{F}^\vee, \mathbb{A}^1) = \mathscr{F},$$

which is inverse to the tautological map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$.

The fifth claim is clear.

The 'moreover' follows from the fact that we can reconstruct $\mathscr{F}$ from $\mathscr{F}^\vee = \operatorname{Spec}_X \operatorname{Sym}_{\mathscr{O}_X} \mathscr{F}$, as the degree 1 part of the graded sheaf of $\mathscr{O}_X$-modules $\pi_*(\mathscr{O}_{\mathscr{F}^\vee})$, where $\pi : \mathscr{F}^\vee \to X$ is the projection morphism. □

Representable sheaves of modules. If $\varphi : E \to F$ is a homomorphism of vector bundles over X, then $\ker \varphi$, constructed in the category of big sheaves, is a representable sheaf of $\mathscr{O}_X$-modules. In fact, $\ker \varphi$ is equal to the fibred product of stacks as shown in the following diagram.

$$\begin{array}{ccc} \ker \varphi & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle 0} \\ E & \xrightarrow{\varphi} & F \end{array}$$

Sheaves such as $\ker \varphi$ belong to a class of $\mathscr{O}_X$-modules *dual* to coherent sheaves.

Proposition 1.2. *Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. The following are equivalent.*

(i) *There exists a coherent sheaf $\mathscr{N}$ such that $\mathscr{F}$ is isomorphic to $\mathscr{N}^\vee$.*

(ii) *There exists a smooth cover $X_i \to X$ of X by finite type affine schemes X_i such that, for every i, the restriction $\mathscr{F}_i$ of $\mathscr{F}$ to the big étale site of X_i is isomorphic to the kernel of a homomorphism of vector bundles over X_i.*

Proof. The fact that (i) implies (ii) follows from the results proved above. So let us indicate how to prove that (ii) implies (i).

Let us first assume that $\mathscr{F}$ is isomorphic to the kernel of a homomorphism of vector bundles $E_0 \to E_1$. One checks that $\mathscr{F}$ is then represented by $\operatorname{Spec}_X \operatorname{Sym}_{\mathscr{O}_X} \operatorname{cok}(E_1^\vee \to E_0^\vee)$. Thus, $\mathscr{F}$ is isomorphic to the dual of the coherent sheaf $\operatorname{cok}(E_1^\vee \to E_0^\vee)$.

Now suppose that $\mathscr{F}$ is locally isomorphic to the kernel of a homomorphism of vector bundles. It suffices to prove that $\mathscr{F}^\vee$ is coherent and that $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism. Both claims can be checked locally and are true for duals of coherent sheaves. □

Definition 1.3. We say a sheaf of $\mathscr{O}_X$-modules is *locally coherent representable* if any of the two equivalent conditions of Proposition 1.2 is satisfied. The terminology is justified by Proposition 1.4 below.

In other words, the category of locally coherent representable sheaves over X is the essential image of the fully faithful functor mentioned in Proposition 1.1. We therefore have an equivalence of categories

$$\begin{array}{c} (\text{coh. sheaves over } X) \longrightarrow (\text{loc. coh. repr. sheaves over } X) \\ \mathscr{F} \longmapsto \mathscr{F}^\vee. \end{array} \tag{5}$$

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The following proposition summarizes facts about locally coherent representable sheaves, which all follow easily from facts mentioned above.

PROPOSITION 1.4. *Let $\mathscr{F}$ be a locally coherent representable sheaf over the algebraic stack X. Then:*

(i) *the sheaf $\mathscr{F}$ is locally coherent;*
(ii) *the sheaf $\mathscr{F}^\vee$ is coherent;*
(iii) *the canonical homomorphism $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism of sheaves of $\mathscr{O}_X$-modules;*
(iv) *the sheaf $\mathscr{F}$ is representable by an algebraic stack $F \to X$, which is of finite type and affine over X;*
(v) *in fact, $\mathscr{F} = \operatorname{Spec}_X \operatorname{Sym}_{\mathscr{O}_X} \mathscr{F}^\vee$.*

Moreover, the functor $\mathscr{F} \mapsto \mathscr{F}^\vee$ is an essential inverse to the functor (5). It maps left exact sequences of locally coherent representable sheaves to right exact sequences of coherent sheaves.

PROPOSITION 1.5. *Let $\mathscr{F}$ be a locally coherent representable sheaf over the finite type algebraic stack X. There is a unique stratification $X' \to X$ with the property that an X-scheme S factors through X' if and only if $\mathscr{F}|_S$ is a vector bundle. More precisely, $X' = \coprod_{n \geqslant 0} X_n$, and $X_n \to X$ is a locally closed immersion of algebraic stacks with the property that $S \to X$ factors through X_n if and only if $\mathscr{F}|_S$ is a vector bundle of rank n.*

Proof. The sought-after stratification is the flattening stratification of the coherent sheaf $\mathscr{F}^\vee$. □

Example 1.6. Consider $X = \mathbb{A}^1$, with coordinate t, and let $\mathscr{C}$ be the cokernel of the homomorphism of vector bundles $t : \mathbb{A}^1_X \to \mathbb{A}^1_X$. It is the skyscraper sheaf of the origin, considered as a coherent sheaf on X and extended to a big sheaf over X in the usual way. The sheaf $\mathscr{C}$ is an example of a coherent sheaf which is not representable.

Let $\mathscr{K}$ be the kernel of $t : \mathbb{A}^1_X \to \mathbb{A}^1_X$. This is locally coherent representable, but not cartesian, and hence not coherent.

Note that $\mathscr{C}^\vee = \mathscr{K}$. This shows that $\mathscr{F}^\vee$ may not be coherent, even if $\mathscr{F}$ is.

Note also that $\mathscr{K}^\vee = \mathscr{C}$, which shows that $\mathscr{F}^\vee$ may not be representable, even if $\mathscr{F}$ is.

Finally, note that $(\mathscr{K}_X)^\vee = 0^\vee = 0$, but $(\mathscr{K}^\vee)_X = \mathscr{C}_X$ is the structure sheaf of the origin in X, considered as a skyscraper sheaf on $X_{\text{ét}}$, which is not zero. This gives an example where $(\mathscr{F}^\vee)_U \neq (\mathscr{F}_U)^\vee$.

Remark 1.7. Of course, the category of coherent sheaves on an algebraic stack X has kernels and internal homs, but they do not agree with those in the category of big sheaves, which we considered above. It is therefore important to specify the context when dealing with kernels or duals in the category of coherent sheaves. Unless specified otherwise, we will always consider sheaves of $\mathscr{O}_X$-modules as big sheaves.

1.2 Linear algebraic stacks

We will review the definition of linear algebraic stacks and some basic constructions. For definitions and basic properties of fibred categories we refer the reader to [SGA1, Exposé VI]. The material here is presumably known, but we could not find a suitable reference.

Suppose $\mathfrak{X} \to \mathscr{S}$ is a category over $\mathscr{S}$. We write $\mathfrak{X}(S)$ for the fibre of $\mathfrak{X}$ over the object S of $\mathscr{S}$. If $f : S' \to S$ is a morphism in $\mathscr{S}$, and $x' \in \mathfrak{X}(S')$ and $x \in \mathfrak{X}(S)$ are $\mathfrak{X}$-objects lying over S' and S, respectively, we write $\operatorname{Hom}_f(x', x)$ for the set of morphisms from x' to x in $\mathfrak{X}$, lying over f. For $S' = S$ and $f = \mathrm{id}_S$, we write $\operatorname{Hom}_S(x', x)$.

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Recall that a morphism $\alpha : x' \to x$ lying over $f : S' \to S$ is *cartesian* if for every object x'' of $\mathfrak{X}(S)$, composition with α induces a bijection $\mathrm{Hom}_S(x'', x') \xrightarrow{\simeq} \mathrm{Hom}_f(x'', x)$. Recall further that $\mathfrak{X} \to \mathscr{S}$ is a *fibred category* if every composition of cartesian morphisms is cartesian and if for every $f : S' \to S$ in $\mathscr{S}$ and every x over S, there exists a cartesian morphism over f with target x. A *cartesian functor* between categories over $\mathscr{S}$ is one that preserves cartesian morphisms.

If $\mathfrak{X}$ is a fibred category over $\mathscr{S}$, the subcategory of $\mathfrak{X}$ consisting of the same objects and all cartesian morphisms is a category fibred in groupoids over $\mathscr{S}$. We denote it by $\mathfrak{X}_{\mathrm{cfg}}$ and call it the *underlying category fibred in groupoids*.

Definition 1.8. A category $\mathfrak{X}$ over $\mathscr{S}$ is an *$\mathscr{O}$-linear category over $\mathscr{S}$* if for every $f : S' \to S$ in $\mathscr{S}$ and all $x' \in \mathfrak{X}(S')$, $x \in \mathfrak{X}(S)$, the set $\mathrm{Hom}_f(x', x)$ is endowed with the structure of an $\mathscr{O}(S')$-module in such a way that for every pair of morphisms $g : S'' \to S'$, $f : S' \to S$ and every triple of objects $x'' \in \mathfrak{X}(S'')$, $x' \in \mathfrak{X}(S')$, $x \in \mathfrak{X}(S)$, the composition

$$\mathrm{Hom}_f(x', x) \times \mathrm{Hom}_g(x'', x') \longrightarrow \mathrm{Hom}_{f\circ g}(x'', x)$$

is $\mathscr{O}(S')$-bilinear.

An *$\mathscr{O}$-linear functor* $F : \mathfrak{X} \to \mathfrak{Y}$ between $\mathscr{O}$-linear categories is a functor of categories over $\mathscr{S}$ such that for every $f : S' \to S$ and all $x' \in \mathfrak{X}(S')$, $x \in \mathfrak{X}(S)$, the map $\mathrm{Hom}_f(x', x) \to \mathrm{Hom}_f(F(x'), F(x))$ is $\mathscr{O}(S')$-linear.

Assume given an $\mathscr{O}$-linear fibred category $\mathfrak{X}$ over $\mathscr{S}$. Pullback in $\mathfrak{X}$ is $\mathscr{O}$-linear, i.e., if $f : S' \to S$ is a morphism in $\mathscr{S}$ and $x, y \in \mathfrak{X}(S)$ are objects with pullbacks $x', y' \in \mathfrak{X}(S')$, the pullback map $f^* : \mathrm{Hom}_S(x, y) \to \mathrm{Hom}_{S'}(x', y')$ is $\mathscr{O}(S)$-linear.

So, if we fix objects $x, y \in \mathfrak{X}(S)$, the presheaf $\underline{\mathrm{Hom}}_S(x, y)$ over the usual small étale site of S, defined by $\underline{\mathrm{Hom}}_S(x, y)(T) = \mathrm{Hom}_T(x|_T, y|_T)$ for every étale $T \to S$, is a presheaf of $\mathscr{O}_{S_{\text{ét}}}$-modules. Moreover, for *any* morphism $f : S' \to S$ in $\mathscr{S}$, we have a natural homomorphism of presheaves of $\mathscr{O}_S$-modules $\underline{\mathrm{Hom}}_S(x, y) \to f_* \underline{\mathrm{Hom}}_{S'}(x', y')$. Put together, the small presheaves $\underline{\mathrm{Hom}}_T(x, y)$, as $T \to S$ varies over the big étale site of the scheme S, form a big presheaf, which we denote by $\underline{\mathrm{Hom}}(x, y)$.

Definition 1.9. A *linear algebraic stack* is an $\mathscr{O}$-linear fibred category $\mathfrak{X}$ over $\mathscr{S}$, such that:

(i) for every object $S \in \mathscr{S}$ and every pair $x, y \in \mathfrak{X}(S)$, the presheaf $\underline{\mathrm{Hom}}(x, y)$ on the big étale site of the scheme S is a locally coherent representable sheaf of $\mathscr{O}_S$-modules;

(ii) the underlying category fibred in groupoids $\mathfrak{X}_{\mathrm{cfg}} \to \mathscr{S}$ is an algebraic stack over R (locally of finite type).

A *morphism* of linear algebraic stacks is an $\mathscr{O}$-linear cartesian functor over $\mathscr{S}$.

Remark 1.10. If $\mathfrak{X}$ is a linear algebraic stack with underling algebraic stack $X = \mathfrak{X}_{\mathrm{cfg}}$, there exists a locally coherent representable sheaf $\mathscr{H}$ over $X \times X$ that represents the sheaf over $X \times X$ whose set of sections over the pair $x, y \in X(S)$ is the $\mathscr{O}(S)$-module $\mathrm{Hom}_S(x, y)$. The sheaf $\mathscr{H}$ is the *universal sheaf of homomorphisms*. The subsheaf $\mathscr{I} \subset \mathscr{H}$ representing isomorphisms is naturally identified with X and the projection to $X \times X$ with the diagonal.

Pulling back $\mathscr{H}$ via the diagonal to X, we obtain the *universal sheaf of endomorphisms* $\mathscr{E} \to X$, which represents the sheaf whose set of sections over $x \in X(S)$ is the $\mathscr{O}(S)$-algebra $\mathrm{End}_S(x)$. Let us emphasize that $\mathscr{E} \to X$ is a representable morphism of algebraic stacks, which is at the same time a sheaf of algebras and a locally coherent representable sheaf of $\mathscr{O}_X$-modules.

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The linear algebraic stack $\mathfrak{X}$ can be reconstructed from its underlying algebraic stack X and the representable sheaf of $\mathscr{O}_{X\times X}$-algebras $\mathscr{H}$. We leave it to the reader to write down axioms for the pair $(X,\mathscr{H})$ that ensure that $(X,\mathscr{H})$ comes from a linear algebraic stack.

Examples.

Example 1.11. Let X be a projective R-scheme. The linear stack $\mathfrak{Coh}_X$ has as objects lying over the R-scheme S the coherent sheaves on $X\times S$, which are flat over S. For a morphism of R-schemes $f:S'\to S$, and $\mathscr{F}'\in\mathfrak{Coh}_X(S')$ and $\mathscr{F}\in\mathfrak{Coh}_X(S)$, we set $\operatorname{Hom}_f(\mathscr{F}',\mathscr{F})=\operatorname{Hom}_{\mathscr{O}_{X\times S'}}(\mathscr{F}', f^*\mathscr{F})$. A morphism $\mathscr{F}'\to\mathscr{F}$ in $\mathfrak{Coh}_X$ over f in $\mathscr{S}$ is cartesian if it induces an isomorphism $\mathscr{F}'\cong f^*\mathscr{F}$.

The linear stack $\mathfrak{Coh}_X$ is algebraic.

To see this, suppose $\mathscr{F}$ and $\mathscr{G}$ are coherent sheaves on $X\times S$, flat over S. The fact that $\underline{\operatorname{Hom}}(\mathscr{F},\mathscr{G})$ is a locally coherent representable sheaf of $\mathscr{O}_X$-modules follows from the fact that there exists a coherent sheaf $\mathscr{N}$ on the big étale site of S such that $\underline{\operatorname{Hom}}(\mathscr{F},\mathscr{G})=\mathscr{N}^\vee$ (see [GD67, EGA III 7.7.8, 7.7.9]). In fact, for a morphism of schemes $T\to S$, we have $\underline{\operatorname{Hom}}_T(\mathscr{F},\mathscr{G})=\pi_{T*}\mathscr{H}om(\mathscr{F}_{X\times T},\mathscr{G}_{X\times T})$. The fact that pushforward does not commute with arbitrary pullbacks means that $\underline{\operatorname{Hom}}(\mathscr{F},\mathscr{G})$ is not in general cartesian and hence not in general coherent. On the other hand, by [GD67, EGA III 7.7.8, 7.7.9], we have $\pi_{T*}\mathscr{H}om(\mathscr{F}_{X\times T},\mathscr{G}_{X\times T})=(\mathscr{N}_T)^\vee$, which proves that, indeed, $\underline{\operatorname{Hom}}(\mathscr{F},\mathscr{G})=\mathscr{N}^\vee$.

The fact that $(\mathfrak{Coh}_X)_{\mathrm{cfg}}$ is algebraic and locally of finite type is proved in [LMB00, 4.6.2.1].

Example 1.12. As a special case of the previous example, consider the case $X=\operatorname{Spec}R$. Then the linear algebraic stack $\mathfrak{Coh}_{\operatorname{Spec}R}$ is the *linear stack of vector bundles*, denoted by $\mathfrak{Vect}$. The underlying algebraic stack $\mathfrak{Vect}_{\mathrm{cfg}}$ is the disjoint union $\coprod_{n\geqslant 0}B\,\mathrm{GL}_n$. The sheaf $\mathscr{H}$ over

$$\coprod_{n\geqslant 0}B\,\mathrm{GL}_n\times\coprod_{n\geqslant 0}B\,\mathrm{GL}_n=\coprod_{n,m\geqslant 0}B(\mathrm{GL}_n\times\mathrm{GL}_m)$$

is given by the natural representation $M(m\times n)$ of $\mathrm{GL}_n\times\mathrm{GL}_m$ over $B(\mathrm{GL}_n\times\mathrm{GL}_m)$.

Example 1.13. A generalization of $\mathfrak{Vect}$ in a different direction is given by quiver representations.

Let Q be a quiver. The stack of representations of Q, denoted by $\mathfrak{Rep}_Q$, has as $\mathfrak{Rep}_Q(S)$ the set of diagrams $(\mathscr{F})$ in the shape of Q of locally free finite rank $\mathscr{O}_S$-modules. For a morphism $f:S'\to S$ of R-schemes, we have that $\operatorname{Hom}_f(\mathscr{F}',\mathscr{F})$ is the $\mathscr{O}(S')$-module of homomorphisms $\mathscr{F}'\to f^*\mathscr{F}$ of diagrams of locally free $\mathscr{O}_{S'}$-modules.

Example 1.14. As a toy example, let A be a vector bundle over $\operatorname{Spec}R$, endowed with the structure of a sheaf of $\mathscr{O}_{\operatorname{Spec}R}$-modules, with smooth group scheme of units $A^\times$, also of finite type. Then we define the linear stack of $A^\times$-torsors to have as objects over the R-scheme S the right $A^\times$-torsors over S, and for $f:S'\to S$ and $A^\times$-torsors P' over S' and P over S, we set $\operatorname{Hom}_f(P',P)=\operatorname{Hom}_{S'}(P',f^*P)=P'\times_{A^\times}A\times_{A^\times}f^*P$. In this example, the underlying algebraic stack is $BA^\times$ and we have $\mathscr{H}=A^\times\backslash A/A^\times$.

The case $A=0$ is not excluded. The associated linear stack is $\mathrm{id}:\mathscr{S}\to\mathscr{S}$. All $\operatorname{Hom}_f(x,y)$ are singletons, endowed with their unique module structure. This stack is represented by $\operatorname{Spec}R$. It can also be thought of as the stack of zero-dimensional vector bundles.

Substacks. Let $\mathfrak{X}$ be a linear algebraic stack with underlying algebraic stack $X=\mathfrak{X}_{\mathrm{cfg}}$. If $Y\subset X$ is a locally closed algebraic substack, there is a canonical linear algebraic stack $\mathfrak{Y}$ with underlying algebraic stack $\mathfrak{Y}_{\mathrm{cfg}}=Y$. In fact, we can define $\mathfrak{Y}$ to be the full subcategory of $\mathfrak{X}$ consisting of objects which are in Y.

In this situation, we call $\mathfrak{Y} \to \mathfrak{X}$ a locally closed linear substack of $\mathfrak{X}$.

Fibred products. Let $F : \mathfrak{X} \to \mathfrak{Z}$ and $G : \mathfrak{Y} \to \mathfrak{Z}$ be cartesian morphisms of $\mathscr{O}$-linear fibred categories. We define a new $\mathscr{O}$-linear fibred category $\mathfrak{W}$ as follows: objects of $\mathfrak{W}$ over the object T of $\mathscr{S}$ are triples (x, α, y), where x is an $\mathfrak{X}$-object over T, y is a $\mathfrak{Y}$-object over T, and α is an isomorphism $\alpha : F(x) \to G(y)$ over T. A morphism from (x', α', y') to (x, α, y) over $T' \to T$ is a pair of morphisms $f : x' \to x$ over $T' \to T$ and $g : y' \to y$ over $T' \to T$ such that $\alpha \circ F(f) = G(g) \circ \alpha'$.

In other words, we can write the set of morphisms from (x', α', y') to (x, α, y) over $\varphi : T' \to T$ as the fibred product

$$\mathrm{Hom}_\varphi(x', x) \times_{\mathrm{Hom}_\varphi(F(x'),G(y))} \mathrm{Hom}_\varphi(y', y),$$

and as each of the sets in this fibred product is an $\mathscr{O}(T')$-module, and the maps are linear, this fibred product is also an $\mathscr{O}(T')$-module. We leave it to the reader to verify that composition is bilinear.

Let us verify that $\mathfrak{W}$ is a fibred category. Suppose that (x, α, y) is a triple over T and $\varphi : T' \to T$ a morphism in $\mathscr{S}$. We construct a triple (x', α', y') over T' by taking as x' a pullback of x via φ and as y' a pullback of y via φ. Then, as G is cartesian, $G(y')$ is a pullback of $G(y)$ via φ. Hence, there exists a unique morphism $\alpha' : F(x') \to G(y')$ covering T' such that $\alpha \circ F(x' \to x) = G(y' \to y) \circ \alpha'$. Then α' is cartesian, because cartesian morphisms satisfy the necessary 'two out of three' property. Then α' is invertible, because cartesian morphisms covering an identity are invertible. The triple (x', α', y') comes with a given morphism to (x, α, y) that covers φ. It is easily verified that this morphism is cartesian.

Therefore, $\mathfrak{W}$ is an $\mathscr{O}$-linear fibred category. By construction, the two projections $\mathfrak{W} \to \mathfrak{X}$ and $\mathfrak{W} \to \mathfrak{Y}$ are cartesian. We call $\mathfrak{W}$ the *fibred product* of $\mathfrak{X}$ and $\mathfrak{Y}$ over $\mathfrak{Z}$.

Suppose $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ are algebraic, with underlying algebraic stacks X, Y, and Z, respectively. For triples (x', α', y') and (x, α, y) over S, the presheaf $\underline{\mathrm{Hom}}((x', \alpha', y'), (x, \alpha, y))$ is equal to the fibred product

$$\underline{\mathrm{Hom}}(x', x) \times_{\underline{\mathrm{Hom}}(Fx',Gy)} \underline{\mathrm{Hom}}(y', y)$$

and is therefore a locally coherent representable sheaf of $\mathscr{O}_S$-modules. We see that $\mathfrak{W}$ is again a linear algebraic stack. Moreover, the underlying algebraic stack of W is the fibred product $X \times_Z Y$.

Lack of locality.

Remark 1.15. Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are linear algebraic stacks with underlying algebraic stacks X and Y. We can construct a disjoint union linear algebraic stack $\mathfrak{X} \amalg \mathfrak{Y}$ whose underlying algebraic stack is $X \amalg Y$ by declaring all homomorphisms between objects of $\mathfrak{X}$ and objects of $\mathfrak{Y}$ to be zero. This concept of disjoint union is not useful for our purposes. For the linear algebraic stacks we are interested in, the underlying algebraic stack often decomposes into a disjoint union, even though the linear algebraic stack does not. An example is given by the linear stack of vector bundles $\mathfrak{Vect}$, Example 1.12.

Thus, linear algebraic stacks exhibit less local behaviour than algebraic stacks and are therefore less geometrical. This is one of the reasons we prefer to work with *algebroids*, rather than linear algebraic stacks.

Special linear stacks. For a linear algebraic stack $\mathfrak{M}$, every fibre category $\mathfrak{M}(S)$ is an R-linear category. By putting special requirements on these linear categories, we get stronger notions of linear algebraic stack.

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For a linear algebraic stack $\mathfrak{M}$, we denote the universal sheaf of endomorphisms by $\mathfrak{A} \to \mathfrak{M}$.

Definition 1.16. A linear algebraic stack $\mathfrak{M}$ has a *zero object* if the R-linear category $\mathfrak{M}(R)$ admits a zero object.

If $\mathfrak{M}$ admits a zero object, then for every R-scheme S, the R-linear category $\mathfrak{M}(S)$ admits a zero object, namely, the pullback of the zero object in $\mathfrak{M}(\operatorname{Spec} R)$ via the unique morphism $S \to \operatorname{Spec} R$.

A zero object for $\mathfrak{M}$ defines a section $\operatorname{Spec} R \xrightarrow{0} \mathfrak{M}$, which is an isomorphism onto the closed substack of $\mathfrak{M}$ defined by the condition $1 = 0$ inside $\mathfrak{A}$.

If $\mathfrak{M}$ admits a zero object, we denote the complement of the zero object in $\mathfrak{M}$ by $\mathfrak{M}_*$. It is a linear open substack of $\mathfrak{M}$.

Definition 1.17. The linear algebraic stack $\mathfrak{M}$ *admits direct sums* if for every R-scheme S, the R-linear category $\mathfrak{M}(S)$ admits all (finite) direct sums.

The pullback functor $\mathfrak{M}(S) \to \mathfrak{M}(S')$ for a morphism of R-schemes $S' \to S$ commutes with direct sums. Hence, if $\mathfrak{M}$ admits direct sums, there is a canonical morphism of linear stacks

$$\begin{aligned} \mathfrak{M} \times \mathfrak{M} &\longrightarrow \mathfrak{M} \\ (x, y) &\longmapsto x \oplus y. \end{aligned}$$

See also Remark 1.63.

1.3 Finite type algebras

None of our algebras are assumed to be commutative, but they are all assumed to be unital.

Definition 1.18. Let X be an algebraic stack. By an *algebra* over X, we mean a sheaf of $\mathscr{O}_X$-algebras over X. If the algebra A over the algebraic stack X is an algebraic stack itself, i.e., if the structure morphism $A \to X$ is a representable morphism of stacks, then we say that A is *representable*. If A is represented by a finite type affine stack of the form $\operatorname{Spec}_X \operatorname{Sym}_{\mathscr{O}_X} \mathscr{F}$ for a coherent sheaf $\mathscr{F}$ over X, we call A a *finite type algebra* over X.

For an automorphism φ of an algebra A, we denote the subalgebra of fixed sections by A^φ. For a section a of A we denote by A^a the subalgebra of sections commuting with a.

The sheaf of $\mathscr{O}_X$-modules underlying a finite type algebra is locally coherent representable.

For an automorphism φ of a finite type algebra A, the fixed algebra A^φ is again a finite type algebra, because the underlying sheaf of modules is the kernel of $A \xrightarrow{1-\varphi} A$.

If $\mathfrak{X}$ is a linear algebraic stack with underlying algebraic stack X, then the universal sheaf of endomorphisms $\mathscr{E} \to X$ is a finite type algebra.

Note that finite type algebras need not have a coherent underlying sheaf of $\mathscr{O}_X$-modules. For example, let $X = \mathbb{A}^1$, with coordinate t, and let $A \to X$ be the centralizer in $(M_{2\times 2})_X$ of the matrix $\left(\begin{smallmatrix} t & 0 \\ 0 & 0 \end{smallmatrix}\right)$. In this case, the big sheaf underlying A is not cartesian.

Remark 1.19. Dually, a finite type algebra A over X corresponds to a coherent sheaf M over X that is endowed with a coalgebra structure (Δ, ε), where $\Delta : M \to M \otimes_{\mathscr{O}_X} M$ is an associative comultiplication with counit $\varepsilon : M \to \mathscr{O}_X$. For any X-scheme U, we have $A_U = M_U^\vee$.

Inertia representation. Whenever $A \to X$ is an algebra over the algebraic stack X, we have a tautological morphism of sheaves of groups over X,

$$I_X \longrightarrow \underline{\operatorname{Aut}}(A). \tag{6}$$

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Here, I_X is the inertia stack of X, i.e., the stack of pairs (x, φ) where x is an object of X and φ an automorphism of x, and $\underline{\mathrm{Aut}}(A)$ is the sheaf of automorphisms of the sheaf of algebras A over X. To construct (6), consider the stack of sheaves of algebras $\mathfrak{Alg}$ over $\mathscr{S}$, which has as objects over the scheme S the sheaves of $\mathscr{O}_S$-algebras on the usual (small) étale site of S. A morphism from the sheaf of $\mathscr{O}_{S'}$-algebras A' over S', covering the morphism of schemes $f : S' \to S$ to the sheaf of $\mathscr{O}_S$-algebras A over S, is, by definition, an isomorphism of sheaves of $\mathscr{O}_{S'}$-algebras $A' \to f^*A$. The sheaf of algebras $A \to X$ gives rise to a morphism of $\mathscr{S}$-stacks $a : X \to \mathfrak{Alg}$. We get an induced morphism on inertia stacks $I_X \to I_{\mathfrak{Alg}}$ and notice that $a^* I_{\mathfrak{Alg}} = \underline{\mathrm{Aut}}(A)$.

With this definition, an automorphism φ of the object x of the stack X is mapped to the *inverse* of the restriction morphism $\varphi^* : A(x) \to A(x)$.

Lemma 1.20. *Suppose X is a gerbe over the algebraic space Y and $A \to X$ is an algebra. Then there exists a sheaf of $\mathscr{O}_Y$-algebras B and an isomorphism $A \cong B|_X$ if and only if the inertia representation $I_X \to \underline{\mathrm{Aut}}_X(A)$ is trivial.*

Similarly, if X is a connected gerbe over the Deligne–Mumford stack Y, then an algebra A over X descends to B over Y if and only if the inertia representation restricts to a trivial homomorphism $I_X^\circ \to \underline{\mathrm{Aut}}_X(A)$.

In either case, A is representable or of finite type if and only if B is.

We can pull back the sheaf of algebras A over X, via the structure morphism $I_X \to X$, to obtain the sheaf of algebras $A|_{I_X}$. This sheaf of algebras is endowed with a tautological automorphism, induced from (6). We shall denote the algebra of invariants for this automorphism by $A^{\mathrm{fix}}_{I_X}$.

The following statement is somewhat tautological and holds more generally than for algebras.

Proposition 1.21. *Suppose that A is a representable algebra over the algebraic stack X. Then the inertia stack of A is naturally identified with $A^{\mathrm{fix}}_{I_X}$. In particular, I_A is a representable algebra over I_X.*

Proof. We have a commutative diagram of algebraic stacks

$$\begin{array}{ccc} I_A & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ I_X & \longrightarrow & X \end{array}$$

which identifies I_A with a substack of $A|_{I_X}$. The algebra $A|_{I_X}$ is the stack of triples (x, φ, a), where x is an object of X, φ is an automorphism of x, and $a \in A(x)$ is an object of A lying over x. Such a triple is in I_A if and only if $\varphi \in \mathrm{Aut}(x)$ is in the subgroup $\mathrm{Aut}(a) \subset \mathrm{Aut}(x)$. This is equivalent to φ fixing a under the action of $\mathrm{Aut}(x)$ on $A(x)$. This is the claim. □

In fact, the fibre of I_A over the object x of X is equal to

$$I_A(x) = \{(\varphi, a) \in \mathrm{Aut}(x) \times A(x) \mid \varphi^*(a) = a\}.$$

The fibre of $I_A(x)$ over $\varphi \in \mathrm{Aut}(x)$ is the subalgebra $A(x)^\varphi \subset A(x)$, and the fibre of $I_A(x)$ over $a \in A(x)$ is the subgroup $\mathrm{Stab}_{\mathrm{Aut}(x)}(a) \subset \mathrm{Aut}(x)$.

Algebra bundles.

Definition 1.22. We call a finite type algebra $A \to X$ an *algebra bundle* if the underlying $\mathscr{O}_X$-module is locally free (necessarily of finite rank).

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When studying finite type algebras over finite type stacks X, we may, after passing to a locally closed stratification of X, assume that the finite type algebra is an algebra bundle.

Definition 1.23. Let $A \to X$ be a finite type algebra over the algebraic stack X. The stratification $X' \subset X$ of Proposition 1.5 is characterized by the property that an X-scheme S factors through X' if and only if $A|_S$ is an algebra bundle. The pullback $A' = A|_{X'}$ is an algebra bundle, and the induced stratification $A' \to A$ of A is called the *rank stratification* of A.

Remark 1.24. By considering the representation of A on itself by left multiplication, we see that every algebra bundle is a sheaf of subalgebras of the algebra $\mathrm{End}(V)$ of endomorphisms of a vector bundle V over the stack X.

Central idempotents.

Lemma 1.25. *The centre of an algebra bundle is a finite type algebra.*

Proof. The centre of A is the kernel of the $\mathscr{O}_X$-linear homomorphism of vector bundles $A \to \underline{\mathrm{End}}_{\mathscr{O}_X}(A)$, given by $a \mapsto [a, \cdot]$. As such, it is a locally coherent representable sheaf. □

Thus, if $A \to X$ is a finite type algebra over a finite type stack, after passing to a locally closed stratification of X, we may assume that A is an algebra bundle whose centre is an algebra bundle.

If $A \to X$ is a commutative algebra bundle, then $\pi : Y = \mathrm{Spec}_X A \to X$ is a finite flat representable morphism and $A = \pi_* \mathscr{O}_Y$. In fact, the category of commutative algebra bundles over X is equivalent to the category of finite flat representable stacks over X.

For a commutative finite type algebra $A \to X$, we denote the stack of idempotents in A by $E(A)$.

Lemma 1.26. *Suppose A is a commutative algebra bundle over the algebraic stack X. Then the structure morphism $E(A) \to X$ is affine, of finite type, and étale. In particular, there exists a non-empty open substack $U \subset X$ such that $E(A)|_U = E(A|_U)$ is finite étale over U.*

Proof. We reduce to the case where X is a scheme and then quote Lemme 18.5.3 from [GD67, EGA IV]. □

By this lemma, when studying finite type algebras over the finite type stack X, we may, after passing to a stratification of X, assume that the stack of central idempotents is finite étale over X.

Primitive idempotents. Recall that a non-zero idempotent e is called *primitive* if whenever $e = e_1 + e_2$ for orthogonal idempotents e_1, e_2, then necessarily $e_1 = 0$ or $e_2 = 0$.

In a finite-dimensional commutative algebra over a field, the following are true.

(i) Every idempotent is in a unique way (up to order of the summands) a sum of orthogonal primitive idempotents; this is the *primitive decomposition*.
(ii) Orthogonal idempotents have disjoint primitive decompositions.
(iii) Distinct primitive idempotents are orthogonal to each other.
(iv) The primitive idempotents add up to 1.

Thus, the idempotents are in bijection with the subsets of the (finite) set of primitive idempotents.

Let $A \to X$ be a finite type algebra.

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Definition 1.27. An idempotent local section $e : U \to A$ of $A \to X$ is *primitive* if it gives rise to a primitive idempotent in the fibre of A over every geometric point of U.

Suppose $A = \pi_* \mathscr{O}_Y$ is a commutative algebra bundle and e an idempotent global section. Let $Y_1 \subset Y$ be the open and closed substack defined by the equation $e = 1$. Then e is primitive if and only if the geometric fibres of $Y_1 \to X$ are connected. As the function counting the number of connected components of the fibres is lower semi-continuous, the subset of X where e is primitive is closed. In general, this subset is not open. Therefore, when studying primitive idempotents, we assume that $E(A) \to X$ is finite étale.

Lemma 1.28. *Let $A \to X$ be a commutative algebra bundle with finite étale stack of idempotents $E(A) \to X$. There is an open and closed substack $PE(A) \subset E(A)$ such that an idempotent local section factors through $PE(A)$ if and only if it is primitive.*

Proof. We may assume that $E(A) \to X$ is constant. Then the multiplication operation and the partially defined addition operation on $E(A)$ are also constant. The claim follows. □

Definition 1.29. Let $A \to X$ be an algebra bundle with centre $Z \to X$. Let $ZE(A)$ be the stack of idempotents in Z, in other words, the stack of central idempotents in A. Assume that $ZE(A) \to X$ is finite étale. The substack of primitive idempotents in $ZE(A)$ is denoted by $PZE(A)$ and called the stack of *primitive central idempotents* of A. It is finite étale over X. The degree of $PZE(A) \to X$ is called the *central rank* of A.

If X is connected, the number of connected components of $PZE(A)$ is the *split central rank* of A. More precisely, the partition of the central rank given by the degrees of the connected components of $PZE(A)$ is called the *central type* of A. (So the split central rank is the length of the type.)

Remark 1.30. Let X be connected, and let $A \to X$ be a commutative finite type algebra with finite étale stack of idempotents $E(A) \to X$. Then there is a one-to-one correspondence between the connected components of $PE(A)$ and the primitive idempotents in the algebra of global sections $\Gamma(X, A)$.

The degree stratification. Let k be a field and A a finite-dimensional k-algebra. The *rank* r of A is the dimension of A as a k-vector space. For an element $a \in A$, we define its *degree* to be the dimension of the commutative subalgebra $k[a] \subset A$. It is equal to the degree of the minimal polynomial of a, i.e., the monic generator of the kernel of the algebra map $k[x] \to A$, defined by $x \mapsto a$.

Now let A be an algebra bundle of rank r over the algebraic stack X, and $a \in A(S)$ a local section of A over an X-scheme S.

Definition 1.31. If the cokernel (as a homomorphism of $\mathscr{O}_S$-modules) of the morphism of $\mathscr{O}_S$-algebras $\mathscr{O}_S[x] \to A_S$ defined by $x \mapsto a$ is flat over $\mathscr{O}_S$, we say that a is *strict*, and we call the rank of the image of $\mathscr{O}_S[x] \to A$ the *degree* of a.

If $f(x) \in \mathscr{O}_S[x]$ is the characteristic polynomial of a, the morphism $\mathscr{O}_S[x] \to A$ factors through $\mathscr{O}_S[x]/(f)$, by the theorem of Caley–Hamilton. Hence, the cokernel of $\mathscr{O}_S[x] \to A$ is actually a cokernel of a homomorphism of vector bundles and hence coherent. The condition that this cokernel be flat is equivalent to it being locally free. It implies that forming the image of $\mathscr{O}_S[x] \to A$ commutes with base change, and that this image, denoted by $\mathscr{O}_S[a]$, is also locally free.

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PROPOSITION 1.32. *For every $n = 1, \dots, r$, there exists a locally closed substack $A_n \subset A$ with the property that a local section $a \in A(S)$ factors through $A_n(S)$ if and only if a is strict of degree n. The A_n are pairwise disjoint and their disjoint union*

$$A^{\mathrm{strat}} = \coprod_{n=1}^{r} A_n$$

maps surjectively to A. The section $a \in A(S)$ factors through $A^{\mathrm{strat}} \to A$ if and only if it is strict.

Proof. Consider the tautological section Δ of the pullback of A via the structure map $A \to X$. It gives rise to a morphism of $\mathcal{O}_A$-algebras $\mathcal{O}_A[x] \to A_A$. Then A^{strat} is given by the flattening stratification of its cokernel, and $A_n \subset A^{\mathrm{strat}}$ is the component where the cokernel has rank $r - n$. □

We call the stratification $A^{\mathrm{strat}} \to A$ the *degree stratification* of A.

Semi-simple elements. Let k be an algebraically closed field and A a finite-dimensional k-algebra. Recall that an element $a \in A$ is *semi-simple* if the following equivalent conditions are satisfied.

(i) The map $A \to A$ given by left multiplication by a is diagonalizable.
(ii) The minimal polynomial $f \in k[x]$ of a is separable, i.e., satisfies $(f, f') = 1$.
(iii) The commutative subalgebra $k[a] \subset A$ is reduced, or, equivalently, étale over k.

DEFINITION 1.33. Let $A \to X$ be an algebra bundle. A local section $a \in A(S)$, for an X-scheme S, is called *semi-simple* if it is strict and for every geometric point $s \in S$, the element induced by a in $A(s)$ is semi-simple.

For example, an idempotent section $e \in A(S)$ is semi-simple over the open subset of S, where e is neither 0 nor 1 (in a commutative algebra bundle, this subset is also closed).

Assuming $a \in A(S)$ is strict, a is semi-simple if and only if the geometric fibres of the finite flat S-scheme $\mathrm{Spec}_S \mathcal{O}_S[a]$ are unramified. This condition is equivalent to $\mathrm{Spec}_S \mathcal{O}_S[a]$ being unramified and hence étale over S.

The semi-simple sections of A form a subsheaf $A^{\mathrm{ss}} \subset A$.

PROPOSITION 1.34. *Let $A \to X$ be an algebra bundle over the algebraic stack X. Then A^{ss} is an algebraic stack with a representable structure morphism of finite type $A^{\mathrm{ss}} \to X$.*

Proof. In fact, $A^{\mathrm{ss}} \subset A^{\mathrm{strat}}$ is the open substack defined by the condition that the finite flat representable morphism $\mathrm{Spec}_A \mathcal{O}_A[\Delta] \to A$ is unramified. (The section Δ is the tautological one, as in the proof of Proposition 1.32.) Thus, we have a factorization of the monomorphism $A^{\mathrm{ss}} \to A$ as

$$A^{\mathrm{ss}} \xrightarrow{\text{open immersion}} A^{\mathrm{strat}} \xrightarrow{\text{stratification}} A\,.$$

Thus, $A^{\mathrm{ss}} \subset A$ is a constructible substack. □

The semi-simple centre. For a commutative finite-dimensional algebra over an algebraically closed field, we have the following facts.

(i) The primitive idempotents are linearly independent.
(ii) An element is semi-simple if and only if it is a linear combination of primitive idempotents.

We need a version of this statement for algebra bundles.

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Proposition 1.35. *Let $A \to X$ be a commutative algebra bundle whose stack of idempotents is finite étale, and let $Y \to X$ be the finite flat cover corresponding to A. There is a canonical finite flat morphism of X-stacks $Y \to PE(A)$. Over every geometric point x of X, this morphism maps each point in the fibre Y_x to the characteristic function of its connected component in Y_x (which is a primitive idempotent in $A|_x$). Dually, we obtain a strict monomorphism of algebra bundles*

$$\pi_*\mathscr{O}_{PE(A)} \longrightarrow A,$$

where $\pi : PE(A) \to X$ is the structure map.

The induced morphism

$$(\pi_*\mathscr{O}_{PE(A)})^{\mathrm{strat}} \longrightarrow A^{\mathrm{strat}}$$

factors through the open substack $A^{\mathrm{ss}} \subset A^{\mathrm{strat}}$ and induces a surjective closed immersion of algebraic stacks $(\pi_*\mathscr{O}_{PE(A)})^{\mathrm{strat}} \to A^{\mathrm{ss}}$.

Proof. Consider the finite étale cover of primitive idempotents $\pi : PE(A) \to X$. We have a tautological global section e of $A|_{PE(A)}$, and $a \mapsto ae$ defines a homomorphism of $\mathscr{O}_{PE(A)}$-modules $\mathscr{O}_{PE(A)} \to A|_{PE(A)}$. Pushing forward with π and composing with the trace map $\pi_*(A|_{PE(A)}) \to A$ defines the morphism of algebra bundles over X,

$$\pi_*\mathscr{O}_{PE(A)} \longrightarrow A.$$

It is a strict monomorphism of vector bundles, because it is injective over every geometric point, by fact (i), above. Dually, we obtain a morphism of X-stacks $Y \to PE(A)$, which is the morphism described in the statement of the proposition. It is flat, because $PE(A)$ is étale over X, and flatness can be checked étale locally.

Passing to the degree stratification commutes with strict monomorphisms of algebra bundles, so we have a cartesian diagram of X-stacks

$$\begin{array}{ccc} (\pi_*\mathscr{O}_{PE(A)})^{\mathrm{strat}} & \longrightarrow & A^{\mathrm{strat}} \\ \downarrow & & \downarrow \\ \pi_*\mathscr{O}_{PE(A)} & \longrightarrow & A \end{array}$$

which shows that $(\pi_*\mathscr{O}_{PE(A)})^{\mathrm{strat}} \to A^{\mathrm{strat}}$ is a closed immersion. That this closed immersion factors through $A^{\mathrm{ss}} \subset A^{\mathrm{strat}}$ and is surjective onto A^{ss} can be checked over the geometric points of X, where it follows from fact (ii), above. □

Permanence of rank and split rank.

Proposition 1.36. *Let $A \hookrightarrow A'$ be a monomorphism of commutative finite type algebras with finite étale stacks of idempotents over the connected stack X. Denote the ranks of A and A' by n and n', and the split ranks by k and k', respectively. Then $n \leqslant n'$ and $k \leqslant k'$. Moreover:*

(i) *if A' admits a semi-simple global section that does not factor through the fibre $A|_x$ for all points of x, then $n < n'$;*
(ii) *if A' admits an idempotent global section that is not in A, then $k < k'$.*

Proof. The monomorphism $A \hookrightarrow A'$ induces an open and closed embedding of finite étale X-stacks $E(A) \hookrightarrow E(A')$. Every idempotent e in A can be decomposed uniquely into a sum of orthogonal primitive idempotents in A'. Let us call this the *primitive decomposition* of e in A'. Consider the correspondence $Q \subset PE(A) \times_X PE(A')$ defined by

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$$\begin{aligned}(e,e') \in Q &\iff e' \text{partakes in the primitive decomposition of } e \text{ in } A' \\ &\iff ee' = e'.\end{aligned}$$

One shows that Q is a finite étale cover of X locally in the étale topology of X, reducing to the case where both $E(A)$ and $E(A')$ are trivial covers. By properties of the primitive decomposition, the projection $Q \to PE(A)$ is surjective and the projection $Q \to PE(A')$ is injective. Thus, we have

$$n = \deg PE(A) \leqslant \deg Q \leqslant \deg PE(A') = n'.$$

If $n = n'$, then both $Q \to PE(A)$ and $Q \to PE(A')$ are isomorphisms, showing that $PE(A) = PE(A')$. By Proposition 1.35, this implies that the morphism $A^{\mathrm{ss}} \to (A')^{\mathrm{ss}}$ is surjective. This proves (i).

We can repeat the argument for the algebras of global sections $\Gamma(X,A) \hookrightarrow \Gamma(X,A')$. We deduce that $k \leqslant k'$, and if $k = k'$, every primitive idempotent in $\Gamma(X,A)$ remains primitive in $\Gamma(X,A')$ and every primitive idempotent of $\Gamma(X,A')$ is in $\Gamma(X,A)$. We deduce that $\Gamma(X,A)$ and $\Gamma(X,A')$ have the same idempotents, which proves (ii). □

Families of idempotents.

Definition 1.37. For a finite type algebra $A \to X$, we denote by $E_n(A) \to X$ the stack of n-tuples of non-zero idempotents in A that are pairwise orthogonal and add up to unity. We call sections of $E_n(A)$ also *complete sets of orthogonal idempotents.*

Note that the family members of the sections of $E_n(A)$ need not be central.

The stack $E_n(A)$ is algebraic and of finite type over X.

For $n = 0$, the stack $E_0(A)$ is empty, unless $A = 0$, in which case it is identified with X. For $n = 1$, the stack $E_1(A)$ contains exactly the unit in A (so is identified with X), unless $A = 0$, in which case $E_1(A)$ is empty.

Group of units. Let $A \to X$ be a finite type algebra over the algebraic stack X. The subsheaf of units $A^\times \subset A$ is defined by

$$A^\times(x) = \{a \in A(x) \mid \exists b \in A(x) \colon ab = ba = 1\}$$

for every object x of X. We can see that $A^\times$ is a (relative) affine group scheme over X by writing it as the fibred product, as shown in the following diagram.

$$\begin{array}{ccc} A^\times & \longrightarrow & X \\ {\scriptstyle u\mapsto(u,u^{-1})}\downarrow & & \downarrow{\scriptstyle (1,1)} \\ A \times A & \xrightarrow{(a,b)\mapsto(ab,ba)} & A \times A \end{array}$$

If A is an algebra bundle, the subsheaf $A^\times \subset A$ is represented by an open substack, because in this case, a local section a is invertible if and only if the determinants of left and right multiplication by a on A do not vanish. We conclude that if A is an algebra bundle, $A^\times$ is smooth over X with geometrically connected fibres. For the general case, this still implies that the fibres of $A^\times \to X$ are smooth and geometrically connected, as the fibres do not change when passing to a stratification of X.

A similar argument using the determinant proves that if $A \to B$ is a strict monomorphism of algebra bundles, we have $A^\times = A \cap B^\times$. Also, if A is an algebra bundle, any morphism $A \to B$ to another algebra bundle is determined by its restriction to $A^\times$.

1.4 Algebroids

Definition 1.38. An *algebroid* is a triple (X, A, ι), where X is an algebraic stack, A is a finite type algebra over X, and $\iota : A^\times \to I_X$ is a homomorphism of sheaves of groups over X, which identifies $A^\times$ with an open substack of I_X. Moreover, we require that the diagram

$$\begin{array}{ccc} A^\times & \xrightarrow{\iota} & I_X \\ & \searrow & \downarrow \\ & & \underline{\mathrm{Aut}}(A) \end{array} \qquad (7)$$

of groups over X commutes. Here, the map $A^\times \to \underline{\mathrm{Aut}}_X(A)$ associates to a unit u of A the inner automorphism $x \mapsto uxu^{-1}$. The vertical map $I_X \to \underline{\mathrm{Aut}}_X(A)$ is the inertia representation (6). If ι is an isomorphism, we call (X, A, ι) a *strict algebroid*.

We will usually abbreviate the triple (X, A, ι) to X and write A_X for A if we need to specify the algebra. We call the commutativity of (7) the *algebroid property*.

For an explanation of the terminology, see Remark 1.44.

The condition that ι is an open immersion implies that $A^\times$ represents the subsheaf $I_X^\circ \subset I_X$ of connected components of the identity. (Over a field, $A^\times$ is connected, and $A^\times \to I_X$ being an open immersion implies that $I_X/A^\times$ is étale. These conditions characterize the connected component over a field.) In particular, if X is a connected gerbe over a Deligne–Mumford stack S, then the relative inertia $I_{X/S}$ is necessarily equal to I_X°, and hence $A^\times$ is identified with $I_{X/S}$.

If A is an algebra bundle, $A^\times$, and therefore also I_X°, is an affine smooth (relative) group scheme over X. Hence, X admits a *coarse Deligne–Mumford stack* $X \to \overline{X}$, which is uniquely determined by being a Deligne–Mumford stack and $X \to \overline{X}$ being a connected gerbe. Moreover, $A^\times$ is then identified with the relative inertia group $I_{X/\overline{X}} \subset I_X$, and we have a short exact sequence of relative group schemes

$$1 \longrightarrow A^\times \xrightarrow{\iota} I_X \longrightarrow I_{\overline{X}}|_X \longrightarrow 1$$

over X. In the case where (X, A) is a strict algebroid, $I_X^\circ = I_X$ and $\overline{X}$ is an algebraic space, in fact the *coarse moduli space* of X.

In many cases, the algebroid property is automatic.

Proposition 1.39. *Consider a triple (X, A, ι), where X is an algebraic stack, A is an algebra bundle over X, and $\iota : A^\times \to I_X$ is a homomorphism of sheaves of groups over X that identifies $A^\times$ with an open substack of I_X. Then (7) commutes, so (X, A, ι) is an algebroid.*

Proof. As $\iota : A^\times \to I_X$ is defined over X, the homomorphism ι is equivariant with respect to the inertia action. The proof now combines the facts that ι is a monomorphism, that the inertia action on I_X is the inner action, and that a morphism of algebra bundles is determined by its restriction to units.

In fact, let $a \in A^\times$. To show that ${}^a(\cdot) = \iota(a)\cdot$, as automorphism of A, it suffices to show that for all $b \in A^\times$ we have ${}^a b = \iota(a) \cdot b$. We can check this after applying ι, so it suffices that $\iota({}^a b) = \iota(\iota(a) \cdot b)$, or ${}^{\iota(a)}\iota(b) = \iota(a) \cdot \iota(b)$, which is true. □

Example 1.40 (Algebroid underlying a linear stack). Let $\mathfrak{X}$ be a linear algebraic stack with underlying algebraic stack X, and let $A \to X$ be the universal sheaf of endomorphisms of

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Remark 1.10. Then automorphisms are invertible endomorphisms, so we use for ι the tautological identification $A^\times = I_X$.

The inertia representation being the inverse of the pullback action, it is, indeed, given by (left) inner automorphisms.

We call (X, A) the *algebroid underlying* the linear algebraic stack $\mathfrak{X}$. It is a strict algebroid.

Example 1.41. Consider the linear stack of vector bundles $\mathfrak{Vect}$, as in Example 1.12. The underlying algebroid consists of the disjoint union of the quotient stacks $\mathrm{GL}_n \backslash M_{n\times n}$, given by the adjoint representations, for $n \geqslant 0$. Thus, in passing from the linear stack to the underlying algebroid, we discard all $M_{m\times n}$ for $m \neq n$, and for $m = n$ we restrict the left-right bi-action of GL_n on $M_{n\times n}$ to the (left only) adjoint action. Thus, we remove exactly the information which we consider non-local, see Remark 1.15.

Example 1.42 (Classifying algebroid). Let $A \to X$ be an algebra bundle over a Deligne–Mumford stack X. Let $A^\times$ act on A from the left by inner automorphisms. Then $A^\times \backslash A$ is an algebra bundle over the relative classifying stack $Y = B_X A^\times$. We have an exact sequence of group schemes over Y,

$$1 \longrightarrow I_{Y/X} \longrightarrow I_Y \longrightarrow I_X|_Y \longrightarrow 1,$$

where $I_{Y/X} = A^\times \backslash A^\times$. As $I_X \to X$ is unramified, $I_{Y/X} \to I_Y$ is an open embedding, and so $(Y, A^\times \backslash A)$ is an algebroid. If X is a space, $(Y, A^\times \backslash A)$ is a strict algebroid.

Remark 1.43 (Algebroids which are trivial gerbes are classifying algebroids). Let (X, A) be an algebroid such that A is an algebra bundle over X, and therefore X is a connected gerbe over the Deligne–Mumford stack S, with $A^\times = I_{X/S}$. Suppose the gerbe $X \to S$ admits a section $x : S \to X$. Via x, we pull back A to a bundle of algebras C over S. We claim that (X, A) is canonically isomorphic to $(BC^\times, C^\times \backslash C)$.

In fact, because X is a gerbe over S, the section $x : S \to X$ is a universal principal $x^* I_{X/S}$-bundle. The pullback diagram

$$\begin{array}{ccc} C & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle x} \\ A & \longrightarrow & X \end{array}$$

shows that C is an $x^* I_{X/S}$-bundle over A. Hence, $A = C/x^* I_{X/S}$. Via the isomorphism $\iota : C^\times \to x^* I_{X/S}$, the action of $x^* I_{X/S}$ on C is identified with the action by left inner automorphisms. This follows from the algebroid property of (X, A) and proves the claim.

Remark 1.44. If $X \to \operatorname{Spec} R$ is a gerbe, any strict algebroid over X can be promoted to a linear algebraic stack whose underlying algebraic stack is X.

More generally, there exists a notion of *relative* linear algebraic stack, where the base R is replaced by an arbitrary scheme (or algebraic space). Then every strict algebroid (X, A) where X is a gerbe over a space S becomes naturally a linear algebraic stack *over* S. These types of linear algebraic stacks occur naturally in the theory of deformation quantization, where they were introduced by Kontsevich under the name of 'stack of algebroids', see [Kon01]. In [Kon01], one can also find a description of these stacks of algebroids in terms of cocycles (compare also [DP05]).

Thus, our notion of algebroid is a natural generalization of Kontsevich's notion of a stack of algebroids. This justifies our terminology.

Loosely speaking, algebroids are stacks which are linear over their coarse Deligne–Mumford stack.

Example 1.45 (Schemes as algebroids). Every Deligne–Mumford stack Z is an algebroid via the definition $A_Z = 0_Z$. There is no natural way to enhance the algebroid $(Z, 0_Z)$ to a linear algebraic stack, unless $Z = \operatorname{Spec} R$ is the final scheme. This exhibits one way in which algebroids are more flexible than linear algebraic stacks.

Example 1.46 (Algebroids over quotient stacks). Let $X = G\backslash Y$ be a quotient stack. A finite type algebra A over X is given by a finite type algebra B over Y, together with a lift of the G-action on X to an action on B by algebra automorphisms. The inertia stack of X is naturally identified with $G\backslash \mathrm{Stab}_G Y$. Thus, (X, A) becomes an algebroid if we specify a G-equivariant open embedding of Y-group schemes $\iota : B^\times \to \mathrm{Stab}_G Y$. If B is not an algebra bundle, we also need to require that ${}^u b = \iota(u) \cdot b$, for all $u \in B^\times$, $b \in B$.

Morphisms of algebroids.

Definition 1.47. We call a morphism of algebraic stacks $f : X \to Y$ *inert*, if the diagram of stacks

$$\begin{array}{ccc} I^\circ_X & \longrightarrow & I^\circ_Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is cartesian. If $I_X \to I_Y|_X$ is an isomorphism, we call f *strictly inert*.

The connected component I°_Y is not necessarily an algebraic stack, but if it is, then so is I°_X, if $X \to Y$ is inert. We will only apply this concept when Y is an algebroid, so that I°_Y is representable over Y.

The basic facts about inert morphisms are as follows.

(i) Every inert morphism of algebraic stacks is Deligne–Mumford representable, because its relative inertia group scheme is unramified.

(ii) Every base change of an inert morphism of algebraic stacks is inert.

(iii) Every monomorphism of algebraic stacks (in particular, every locally closed immersion and every stratification) is (strictly) inert.

(iv) Being inert is local in the étale topology of the target.

(v) Every morphism of Deligne–Mumford stacks is inert.

Base changes of morphisms of Deligne–Mumford stacks are, in fact, the only inert morphisms, at least up to stratifications.

Proposition 1.48. *Suppose $X \to Y$ is an inert morphism of algebraic stacks, and suppose I°_Y (and hence also I°_X) is smooth and representable over X, so that we have coarse Deligne–Mumford stacks $\overline{X}$, $\overline{Y}$ and an induced morphism $\overline{X} \to \overline{Y}$. Then the diagram*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \overline{X} & \longrightarrow & \overline{Y} \end{array}$$

is cartesian.

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Proof. To prove that the morphism of gerbes $X \to Y|_{\overline{X}}$ over $\overline{X}$ is an isomorphism, we may pass to an étale cover $\overline{X}'$ of X and assume that the gerbe X is trivial. Then $X = B_{\overline{X}}G$, where G is the pullback of I_X° to $\overline{X}$ via a trivializing section. Moreover, $Y|_{\overline{X}} = B_{\overline{X}}H$, where H is the pullback of $I_Y^\circ|_X$ to $\overline{X}$ via the same trivializing section. Since $I_X^\circ \to I_Y^\circ|_X$ is an isomorphism, so is $G \to H$, and hence $X \to Y|_{\overline{X}}$. □

DEFINITION 1.49. A *morphism of algebroids* $X \to Y$ is a pair (f, φ) where $f : X \to Y$ is a morphism of algebraic stacks and $\varphi : A_X \to A_Y$ is a morphism of algebras over f such that the diagram

$$\begin{array}{ccc} A_X^\times & \xrightarrow{\varphi} & A_Y^\times \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\ I_X & \xrightarrow{I_f} & I_Y \end{array} \tag{8}$$

commutes.

The morphism (f, φ) is a *representable* morphism of algebroids if $\varphi : A_X \to A_Y|_X$ is a monomorphism of sheaves of algebras over X. (This implies that f is Deligne–Mumford representable.)

The morphism (f, φ) is *inert* if $\varphi : A_X \to A_Y|_X$ is an isomorphism of finite type algebras. (This implies that f is inert.)

There is a natural notion of 2-morphism of algebroid, which makes algebroids into a 2-category.

Remark 1.50 (Inert morphisms in the case of algebra bundles). Suppose $(X, A_X) \to (Y, A_Y)$ is a morphism of algebroids, where A_X and A_Y are algebra bundles. If $X \to Y$ is an inert morphism of algebraic stacks, then we automatically have $A_X^\times = A_Y^\times|_X$ and hence $A_X = A_Y|_X$, and so $(X, A_X) \to (Y, A_Y)$ is an inert morphism of algebroids.

Remark 1.51 (Strict algebroids and representable morphisms). Suppose (X, A) is a strict algebroid and $f : Y \to X$ a representable morphism of algebraic stacks. If $B \subset A|_Y$ is a finite type subalgebra such that $\iota(B^\times) = I_Y \cap \iota(A^\times|_Y)$ inside $I_X|_Y$, then (Y, B) is a strict algebroid with a representable morphism $(Y, B) \to (X, A)$. (The algebroid condition for (Y, B) is automatic.)

Every strict algebroid over $f : Y \to X$ and (X, A) comes about in this way.

Remark 1.52 (Pullbacks). Suppose (X, A_X) is an algebroid and $Y \to X$ an inert morphism of algebraic stacks. In this case, Y admits a unique algebroid A_Y endowed with an inert morphism of algebroids $(Y, A_Y) \to (X, A_X)$. In fact, $A_Y = A_X|_Y$.

If (X, A_X) is a strict algebroid, then the morphism $Y \to X$ is necessarily strictly inert, and (Y, A_Y) is necessarily a strict algebroid. We call $(Y, A_Y) \to (X, A_X)$ a *strictly inert morphism* of strict algebroids.

DEFINITION 1.53. We call a morphism of algebroids $(X, A_X) \to (Y, A_Y)$

(i) a vector bundle,
(ii) a principal homogeneous G-bundle, for an algebraic group G,
(iii) a fibre bundle with group G and fibre F,
(iv) a locally closed immersion,
(v) a stratification,

if it is inert and the underlying morphism of algebraic stacks $X \to Y$ has the indicated property.

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Remark 1.54 (Fibred products). Fibred products of algebroids exist and commute with fibred products of underlying stacks and underlying algebras. Fibred products of strict algebroids are strict algebroids. The algebroid underlying a fibred product of linear algebraic stacks is equal to the fibred product of the underlying algebroids.

Proposition 1.55. *Suppose $(X, A) \to (Y, B)$ is a morphism of algebroids where $X \to Y$ is a stratification of algebraic stacks. Then there exists a stratification of algebroids $(X', A') \to (X, A)$ such that the composition $(X', A') \to (Y, B)$ is also a stratification of algebroids.*

If (Y, B) is strict, then $(X', A') \to (X, A)$ and $(X', A') \to (Y, B)$ are stratifications of strict algebroids.

Proof. Passing to the rank stratifications of A and B, we obtain a commutative diagram of algebroids.

$$\begin{array}{ccc} (X', A') & \longrightarrow & (Y', B') \\ \downarrow & & \downarrow \\ (X, A) & \longrightarrow & (Y, B) \end{array}$$

The upper horizontal morphism is an inert morphism of algebroids, because A' and B' are algebra bundles, by Remark 1.50. The claim follows. □

Algebroid inertia.

Remark 1.56 (Inertia). Let (X, A) be an algebroid. Then (I_X, I_A) is another algebroid, which we call the *algebroid inertia* of (X, A). In fact, $I_A = (A|_{I_X})^{\mathrm{fix}}$, the subalgebra of $A|_{I_X}$ of elements invariant under the tautological automorphism induced by the inertia action of I_X on A. The subgroup of units is $(A^\times|_{I_X})^{\mathrm{fix}}$, and we have the following cartesian diagram.

$$\begin{array}{ccc} (A^\times|_{I_X})^{\mathrm{fix}} & \longrightarrow & A^\times \\ \downarrow & & \downarrow \iota \\ I_{I_X} & \longrightarrow & I_X \end{array}$$

If (X, A) is a strict algebroid, then so is (I_X, I_A).

Remark 1.57 (Induced algebroid structure on the algebra). Let (X, A) be an algebroid. Let $(A|_A)^{\mathrm{fix}}$ be the subalgebra of the pullback $A|_A$ of elements commuting with the tautological section of $A|_A$. (This is equal to the space of commuting pairs in $A \times_X A$). The subgroup of units is $(A^\times|_A)^{\mathrm{fix}}$, and we have the following cartesian diagram.

$$\begin{array}{ccc} (A^\times|_A)^{\mathrm{fix}} & \longrightarrow & A^\times \\ \downarrow & & \downarrow \iota \\ I_A & \longrightarrow & I_X \end{array}$$

This proves that $(A, (A|_A)^{\mathrm{fix}})$ is an algebroid over A.

In fact, we have a commutative diagram of algebroids.

$$\begin{array}{ccc} (A^\times, (A|_{A^\times})^{\mathrm{fix}}) & \xrightarrow{\iota} & (I_X, I_A) \\ \downarrow & & \downarrow \\ (A, (A|_A)^{\mathrm{fix}}) & \longrightarrow & (X, A) \end{array}$$

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If (X, A) is a strict algebroid, this is a diagram of strict algebroids.

Remark 1.58 (Semi-simple algebroid inertia). There is also a semi-simple version of the algebroid inertia. To define it, let (X, A) be an algebroid, and consider the following diagram of algebroids.

$$\begin{array}{ccccc} A^{\times,\mathrm{ss}} & \longrightarrow & A^{\times} & \xrightarrow{\iota} & I_X \\ \downarrow & & \downarrow & & \downarrow \\ A^{\mathrm{ss}} & \longrightarrow & A & \longrightarrow & X \end{array} \tag{9}$$

The square on the right was constructed in Remark 1.57. The morphism $A^{\mathrm{ss}} \to A$ is the composition of a stratification and an open immersion (see Proposition 1.34); in particular, it is inert. Hence, we get an induced algebroid structure on A^{ss}. Requiring the square on the left to be a cartesian diagram of algebroids defines the algebraic stack $A^{\times,\mathrm{ss}}$ and the algebroid structure over it.

If (X, A) is a strict algebroid, all objects in Diagram (9) are strict, and ι is an isomorphism. We are then justified in defining $I_X^{\mathrm{ss}} = A^{\times,\mathrm{ss}}$ and calling it the *semi-simple algebroid inertia* of X. In the general case we define $I_X^{\circ,\mathrm{ss}} = A^{\times,\mathrm{ss}}$ and also call it the *semi-simple algebroid inertia*, by a slight abuse of language.

Idempotents and algebroids.

Lemma 1.59. *Let $a \in A^{\times,\mathrm{ss}}$ be a semi-simple invertible global section of an algebra bundle $A \to X$. Let $\widetilde{A}^{\times,\mathrm{ss}} = A^{\times,\mathrm{ss}}/\mathbb{G}_m$ be the quotient of $A^{\times,\mathrm{ss}}$ by the subgroup of scalars. Consider the action of the group $A^{\times}$ on $A^{\times,\mathrm{ss}} \to \widetilde{A}^{\times,\mathrm{ss}}$ by conjugation. The induced group homomorphism*

$$\mathrm{Stab}_{A^{\times}}(a) \longrightarrow \mathrm{Stab}_{A^{\times}}[a],$$

where $[a]$ is the class of a in $\widetilde{A}^{\times,\mathrm{ss}}$, is an open immersion.

Proof. Let Y_a be the relative spectrum of $\mathscr{O}_X[a]$ over X. The epimorphism of commutative algebras $\mathscr{O}_X[x] \to \mathscr{O}_X[a]$ gives rise to a closed immersion $\varphi : Y_a \to (\mathbb{G}_m)_X$, because a is invertible. We get an induced proper morphism

$$\begin{aligned} Y_a \times_X Y_a &\longrightarrow (\mathbb{G}_m)_X \\ (\lambda, \mu) &\longmapsto \varphi(\lambda)/\varphi(\mu). \end{aligned} \tag{10}$$

As $Y_a \to X$ is unramified, the diagonal $Y_a \to Y_a \times_X Y_a$ is an open immersion, so the complement, denoted by $(Y_a \times Y_a)^{\neq}$, is closed in $Y_a \times Y_a$ and hence proper over X. Hence, the image of $(Y_a \times Y_a)^{\neq}$ in $(\mathbb{G}_m)_X$, denoted by Z, is closed. The complement of Z in $(\mathbb{G}_m)_X$ is hence an open neighbourhood U of the identity section.

We have the following cartesian diagram.

$$\begin{array}{ccc} \mathrm{Stab}_{A^{\times}}(a) & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Stab}_{A^{\times}}[a] & \longrightarrow & (\mathbb{G}_m)_X \\ \downarrow & & \downarrow \\ A^{\times} & \xrightarrow[b \mapsto bab^{-1}a^{-1}]{} & A^{\times} \end{array}$$

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The lower square is cartesian by the definition of $\mathrm{Stab}_{A^\times}[a]$. The fact that the upper square is cartesian follows from the fact that the image of $\mathrm{Stab}_{A^\times}[a]$ in $(\mathbb{G}_m)_X$ is contained in the image of (10). The latter claim follows from the fact that if ${}^b a = \lambda a$, for a scalar $\lambda \in (\mathbb{G}_m)_X$, then λ is necessarily a quotient of eigenvalues of a. (Use Remark 1.24 to assume that a is a matrix. Then ${}^b a = \lambda a$ implies that a and ${}^b a$ are simultaneously diagonalizable. Once we have diagonalized, a and ${}^b a$ are the same up to permutation of the diagonal entries.) □

Remark 1.60. Let (X, A) be an algebroid, and let $Y = E_k(A)$ be the stack of complete labelled sets of k orthogonal idempotents in A. Let us write $B = (A|_Y)^{e_1,\dots,e_k}$ for the subalgebra of $A|_Y$ consisting of elements commuting with each of the k tautological idempotents in $A|_Y$. The homomorphism $\iota : A^\times \to I_X$ restricts to a homomorphism $B^\times \to I_Y$ and defines an algebroid structure over Y. The algebra B is endowed with a canonical complete set of orthogonal *central* idempotents and hence decomposes as a product $B = B_1 \times \cdots \times B_k$.

Let (X, A) be an algebroid, where A is an algebra bundle, and let $e_1, \dots, e_k$ be orthogonal central idempotents in A, decomposing A into a product of algebra bundles $A = A_1 \times \cdots \times A_k$. We get an induced decomposition

$$A^{\times,\mathrm{ss}} = A_1^{\times,\mathrm{ss}} \times \cdots \times A_k^{\times,\mathrm{ss}} \subset A,$$

and the algebroid structure on $A^{\times,\mathrm{ss}}$ is the fibred product over X of the algebroid structures on the $A_i^{\times,\mathrm{ss}}$, $i = 1, \dots, k$.

We obtain an embedding of algebras $\alpha : \mathbb{A}^k_X \to A$ mapping $(a_1, \dots, a_k)$ to $\sum a_i e_i \in A$. Via α, the torus $\mathbb{G}_m^k$ acts on A by left multiplication. The action of $\mathbb{G}_m^k$ on A preserves the semi-simple units, and the restricted action of $\mathbb{G}_m^k$ on $A^{\times,\mathrm{ss}}$ is fibrewise free, so the quotient

$$\widetilde{A}^{\times,\mathrm{ss}} = A^{\times,\mathrm{ss}}/\mathbb{G}_m^k = A_1^{\times,\mathrm{ss}}/\mathbb{G}_m \times \cdots \times A_k^{\times,\mathrm{ss}}/\mathbb{G}_m$$

is representable over X.

Proposition 1.61. *We claim that $\widetilde{A}^{\times,\mathrm{ss}}$ has a canonical algebroid structure, and we get an induced commutative diagram*

$$\begin{array}{ccc} A^{\times,\mathrm{ss}} & \longrightarrow & \widetilde{A}^{\times,\mathrm{ss}} \\ & \searrow & \downarrow \\ & & X \end{array}$$

of algebroids where the horizontal map is inert, and hence a principal $\mathbb{G}_m^k$-bundle of algebroids.

Proof. Because inert morphisms of algebroids are stable under composition and pullback, this claim reduces to the case $k = 1$ and $e_1 = 1$, which we will now consider.

Let us denote the tautological section of $A^\times$ over $A^{\times,\mathrm{ss}}$ by a. Then the algebra over $A^{\times,\mathrm{ss}}$ is given by the centralizer algebra $(A|_{A^{\times,\mathrm{ss}}})^a$. This algebra descends to the quotient $\widetilde{A}^{\times,\mathrm{ss}}$, because the centralizer of an algebra element does not depend on its equivalence class. Let us denote this descended algebra by $(A|_{\widetilde{A}^{\times,\mathrm{ss}}})^{[a]}$. The units in this algebra are identified with

$$(A^\times|_{\widetilde{A}^{\times,\mathrm{ss}}})^{[a]} = \mathrm{Stab}_{(A^\times|_{\widetilde{A}^{\times,\mathrm{ss}}})}(a),$$

which is an open subgroup of

$$\mathrm{Stab}_{(A^\times|_{\widetilde{A}^{\times,\mathrm{ss}}})}[a],$$

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by Lemma 1.59. We also have a cartesian diagram

$$\begin{array}{ccc} \mathrm{Stab}_{(A^\times|_{\widetilde{A}^{\times,\mathrm{ss}}})}[a] & \longrightarrow & A^\times|_{\widetilde{A}^{\times,\mathrm{ss}}} \\ \downarrow & & \downarrow{\scriptstyle \iota} \\ I_{\widetilde{A}^{\times,\mathrm{ss}}} & \longrightarrow & I_X|_{\widetilde{A}^{\times,\mathrm{ss}}} \end{array}$$

because the inertia stack $I_{\widetilde{A}^{\times,\mathrm{ss}}}$ can be identified as

$$I_{\widetilde{A}^{\times,\mathrm{ss}}} = \{([a],\varphi) \in \widetilde{A}^{\times,\mathrm{ss}} \times_X I_X \mid {}^{\varphi}[a] = [a]\}.$$

This proves that $\mathrm{Stab}_{(A^\times|_{\widetilde{A}^{\times,\mathrm{ss}}})}[a]$ is an open subgroup of $I_{\widetilde{A}^{\times,\mathrm{ss}}}$. Composing our two open immersions, we obtain an open immersion of groups over $\widetilde{A}^{\times,\mathrm{ss}}$ from $(A^\times|_{\widetilde{A}^{\times,\mathrm{ss}}})^{[a]}$ to $I_{\widetilde{A}^{\times,\mathrm{ss}}}$, endowing $\widetilde{A}^{\times,\mathrm{ss}}$ with the structure of an algebroid, as required.

We get an induced morphism of algebroids $A^{\times,\mathrm{ss}} \to \widetilde{A}^{\times,\mathrm{ss}}$, which is inert, by construction. □

Remark 1.62. Even if (X, A) is strict, $\widetilde{A}^{\times,\mathrm{ss}}$ is not necessarily strict. It is this construction, in fact, which makes it impossible for us to restrict attention to strict algebroids.

Remark 1.63. Let $(\mathfrak{M}, \mathfrak{A})$ be the algebroid underlying a linear algebraic stack. Suppose that $\mathfrak{M}$ admits direct sums. Let $\mathfrak{M}_*$ be the complement of the zero object in $\mathfrak{M}$. We obtain a canonical morphism of algebroids

$$\begin{aligned} \underbrace{\mathfrak{M}_* \times \cdots \times \mathfrak{M}_*}_{n \text{ times}} &\longrightarrow E_n(\mathfrak{A}) \\ (x_1, \dots, x_n) &\longmapsto (x_1 \oplus \cdots \oplus x_n; \pi_1, \dots, \pi_n), \end{aligned} \tag{11}$$

where $\pi_1, \dots, \pi_n$ are the projectors corresponding to the factors $x_1, \dots, x_n$ of $x_1 \oplus \cdots \oplus x_n$. Over every R-scheme S, this morphism is fully faithful. The underlying morphism of algebraic stacks (11) is a monomorphism.

If we require all fibres $\mathfrak{M}(S)$ to be Karoubian, i.e., we require all idempotents to admit the corresponding direct summands, (11) is an isomorphism of algebroids.

We say a linear algebraic stack $\mathfrak{M}$ is *Karoubian* if it admits direct sums and all fibres are Karoubian.

Algebroid representations.

Definition 1.64. Let X be an algebroid. A *representation* of X is a morphism of algebroids $\rho : X \to \mathfrak{Vect}$ to the algebroid underlying the linear stack $\mathfrak{Vect}$ of vector bundles. If ρ factors through vector bundles of rank n, i.e., defines a morphism $X \to B\,\mathrm{GL}_n$ with its natural algebroid structure (see Example 1.41), we say that ρ has rank n.

If the algebroid morphism $X \to B\,\mathrm{GL}_n$ is representable, we say the representation ρ is *faithful*.

To give a representation of the algebroid (X, A) is equivalent to specifying a vector bundle V over X together with a morphism of algebras $A \to \mathrm{End}(V)$ such that the induced morphism on unit groups $A^\times \to \mathrm{GL}(V)$ makes the diagram

$$\begin{array}{ccc} A_X^\times & \xrightarrow{\ \iota\ } & I_X \\ & \searrow & \downarrow \\ & & \mathrm{GL}(V) \end{array}$$

commute, where $I_X \to \mathrm{GL}(V)$ is the inertia representation given by the vector bundle V/X.

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The representation V of rank n is faithful if and only if $A \to \operatorname{End}(V)$ is a monomorphism of algebras over X. If this is the case, the underlying morphism of stacks $X \to B\,\mathrm{GL}_n$ is Deligne–Mumford representable, and the GL_n-bundle of frames $Y = \underline{\mathrm{Isom}}(V, \mathscr{O}^n)$ is represented by a Deligne–Mumford stack.

So, a faithful representation identifies X as a quotient stack $X = \mathrm{GL}_n \backslash Y$, where Y is a Deligne–Mumford stack. The algebroid structure on X is then given by a GL_n-invariant subalgebra $B \hookrightarrow M_{n\times n} \times Y$ such that the subgroup $B^\times \hookrightarrow \mathrm{GL}_n \times Y$ is equal to the subgroup $\mathrm{Stab}_{\mathrm{GL}_n} Y \hookrightarrow \mathrm{GL}_n \times Y$.

Remark 1.65. Every algebroid (X, A), with A an algebra bundle, admits the tautological adjoint representation given by the adjoint representation of the algebra A on itself. By contrast, the representation of A on itself by left multiplication is not an algebroid representation, unless $A = 0$.

Proposition 1.66. *Every algebroid (X, A) admits a stratification $X' \subset X$ such that the restricted algebroid $(X', A|_{X'})$ admits a faithful representation.*

Proof. Without loss of generality, assume that A is an algebra bundle with smooth unit group $A^\times$ and that X is a connected isotrivial gerbe over the Deligne–Mumford stack S. Thus, $X \to S$ admits a section over a finite étale cover $S' \to S$.

To begin with, we consider the case where $S' = S$, i.e., the case where X is a trivial gerbe over S. By Remark 1.43, we can assume that A descends to S and that we are dealing with the algebroid $(BA^\times, A^\times\backslash A)$. Then we can consider the representation of A on itself by left multiplication $\ell : A \to \operatorname{End}(A)$. It restricts to a representation of S-group schemes $\ell : A^\times \to \mathrm{GL}(A)$. We get an induced morphism of algebraic stacks $BA^\times \to B\,\mathrm{GL}(A)$, which is covered by the morphism of algebras $A^\times\backslash A \to B\,\mathrm{GL}(A)\backslash \operatorname{End}(A)$. Since $\ell : A \to \operatorname{End}(A)$ is injective, this gives the required faithful representation of $(BA^\times, A^\times\backslash A)$.

(Note that this construction does not contradict Remark 1.65. The vector bundle over X defined by the left representation of $A^\times$ on itself is different from the vector bundle underlying the algebroid A over X, which is given by the adjoint representation of $A^\times$ on itself.)

Now consider the general case. The pullback (X', A') of (X, A) to S' is again an algebroid, as $X' \to X$ is inert. Since $X' \to S'$ is a trivial gerbe, (X', A') admits a faithful representation $A' \to \operatorname{End}(V)$ on a vector bundle V over X'. Let $\pi : X' \to X$ be the projection. Then, $\pi_* V$ is a faithful representation of A. In fact, by adjunction, the embedding $\pi^* A \to \operatorname{End}(V)$ gives rise to an embedding $A \to \pi_* \operatorname{End}(V) \to \operatorname{End}(\pi_* V)$. □

Clear algebroids. Suppose that (X, A) is an algebroid and that X is a connected gerbe over the Deligne–Mumford stack $\overline{X}$. Then the centre $Z(A)$ descends to a commutative finite type algebra over $\overline{X}$, by Lemma 1.20.

Definition 1.67. We say an algebroid (X, A) is *clear*, if:

(i) A and $Z(A)$ are algebra bundles over X;
(ii) X is a connected isotrivial gerbe over $\overline{X}$;
(iii) the Deligne–Mumford stack $\overline{X}$ is connected;
(iv) $ZE(A) \to X$ is finite étale.

For a clear algebroid, $ZE(A)$ and $PZE(A)$ descend to finite étale $\overline{X}$-schemes. The definitions of *central rank*, *split central rank*, and *central type* apply to clear algebroids.

For every algebroid (X, A) over a finite type algebraic stack X, there exists a stratification of X such that the restricted algebroids over the pieces of the stratification are all clear. This follows from Proposition 1.5 and Lemma 1.26.

2. The spectrum of semi-simple inertia

Let $K(\mathrm{DM})$ be the $\mathbb{Q}$-vector space on (isomorphism classes of) finite type Deligne–Mumford stacks, modulo scissor relations and bundle relations, i.e., equations of the form $[Y] = [F \times X]$, whenever $Y \to X$ is a fibre bundle with special structure group and fibre F. The product over the base scheme $\operatorname{Spec} R$ makes $K(\mathrm{DM})$ a $\mathbb{Q}$-algebra. We write q for the class of the affine line in $K(\mathrm{DM})$.

Let $\mathfrak{M}$ be a linear algebraic stack and $\mathfrak{A} \to \mathfrak{M}$ its universal endomorphism algebra. Recall that $(\mathfrak{M}, \mathfrak{A})$ is an algebroid (cf. Example 1.40).

Stack functions.

Definition 2.1. A *stack function* is a representable morphism of algebroids $(X, A) \to (\mathfrak{M}, \mathfrak{A})$ such that X is of finite type.

The K-module of $\mathfrak{M}$, denoted by $K(\mathfrak{M})$, is the free $\mathbb{Q}$-vector space on (isomorphism classes of) stack functions, modulo the scissor and bundle relations relative to $(\mathfrak{M}, \mathfrak{A})$. The class in $K(\mathfrak{M})$ defined by a stack function $X \to \mathfrak{M}$ will be denoted by $[X \to \mathfrak{M}]$.

A *scissor relation relative to* $\mathfrak{M}$ is

$$[X \to \mathfrak{M}] = [Z \to X \to \mathfrak{M}] + [X \backslash Z \to X \to \mathfrak{M}],$$

for any closed immersion of algebroids $Z \hookrightarrow X$ and any stack function $X \to \mathfrak{M}$. The substacks Z and $X \backslash Z$ are endowed with their respective pullback algebroids.

A *bundle relation relative to* $\mathfrak{M}$ is

$$[Y \to X \to \mathfrak{M}] = [F \times X \to X \to \mathfrak{M}],$$

for any fibre bundle $Y \to X$ of algebroids with special structure group and fibre F, see Definition 1.53.

There is an action of $K(\mathrm{DM})$ on $K(\mathfrak{M})$, given by

$$[Z] \cdot [X \to \mathfrak{M}] = [Z \times X \to X \to \mathfrak{M}].$$

This action makes $K(\mathfrak{M})$ into a $K(\mathrm{DM})$-module.

The additive zero in $K(\mathfrak{M})$ is given by the empty algebroid

$$0 = [\emptyset \to \mathfrak{M}].$$

If $\mathfrak{M}$ admits a zero object (Definition 1.16), we denote the corresponding stack function by $1 = [\operatorname{Spec} R \xrightarrow{0} \mathfrak{M}]$. We can use it to embed $K(\mathrm{DM})$ into $K(\mathfrak{M})$ via $[X] \mapsto [X] \cdot 1 = [X \to \operatorname{Spec} R \xrightarrow{0} \mathfrak{M}]$. We will always assume that $\mathfrak{M}$ admits a zero object.

Remark 2.2. In the definition of $K(\mathfrak{M})$, one can restrict to stack functions $X \to \mathfrak{M}$, where X is an algebroid bundle (by which we mean that A_X is an algebra bundle). For algebroid bundles, the definitions of locally closed immersion and fibre bundle simplify, because 'inert' is then a property of the underlying morphism of stacks (Remark 1.50), namely, that it pulls back from a morphism of Deligne–Mumford stacks (Proposition 1.48). Even the definition of algebroid itself simplifies in the case of algebroid bundles (Proposition 1.39).

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The filtration by split central rank. We say a stack function $X \to \mathfrak{M}$ is *clear* if X is a clear algebroid (Definition 1.67).

Definition 2.3. We introduce the *filtration by split central rank* $K^{\geqslant k}(\mathfrak{M})$ on $K(\mathfrak{M})$ by declaring $K^{\geqslant k}(\mathfrak{M})$ to be generated as a $\mathbb{Q}$-vector space by clear stack functions $[X \to \mathfrak{M}]$ such that A_X admits k orthogonal central non-zero idempotents (globally).

Alternatively, $K^{\geqslant k}(\mathfrak{M})$ is generated by $[X \to \mathfrak{M}]$, where X is a clear algebroid such that $PZE(A_X)$ has at least k components.

Each filtered piece $K^{\geqslant k}(\mathfrak{M})$ is stable under scalar multiplication by $K(\mathrm{DM})$. Let us introduce the abbreviation

$$K^{\geqslant k}/K^{>k}(\mathfrak{M}) = K^{\geqslant k}(\mathfrak{M})/K^{>k}(\mathfrak{M}).$$

Remark 2.4. Trying to define a direct sum decomposition of $K(\mathfrak{M})$ by split central rank would not work, because a clear algebroid X of split central rank k may very well admit a closed substack $Z \subset X$ whose restricted algebroid is again clear, but of split central rank larger than k. Similarly, the bundle relations do not respect split central rank.

The zero ring has no non-zero central idempotents, but any non-zero ring has at least one (namely, 1). For every stack function $X \to \mathfrak{M}$, we have the closed substack $Z \subset X$ defined as the locus where $1 \in A_X$ vanishes. Then, $[X \to \mathfrak{M}] = [Z \to \mathfrak{M}] + [X\backslash Z \to \mathfrak{M}]$, and $[Z \to \mathfrak{M}] \in K(\mathrm{DM}) \subset K(\mathfrak{M})$ and $[X\backslash Z \to \mathfrak{M}] \in K^{>0}(\mathfrak{M})$. Therefore, $K(\mathrm{DM}) \subset K(\mathfrak{M})$ is a complement for $K^{>0}(\mathfrak{M})$ in $K(\mathfrak{M}) = K^{\geqslant 0}(\mathfrak{M})$, i.e., $K(\mathfrak{M}) = K(\mathrm{DM}) \oplus K^{>0}(\mathfrak{M})$. In particular, we have

$$K^{\geqslant 0}(\mathfrak{M})/K^{>0}(\mathfrak{M}) = K(\mathrm{DM}).$$

2.1 The idempotent operators E_r

Let E_r denote the operator on $K(\mathfrak{M})$ which maps a stack function $[X \to \mathfrak{M}]$ to $[E_r(X) \to X \to \mathfrak{M}]$, where $E_r(X) = E_r(A_X)$ is the stack of r-tuples of non-zero orthogonal idempotents adding to unity in A_X, see Definition 1.37. The algebroid structure on $E_r(X)$ is described in Remark 1.60.

The operators E_r are well defined, because applying E_r to a stratification or a fibre bundle of algebroids gives rise to another inert morphism of algebroids of the same type. The operators $E_r : K(\mathfrak{M}) \to K(\mathfrak{M})$ are $K(\mathrm{DM})$-linear.

This definition applies also to $r = 0$. The stack $E_0(X)$ is empty if $A_X \neq 0$, and $E_0(X) = X$ if X is a Deligne–Mumford stack. Hence, E_0 is diagonalizable and has eigenvalues 0 and 1. The kernel (0-eigenspaces) is $K^{>0}(\mathfrak{M}) \subset K(\mathfrak{M})$; the image (1-eigenspaces) is denoted by $K^0(\mathfrak{M}) \subset K(\mathfrak{M})$ and is generated by all stack functions $[X \to \mathfrak{M}]$, where X is a Deligne–Mumford stack. In fact, $K^0(\mathfrak{M}) = K(\mathrm{DM}) \subset K(\mathfrak{M})$.

For $r = 1$, the operator E_1 vanishes on stack functions $[X \to \mathfrak{M}]$, where X is a Deligne–Mumford stack, and acts as identity on stack functions for which $A_X \neq 0$. Hence, E_1 is also diagonalizable with eigenvalues 0 and 1. The kernel of E_1 is $K^0(\mathfrak{M})$ and the image is $K^{>0}(\mathfrak{M})$. Hence, E_0 and E_1 are complementary idempotent operators on $K(\mathfrak{M})$, i.e., they are orthogonal to each other and add up to the identity.

Recall the Stirling number of the second kind, $S(k, r)$, which is defined in such a way that $r!\, S(k, r)$ is the number of surjections from $\underline{k}$ to $\underline{r}$. Here, and elsewhere, we write $\underline{n} = \{1, \ldots, n\}$.

Theorem 2.5. *The operators E_r, for all $r \geqslant 0$, preserve the filtration $K^{\geqslant k}(\mathfrak{M})$ by split central rank. On the subquotient $K^{\geqslant k}/K^{>k}(\mathfrak{M})$, the operator E_r acts as multiplication by $r!\, S(k, r)$.*

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Proof. Consider a clear algebroid (X, A) with a morphism $X \to \mathfrak{M}$ defining the stack function $[X \to \mathfrak{M}]$ in $K(\mathfrak{M})$. Let n be the central rank of X and k the split central rank of X. The filtered piece $K^{\geqslant k}(\mathfrak{M})$ is generated by such $[X \to \mathfrak{M}]$.

Denote by $X \to \overline{X}$ the coarse Deligne–Mumford stack of X. By assumption, both X and $\overline{X}$ are connected and hence admit Grothendieck-style Galois theories (see [Noo04, § 4]).

Let $\widetilde{X} \to X$ be a connected Galois cover with Galois group Γ, which trivializes $PZE(A) \to X$. As $PZE(A)$ descends to $\overline{X}$, this Galois cover can be constructed as a pullback from $\overline{X}$. Therefore, the morphism $\widetilde{X} \to X$ is inert, and hence $\widetilde{X}$ inherits, via pullback, the structure of an algebroid, and hence $[\widetilde{X} \to X \to \mathfrak{M}]$ is a stack function.

Recall that the degree of the cover $PZE(A) \to X$ is n, and the number of components of $PZE(A)$ is k.

By labelling the components of the pullback of $PZE(A)$ to $\widetilde{X}$, we obtain an action of Γ on the set $\underline{n} = \{1, \dots, n\}$ and an isomorphism of finite étale covers of X

$$\begin{aligned} \widetilde{X} \times_\Gamma \underline{n} &\xrightarrow{\simeq} PZE(A) \\ [x, \nu] &\longmapsto e_{[x,\nu]}. \end{aligned}$$

Both source and target of this isomorphism support natural algebroids and the isomorphism preserves them. The number of orbits of Γ on $\underline{n}$ is k.

Then we also have an isomorphism

$$\begin{aligned} \widetilde{X} \times_\Gamma \operatorname{Epi}(\underline{n}, \underline{r}) &\xrightarrow{\simeq} ZE_r(A) \\ [x, \varphi] &\longmapsto \Bigg(\sum_{\varphi(\nu)=\rho} e_{[x,\nu]} \Bigg)_{\rho=1,\dots,r}, \end{aligned}$$

where ZE_r denotes the stack of labelled complete sets of r orthogonal central idempotents. Again, both stacks involved are in fact algebroids, and this isomorphism is an isomorphism of algebroids.

Hence, we may calculate as follows (all stacks involved are endowed with their natural algebroid structures):

$$\begin{aligned} ZE_r[X \to \mathfrak{M}] &= [\widetilde{X} \times_\Gamma \operatorname{Epi}(\underline{n}, \underline{r}) \to \mathfrak{M}] \\ &= \Bigg[\widetilde{X} \times_\Gamma \coprod_{\varphi \in \operatorname{Epi}(\underline{n}, \underline{r})/\Gamma} \Gamma / \operatorname{Stab}_\Gamma \varphi \to \mathfrak{M} \Bigg] \\ &= \sum_{\varphi \in \operatorname{Epi}(\underline{n}, \underline{r})/\Gamma} [\widetilde{X} / \operatorname{Stab}_\Gamma \varphi \to \mathfrak{M}] \\ &= \sum_{\varphi \in \operatorname{Epi}(\underline{n}, \underline{r})^\Gamma} [X \to \mathfrak{M}] + \sum_{\substack{\varphi \in \operatorname{Epi}(\underline{n}, \underline{r})/\Gamma \\ \operatorname{Stab}_\Gamma \varphi \neq \Gamma}} [\widetilde{X} / \operatorname{Stab}_\Gamma \varphi \to \mathfrak{M}]. \end{aligned}$$

Now, we have $\operatorname{Epi}(\underline{n}, \underline{r})^\Gamma = \operatorname{Epi}(\underline{n}/\Gamma, \underline{r})$, and hence

$$\# \operatorname{Epi}(\underline{n}, \underline{r})^\Gamma = r!\, S(k, r).$$

Thus, we conclude,

$$ZE_r[X \to \mathfrak{M}] = r!\, S(k, r)\, [X \to \mathfrak{M}] + \sum_{\substack{\varphi \in \operatorname{Epi}(\underline{n}, \underline{r})/\Gamma \\ \operatorname{Stab}_\Gamma \varphi \neq \Gamma}} [\widetilde{X} / \operatorname{Stab}_\Gamma \varphi \to \mathfrak{M}].$$

For any proper subgroup $\Gamma' \subset \Gamma$, the quotient $X' = \widetilde{X}/\Gamma'$ is an intermediate cover $\widetilde{X} \to X' \to X$ such that $X' \neq X$. The pullback of $PZE(A)$ to X' has more than k components, because the number of orbits of Γ' on $\underline{n}$ is larger than k. Thus, we have proved the theorem for ZE_r, instead of E_r.

Now, observe that $ZE_r(A) \subset E_r(A)$ is a closed substack, because $ZE_r(A) \to X$ is proper and $E_r(A) \to X$ is separated. So we can write

$$E_r[X \to \mathfrak{M}] = ZE_r[X \to \mathfrak{M}] + [NZE_r(A) \to X \to \mathfrak{M}],$$

where $NZE_r(A)$ is the complement of $ZE_r(A)$ in $E_r(A)$. To prove that $[NZE_r(A) \to \mathfrak{M}] \in K^{>k}(\mathfrak{M})$, let $Y \hookrightarrow NZE_r(A)$ be a locally closed embedding such that $(E_r(A), A^{\mathrm{fix}})|_Y$ is a clear algebroid.

Consider the embedding of algebras $A^{\mathrm{fix}}|_Y \hookrightarrow A|_Y$. It induces an embedding of commutative algebras $Z(A|_Y) \hookrightarrow Z(A^{\mathrm{fix}}|_Y)$, because $Z(A|_Y) \subset A^{\mathrm{fix}}|_Y$. The algebra $A|_Y$ comes with r tautological idempotent sections, all of which are contained in $Z(A^{\mathrm{fix}}|_Y)$, but at least one of which is not contained in $Z(A|_Y)$. So, by Proposition 1.36(ii), the split central rank of $A^{\mathrm{fix}}|_Y$ is strictly larger than the split central rank of $A|_Y$. The latter is at least as big as k, the split central rank of A, because the split central rank cannot decrease under base extension. This shows that $[Y \to \mathfrak{M}] \in K^{>k}(\mathfrak{M})$ and finishes the proof. □

Corollary 2.6. *The operators E_r, for $r \geqslant 0$ are simultaneously diagonalizable. The common eigenspaces form a family $K^k(\mathfrak{M})$ of subspaces of $K(\mathfrak{M})$ indexed by non-negative integers $k \geqslant 0$, and*

$$K(\mathfrak{M}) = \bigoplus_{k\geqslant 0} K^k(\mathfrak{M}). \tag{12}$$

Moreover, for every $r \geqslant 0$,

$$K^{\geqslant r}(\mathfrak{M}) = \bigoplus_{k\geqslant r} K^k(\mathfrak{M}).$$

Let π_k denote the projection onto $K^k(\mathfrak{M})$. We have

$$E_r\pi_k = r!\, S(k,r)\pi_k,$$

for all $r \geqslant 0$, $k \geqslant 0$.

Proof. First remark that for given r, the numbers $r!\,S(k,r)$ form a monotone increasing sequence of integers.

Then note that the operators E_r pairwise commute: the composition $E_r \circ E_{r'}$ associates to an algebroid (X, A) the stack of pairs (e, e'), where both e and e' are complete families of non-zero orthogonal idempotents in A, the length of e being r, the length of e' being r', and the members of e commuting with the members of e'.

Finally, let us prove that for every k and every r, the $\mathbb{Q}$-vector space $K^{\geqslant k}(\mathfrak{M})$ is a union of finite-dimensional subspaces invariant by E_r.

For this, define $K(\mathfrak{M})_{\leqslant N}$ to be generated as a $\mathbb{Q}$-vector space by stack functions $[X \to \mathfrak{M}]$, where X is a clear algebroid, such that the rank of the vector bundle underlying the algebra $A_X \to X$ is bounded above by N. This is an ascending filtration of $K(\mathfrak{M})$, which is preserved by E_r. Set

$$K^{\geqslant k}(\mathfrak{M}) \cap K(\mathfrak{M})_{\leqslant N} = K^{\geqslant k}(\mathfrak{M})_{\leqslant N}.$$

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Suppose $x = [X \to \mathfrak{M}]$ is a stack function with X a clear algebroid of split central rank k, and let N be the rank of the vector bundle underlying A_X. Note that $k \leqslant N$, because for a commutative algebra, the number of primitive idempotents is bounded by the rank of the underlying vector bundle. We deduce that for $k > N$, we have $K^{\geqslant k}(\mathfrak{M})_{\leqslant N} = 0$.

On the other hand, Theorem 2.5 implies by induction that

$$E_r^i(x) \in \mathbb{Q}\, x + \mathbb{Q}\, E_r(x) + \cdots + \mathbb{Q}\, E_r^{i-1}(x) + K^{\geqslant k+i}(\mathfrak{M}).$$

Applying this for $i = N - k + 1$, we see that

$$E_r(E_r^{N-k}(x)) \in \mathbb{Q}\, x + \mathbb{Q}\, E_r(x) + \cdots + \mathbb{Q}\, E_r^{N-k}(x)$$

and hence that $\mathbb{Q}\, x + \mathbb{Q}\, E_r(x) + \cdots + \mathbb{Q}\, E_r^{N-k}(x)$ is invariant under E_r.

This proves that any $x \in K^{\geqslant k}(\mathfrak{M})$ is contained in a finite-dimensional subspace invariant under E_r. Standard techniques from finite-dimensional linear algebra over $\mathbb{Q}$ now imply the result. □

Remark 2.7. The proof of Theorem 2.5 and its corollary show that the central versions ZE_r of the E_r are also diagonalizable. On the other hand, the ZE_r do not commute with each other, and so are less useful.

COROLLARY 2.8. *For $r \geqslant 1$, we have*

$$\ker E_r = \bigoplus_{k<r} K^k(\mathfrak{M}).$$

In particular, for any $x \in K(\mathfrak{M})$, we have $E_r x = 0$ for $r \gg 0$.

COROLLARY 2.9. *For every $k \geqslant 0$, we have*

$$\pi_k = \sum_{r=k}^{\infty} \frac{s(r,k)}{r!}\, E_r,$$

where the $s(n,k)$ are the Stirling numbers of the first kind. In particular, $\pi_0 = E_0$, and

$$\pi_1 = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r}\, E_r.$$

Proof. We have

$$\mathrm{id} = \sum_{\ell \geqslant 0} \pi_\ell,$$

and hence

$$E_r = \sum_{\ell \geqslant 0} E_r \pi_\ell = \sum_{\ell \geqslant 0} r!\, S(\ell, r)\, \pi_\ell,$$

and therefore

$$\begin{aligned}\sum_{r \geqslant 0} \frac{s(r,k)}{r!}\, E_r &= \sum_{r \geqslant 0} \frac{s(r,k)}{r!} \sum_{\ell \geqslant 0} r!\, S(\ell, r)\, \pi_\ell \\ &= \sum_{\ell \geqslant 0} \Bigg(\sum_{r \geqslant 0} S(\ell, r) s(r,k) \Bigg) \pi_\ell = \sum_{\ell \geqslant 0} \delta_{\ell,k}\, \pi_\ell = \pi_k,\end{aligned}$$

by the inverse relationship between the Stirling numbers of the first and second kind. □

Remark 2.10. The Stirling numbers of the first kind appear in the Taylor expansions of the powers of the logarithm:

$$\sum_{r=k}^{\infty} \frac{s(r,k)}{r!}\, t^r = \frac{1}{k!} \log(1+t)^k.$$

Definition 2.11. Let t be a formal variable. We define the operator

$$\pi_t : K(\mathfrak{M})[t] \longrightarrow K(\mathfrak{M})[t]$$

by the formula

$$\pi_t(\xi) = \sum_k \pi_k(\xi)\, t^k$$

and extending $K(\mathrm{DM})[t]$-linearly. We can write, formally,

$$\pi_t = \sum_k \pi_k\, t^k.$$

Remark 2.12. We have the following convenient formula:

$$\pi_t = \sum_n \binom{t}{n} E_n.$$

It follows from Corollary 2.9, using the identity

$$\sum_k \frac{s(n,k)}{n!} t^k = \binom{t}{n}.$$

Example 2.13. The universal rank 2 vector bundle $\mathrm{GL}_2 \backslash \mathbb{A}^2 \to B\,\mathrm{GL}_2$ and its classifying morphism to $\mathfrak{Vect}$ define a Hall algebra element $[B\,\mathrm{GL}_2 \to \mathfrak{Vect}] \in K(\mathfrak{Vect})$, which we will abbreviate to $[B\,\mathrm{GL}_2]$. To decompose $[B\,\mathrm{GL}_2]$ into its pieces according to (12), we consider the action of E_2, as we have $E_r[B\,\mathrm{GL}_2] = 0$ for all $r > 2$. In fact,

$$E_2[B\,\mathrm{GL}_2] = [BT] \quad \text{and} \quad E_2[BT] = 2[BT],$$

where T is a maximal torus in GL_2. Thus, $\mathbb{Q}[B\,\mathrm{GL}_2] + \mathbb{Q}[BT]$ is a subspace of $K(\mathfrak{Vect})$ invariant under E_2, and the matrix of E_2 acting on this subspace is

$$\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}. \tag{13}$$

This matrix is lower triangular, with different numbers on the diagonal, and hence diagonalizable over $\mathbb{Q}$. In fact, the diagonal entries are $2\,S(1,2) = 0$ and $2\,S(2,2) = 2$, as predicted by Theorem 2.5, with $r = 2$ and $k = 1, 2$. Diagonalizing (13) gives the eigenvectors

(i) $v_1 = [B\,\mathrm{GL}_2] - \frac{1}{2}[BT]$ with eigenvalue 0,
(ii) $v_2 = \frac{1}{2}[BT]$ with eigenvalue 2.

Therefore, we have $v_1 \in K^1(\mathfrak{Vect})$ and $v_2 \in K^2(\mathfrak{Vect})$, and since $[B\,\mathrm{GL}_2] = v_1 + v_2$, we have found the required decomposition of $[B\,\mathrm{GL}_2]$, as noted in Corollary 2.9.

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2.2 The spectrum of semi-simple inertia

The connected semi-simple inertia operator on $K(\mathfrak{M})$ is the $\mathbb{Q}$-linear endomorphism

$$\begin{aligned} I^{\circ,\mathrm{ss}} : K(\mathfrak{M}) &\longrightarrow K(\mathfrak{M}) \\ [X \to \mathfrak{M}] &\longmapsto [I_X^{\circ,\mathrm{ss}} \to X \to \mathfrak{M}]. \end{aligned}$$

Here, $I_X^{\circ,\mathrm{ss}} = A_X^{\times,\mathrm{ss}}$ denotes the semi-simple algebroid inertia of the algebroid X, see Remark 1.58. Note that $I^{\circ,\mathrm{ss}}$ respects the scissor and bundle relations defining $K(\mathfrak{M})$ and is linear over $K(\mathrm{DM})$, because passing to connected inertia commutes with inert pullbacks.

Note that $I^{\circ,\mathrm{ss}}$ commutes with E_r for every r. Both compositions $E_r \circ I^{\circ,ss}$ and $I^{\circ,\mathrm{ss}} \circ E_r$ associate to an algebroid (X, A) the stack of pairs (e, a), where a is a semi-simple unit in A and e is a labelled complete set of r orthogonal idempotents in A, all commuting with a. In particular, $I^{\circ,\mathrm{ss}}$ preserves the filtration of $K(\mathfrak{M})$ by split central rank.

In the following, we will need an auxiliary construction. Fix $k \geqslant 0$. For an algebroid (X, A), let $Y = E_k(A)$ and (Y, B) be the algebroid structure on Y (see 1.60). The algebra B comes with a complete set of k orthogonal idempotents, and we divide $B^{\times,\mathrm{ss}}$ by the induced action of $\mathbb{G}_m^k$, as treated in Proposition 1.61, to get $\widetilde{B}^{\times,\mathrm{ss}}$. Let us write

$$\widetilde{I}_k^{\circ,\mathrm{ss}} X = \widetilde{B}^{\times,\mathrm{ss}}.$$

We may think of $\widetilde{I}_k^{\circ,\mathrm{ss}} X$ as the stack of triples $(x, e, [a])$, where x is a point of X, $e = (e_1, \ldots, e_k)$ is a labelled complete set of orthogonal idempotents in A_x, and $[a]$ is an equivalence class of semi-simple units in $A_x^{e_1,\ldots,e_k}$, where $a \sim \sum_{i=1}^k \lambda_i e_i\, a$, for $\lambda_i \in \mathscr{O}_X|_x$ (although this neglects the sheafification inherent in the quotient construction). We have structure morphisms

$$\widetilde{I}_k^{\circ,\mathrm{ss}} X \longrightarrow E_k X \longrightarrow X.$$

This construction gives rise to a $K(\mathrm{DM})$-linear operator

$$\widetilde{I}_k^{\circ,\mathrm{ss}} : K(\mathfrak{M}) \longrightarrow K(\mathfrak{M}).$$

Recall that $q = [\mathbb{A}^1]$. The composition $I^{\circ,\mathrm{ss}} \circ E_r$ is divisible by $(q-1)^r$, as shown by the following proposition.

Proposition 2.14. *For every $k \geqslant 0$, we have*

$$I^{\circ,\mathrm{ss}} \circ E_k = (q-1)^k\, \widetilde{I}_k^{\circ,\mathrm{ss}},$$

as $K(\mathrm{DM})$-linear operators on $K(\mathfrak{M})$.

Proof. We have

$$[B^{\times,\mathrm{ss}}] = (q-1)^k [\widetilde{B}^{\times,\mathrm{ss}}],$$

because of the bundle relations in $K(\mathfrak{M})$, see Proposition 1.61. □

Corollary 2.15. *The map which $I^{\circ,\mathrm{ss}}$ induces on the subquotient $K^{\geqslant k}/K^{>k}(\mathfrak{M})$ is divisible by $(q-1)^k$:*

$$I^{\circ,\mathrm{ss}}|_{K^{\geqslant k}/K^{>k}(\mathfrak{M})} = \frac{1}{k!}(q-1)^k\, \widetilde{I}_k^{\circ,\mathrm{ss}}|_{K^{\geqslant k}/K^{>k}(\mathfrak{M})}.$$

Proof. This is because on $K^{\geqslant k}/K^{>k}(\mathfrak{M})$, the operator E_k acts as multiplication by $k!$. □

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We will use as scalars the localization of $\mathbb{Q}[q]$ at the maximal ideal $(q-1)$, denoted by $\mathbb{Q}[q]_{(q-1)}$, thus inverting all rational polynomials in q that do not vanish at $q=1$. We extend scalars on $K(\mathfrak{M})$ as well, and consider

$$K(\mathfrak{M})_{(q-1)} = \mathbb{Q}[q]_{(q-1)} \cdot K(\mathfrak{M}) \subset K(\mathfrak{M})(q) = K(\mathfrak{M}) \otimes_{\mathbb{Q}[q]} \mathbb{Q}(q).$$

Note that this definition ensures that $K(\mathfrak{M})_{(q-1)}$ is $(q-1)$-torsion free. The direct sum decomposition (12) extends to $K(\mathfrak{M})_{(q-1)}$, and the operator $I^{\circ,\mathrm{ss}}$ extends to a $\mathbb{Q}[q]_{(q-1)}$-linear operator

$$I^{\circ,\mathrm{ss}} : K(\mathfrak{M})_{(q-1)} \longrightarrow K(\mathfrak{M})_{(q-1)}.$$

For a partition $\lambda \vdash n$, we denote its length (i.e., the number of its parts) by $\ell(\lambda)$. We define

$$\mathfrak{Q}_\lambda = \prod_{i\in\lambda}(q^i - 1).$$

This is a polynomial in q of degree n that vanishes to order $\ell(\lambda)$ at $q=1$. We also define

$$\widetilde{\mathfrak{Q}}_\lambda = k! \prod_{i\in\lambda} \frac{q^i - 1}{q-1}.$$

This is a polynomial in q that is invertible in $\mathbb{Q}[q]_{(q-1)}$.

Theorem 2.16. *The operator*

$$\widetilde{I}_k^{\circ,\mathrm{ss}} : K^{\geqslant k}/K^{>k}(\mathfrak{M})_{(q-1)} \longrightarrow K^{\geqslant k}/K^{>k}(\mathfrak{M})_{(q-1)}$$

is diagonalizable. Its eigenvalue spectrum consists of all $\widetilde{\mathfrak{Q}}_\lambda$, for partitions λ of length $\ell(\lambda) = k$.

Proof. We will fix k and work throughout in the subquotient $K^{\geqslant k}/K^{>k}(\mathfrak{M})_{(q-1)}$, restricting all operators tacitly to this subquotient. Note that, as a $\mathbb{Q}[q]_{(q-1)}$-module, $K^{\geqslant k}/K^{>k}(\mathfrak{M})_{(q-1)}$ is isomorphic to $K^k(\mathfrak{M})_{(q-1)}$ and is hence $(q-1)$-torsion free.

We order partitions of length k by *divisibility*. If λ and μ are partitions with $\ell(\lambda)=k$ and $\ell(\mu)=k$, we write $\lambda \mid \mu$ if there exists a permutation σ of $\underline{k}$ such that $\lambda_i \mid \mu_{\sigma(i)}$ for all $i=1,\ldots,k$. This is a partial ordering on the partitions of length k. We write

$$K^{\geqslant\lambda}(\mathfrak{M})_{(q-1)} \tag{14}$$

for the $\mathbb{Q}[q]_{(q-1)}$-subspace of $K^{\geqslant k}/K^{>k}(\mathfrak{M})_{(q-1)}$ generated by clear stack functions of central type divisible by λ.

We will prove the following.

(i) The operator $\widetilde{I}_k^{\circ,\mathrm{ss}}$ preserves the filtration (14) by divisibility of partitions.

(ii) On the quotient $K^{\geqslant\lambda}(\mathfrak{M})_{(q-1)}/K^{>\lambda}(\mathfrak{M})_{(q-1)}$, the operator $\widetilde{I}_k^{\circ,\mathrm{ss}}$ acts as multiplication by $\widetilde{\mathfrak{Q}}_\lambda$.

(iii) The operator $\widetilde{I}_k^{\circ,\mathrm{ss}}$ is locally finite.

These facts will imply the claims concerning diagonalizability of $\widetilde{I}_k^{\circ,\mathrm{ss}}$. This is because for a lower triangular matrix with distinct diagonal entries over a discrete valuation ring to be diagonalizable, it suffices that the differences between the diagonal entries are units. The latter condition is satisfied, because if $\lambda \mid \mu$, then $\widetilde{Q}_\mu - \widetilde{Q}_\lambda$ does not vanish at $q=1$. (This argument does not apply

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directly, because our eigenvalues are not linearly ordered, but only partially. Nevertheless, the conclusion remains true in this larger generality.)

Let us fix a partition λ of length k and consider a clear stack function $X \to \mathfrak{M}$ of central type λ with algebra $A \to X$. Abbreviate the induced element of $K^{\geqslant k}/K^{>k}(\mathfrak{M})$ by $[X]$. Denote the central rank of X by n, so that $\lambda \vdash n$. As A has k central idempotents, $E_k X \to X$ has $k!$ canonical sections, each given by a labelling σ of these k idempotents. Denote the images of these sections by $\{X_\sigma\}$. By the proof of Theorem 2.5, the algebroid $E_k X$ can be stratified as

$$E_k X = \bigsqcup_\sigma X_\sigma \sqcup \bigsqcup_\tau Y_\tau,$$

where the Y_τ are clear algebroids of split central rank larger than k. The part of $\widetilde{I}_k^{\circ,\mathrm{ss}} X \to E_k X$ lying over Y_τ then also has split central rank larger than k. Hence, when calculating $\widetilde{I}_k^{\circ,\mathrm{ss}}[X]$, we can discard all Y_τ. Every X_σ is isomorphic to X, and so we will fix a labelling σ and replace X_σ by X in the following arguments, remembering to multiply the final result by $k!$.

We need to consider $A^{\times,\mathrm{ss}}$ and its quotient $\widetilde{A}^{\times,\mathrm{ss}}$. We write

$$A^{\times,\mathrm{ss}} = Z^{\times,\mathrm{ss}} \sqcup NZ^{\times,\mathrm{ss}}, \tag{15}$$

where $Z \subset A$ is the centre of A (which is a strict subbundle and hence a closed substack) and NZ is its complement. We start by examining $Z^{\times,\mathrm{ss}}$ and its quotient $\widetilde{Z}^{\times,\mathrm{ss}} = Z^{\times,\mathrm{ss}}/\mathbb{G}_m^k$. Note that $Z^{\times,\mathrm{ss}}$ and $\widetilde{Z}^{\times,\mathrm{ss}}$ are pullbacks from the coarse Deligne–Mumford stack $\overline{X}$ of X and hence are inert over X, and their algebroid structures are hence the canonical algebroid structures as inert X-stacks.

CLAIM. *In $K^{\geqslant k}/K^{>k}(\mathfrak{M})$, we have*

$$[\widetilde{Z}^{\times,\mathrm{ss}}] = \frac{1}{k!}\widetilde{\mathfrak{Q}}_\lambda(q)\,[X].$$

As in the proof of Theorem 2.5, let $\widetilde{X} \to X$ be a connected Galois cover with Galois group Γ acting on the set $\underline{n}$ such that

$$\widetilde{X} \times_\Gamma \underline{n} \xrightarrow{\simeq} PZE(A).$$

We get induced isomorphisms

$$\widetilde{X} \times_\Gamma \mathbb{A}^n \xrightarrow{\simeq} \pi_* \mathscr{O}_{PZE(A)}$$

and

$$\widetilde{X} \times_\Gamma \mathbb{G}_m^n \xrightarrow{\simeq} (\pi_* \mathscr{O}_{PZE(A)})^\times.$$

By Proposition 1.35, we have a surjective closed immersion

$$(\pi_* \mathscr{O}_{PZE(A)})^{\times,\mathrm{strat}} \longrightarrow Z^{\times,\mathrm{ss}}.$$

It follows that we have a surjective closed immersion

$$(\widetilde{X} \times_\Gamma \mathbb{G}_m^n)^{\mathrm{strat}} \longrightarrow Z^{\times,\mathrm{ss}},$$

and by passing to the quotient, another surjective closed immersion

$$((\widetilde{X} \times_\Gamma \mathbb{G}_m^n)/\mathbb{G}_m^k)^{\mathrm{strat}} \longrightarrow \widetilde{Z}^{\times,\mathrm{ss}}.$$

So, in $K^{\geqslant k}/K^{>k}(\mathfrak{M})$, we can replace $\widetilde{Z}^{\times,\mathrm{ss}}$ by $(\widetilde{X} \times_\Gamma \mathbb{G}_m^n)/\mathbb{G}_m^k$.

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Our labelling σ of the central idempotents in A_X corresponds to a labelling of the orbits of Γ on $\underline{n}$ and a labelling of the parts of $\lambda \vdash n$. Let us denote these orbits by $I_1, \ldots, I_k$ and λ by $(\lambda_1, \ldots, \lambda_k)$ such that $\lambda_i = |I_i|$ for $i = 1, \ldots, k$. We write $\mathbb{P}(\mathbb{A}^\lambda)$ for the product of projective spaces $\mathbb{P}(\mathbb{A}^{\lambda_1}) \times \cdots \times \mathbb{P}(\mathbb{A}^{\lambda_k})$. Moreover, for a sequence of subsets $J_i \subset I_i$, we write $\mathbb{P}^*(J_1, \ldots, J_k) \subset \mathbb{P}(\mathbb{A}^\lambda)$ for the locally closed subspace defined by the entries in $J_1 \cup \cdots \cup J_k$ being non-zero and all others being zero. We have

$$\mathbb{P}(\mathbb{A}^\lambda) = \bigsqcup_{\substack{(J_1,\ldots,J_k)\in \\ \mathfrak{P}(I_1)\times\cdots\times\mathfrak{P}(I_k)}} \mathbb{P}^*(J_1, \ldots, J_k),$$

where the disjoint union is over all sequences of subsets $J_i \subset I_i$.

The group Γ acts linearly on $\mathbb{P}(\mathbb{A}^\lambda)$, respecting this stratification (although not the individual strata), and we have

$$(\widetilde{X} \times_\Gamma \mathbb{G}_m^n)/\mathbb{G}_m^k = \widetilde{X} \times_\Gamma \mathbb{P}^*(I_1, \ldots, I_k).$$

Moreover,

$$\begin{aligned}
\widetilde{X} \times_\Gamma \mathbb{P}(\mathbb{A}^\lambda) &= \widetilde{X} \times_\Gamma \bigsqcup_{\mathfrak{P}(I_1)\times\cdots\times\mathfrak{P}(I_k)} \mathbb{P}^*(J_1, \ldots, J_k) \\
&= \bigsqcup_{\mathfrak{P}(I_1)\times\cdots\times\mathfrak{P}(I_k)/\Gamma} \widetilde{X} \times_{\mathrm{Stab}(J_1,\ldots,J_n)} \mathbb{P}^*(J_1, \ldots, J_k) \\
&= \widetilde{X} \times_\Gamma \mathbb{P}^*(I_1, \ldots, I_k) \\
&\quad \sqcup \bigsqcup_{\substack{\mathfrak{P}(I_1)\times\cdots\times\mathfrak{P}(I_k)/\Gamma \\ \mathrm{Stab}(J_1,\ldots,J_n)\subsetneq\Gamma}} \widetilde{X} \times_{\mathrm{Stab}(J_1,\ldots,J_n)} \mathbb{P}^*(J_1, \ldots, J_k).
\end{aligned}$$

Every subgroup $\Gamma' \subsetneq \Gamma$ that is the stabilizer of a sequence $(J_1, \ldots, J_k)$ has more than k orbits on $\underline{n}$. As in the proof of Theorem 2.5, this implies that $\widetilde{X}/\Gamma'$ is in $K^{>k}(\mathfrak{M})$. The same is then true for $\widetilde{X} \times_{\Gamma'} \mathbb{P}^*(J_1, \ldots, J_k)$, as the projection $\widetilde{X} \times_{\Gamma'} \mathbb{P}^*(J_1, \ldots, J_k) \to \widetilde{X}/\Gamma'$ is inert (being the pullback of a corresponding morphism of course Deligne–Mumford stacks). We deduce that in $K^{\geqslant k}/K^{>k}(\mathfrak{M})$, we have

$$\begin{aligned}
[\widetilde{Z}^{\times,\mathrm{ss}}] &= [(\widetilde{X} \times_\Gamma \mathbb{G}_m^n)/\mathbb{G}_m^k] = [\widetilde{X} \times_\Gamma \mathbb{P}^*(I_1, \ldots, I_k)] = [\widetilde{X} \times_\Gamma \mathbb{P}(\mathbb{A}^\lambda)] \\
&= [\mathbb{P}(\mathbb{A}^\lambda)]\,[X] = \frac{1}{k!}\widetilde{\mathfrak{Q}}_\lambda(q)\,[X]. \qquad (16)
\end{aligned}$$

In the last step, we used the bundle relations in $K(\mathfrak{M})$. The bundle $\widetilde{X} \times_\Gamma \mathbb{P}(\mathbb{A}^\lambda)$ is a product of projective bundles associated to vector bundles, whose structure groups are special (as they are general linear groups). This finishes the proof of this claim.

Now consider a locally closed embedding $Y \hookrightarrow NZ^{\times,ss}/\mathbb{G}_m^k$ such that Y is a clear algebroid.

Claim. *The central type of Y strictly divides λ.*

Over Y, we then consider the inclusion of commutative algebra bundles $ZA_X|_Y \hookrightarrow ZA_Y$. By Proposition 1.36, the split rank of ZA_Y (which is the split central rank of Y) is at least as large as the split rank of $ZA_X|_Y$, which, in turn, is at least as large as the split rank of ZA_X, which is k. Since we are working modulo $K^{>k}(\mathfrak{M})$, we may assume that the split central rank of Y is k and hence that the split rank of ZA_Y and of $ZA_X|_Y$ are both equal to k. Consider the correspondence Q which we used in the proof of Proposition 1.36:

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$$\begin{array}{ccc} Q & \hookrightarrow & PZE(A_Y) \\ \downarrow & & \\ PZE(A_X|_Y). & & \end{array}$$

All three stacks in this diagram are representable finite étale covers of Y. By assumption, both $PZE(A_Y)$ and $PZE(A_X|_Y)$ have k connected components. This implies that the horizontal inclusion in the diagram is an isomorphism and that we have a surjective representable finite étale cover

$$PZE(A_Y) \twoheadrightarrow PZE(A_X|_Y). \tag{17}$$

Since $PZE(A_X|_Y)$ and $PZE(A_X)$ have the same number of components, the degrees of these components are equal as well, which means that the central type of $A_X|_Y$ is equal to the central type of A_X, which is λ. The existence of (17) then implies that λ divides the central type of A_Y.

The surjection (17) is not an isomorphism, because otherwise, by Proposition 1.36, we would have a surjection $ZA_Y^{\times,\mathrm{ss}} \to ZA_X^{\times,\mathrm{ss}}|_Y$; but this would force the tautological section class of $ZA_Y^{\times,\mathrm{ss}}$, given by the structure map $Y \to A_X^{\times,\mathrm{ss}}/\mathbb{G}_m^k$, to be central in A_X (at least pointwise), which it is not.

This shows that the central type of Y *strictly* divides λ and proves our second claim.

We have thus completed the proof of (i) and (ii). For the local finiteness of $\widetilde{I}_k^{\circ,\mathrm{ss}}$, we proceed as in the proof of Corollary 2.6. Every time we apply $\widetilde{I}_k^{\circ,\mathrm{ss}}$, we produce only clear algebroids whose central type is a multiple of λ, but as we can bound the central rank by the rank, which does not increase by applying $\widetilde{I}_k^{\circ,\mathrm{ss}}$, after finitely many steps, this process stops. □

COROLLARY 2.17. *The operator $I^{\circ,\mathrm{ss}} : K(\mathfrak{M})_{(q-1)} \to K(\mathfrak{M})_{(q-1)}$ is diagonalizable. Its eigenvalue spectrum consists of the $\mathcal{Q}_\lambda \in \mathbb{Q}[q]$, for all partitions λ. Denote the eigenspace corresponding to the eigenvalue $\mathcal{Q}_\lambda$ by $K^\lambda(\mathfrak{M})_{(q-1)}$. We have*

$$K^k(\mathfrak{M})_{(q-1)} = \bigoplus_{\ell(\lambda)=k} K^\lambda(\mathfrak{M})_{(q-1)}.$$

Example 2.18. Consider, as in Example 2.13, the stack function of rank 2 vector bundles. It defines an element $[B\,\mathrm{GL}_2]$ of $K^{\geqslant 1}/K^{>1}(\mathfrak{Vect})$, which we are going to decompose into its eigencomponents with respect to the operator $\widetilde{I}_1^{\circ,\mathrm{ss}}$.

The stack function $[B\,\mathrm{GL}_2]$ is clear; its central rank is 1. The decomposition (15) is given in this case as

$$I^{\circ,\mathrm{ss}}_{B\,\mathrm{GL}_2} = \Delta/\mathrm{GL}_2 \sqcup T^*/N,$$

where Δ is the central torus of GL_2 and $T^* = T \backslash \Delta$. Also, N is the normalizer of T in GL_2. We get the corresponding decomposition

$$\begin{aligned} \widetilde{I}_1^{\circ,\mathrm{ss}} B\,\mathrm{GL}_2 &= (\Delta/\mathbb{G}_m)/\mathrm{GL}_2 \sqcup (T^*/\mathbb{G}_m)/N \\ &= B\,\mathrm{GL}_2 \sqcup \widetilde{T}^*/N, \end{aligned}$$

where $\widetilde{T}^* = T^*/\mathbb{G}_m$, and we have

$$\widetilde{I}_1^{\circ,\mathrm{ss}}[B\,\mathrm{GL}_2] = [B\,\mathrm{GL}_2] + [\widetilde{T}^*/N].$$

Note that $\widetilde{T}^*/N$ is not a strict algebroid. In fact, let $T' \subset T^*$ be the closed subscheme consisting of elements of trace zero and write $T^* = T' \sqcup T^{**}$. Then (at least if $2 \in R^\times$) we have $T'/\mathbb{G}_m = \operatorname{Spec} R$, and $[\widetilde{T}^*/N] = [BN] + [\widetilde{T}^{**}/N]$, and BN is not a strict algebroid, as N is not connected. But $\widetilde{T}^*/N$ is a clear algebroid. Its connected inertia stack is $\widetilde{T}^* \times T/N$, and its associated coarse Deligne–Mumford stack is $\widetilde{T}^*/\mathbb{Z}_2$, which is only generically a scheme. The central rank of $\widetilde{T}^*/N$ is 2, and the split central rank 1.

Now we consider $\widetilde{I}_1^{\circ,\mathrm{ss}}[\widetilde{T}^*/N]$. We start by noting that all connected inertia of $\widetilde{T}^*/N$ is central. Hence, modulo $K^{>1}(\mathfrak{Vect})$, we have

$$\widetilde{I}_1^{\circ,\mathrm{ss}}[\widetilde{T}^*/N] = (q+1)[\widetilde{T}^*/N],$$

by (16).

We see that $\mathbb{Q}[q]_{(q-1)}[B\,\mathrm{GL}_2] + \mathbb{Q}[q]_{(q-1)}[\widetilde{T}^*/N]$ is invariant under $\widetilde{I}_1^{\circ,\mathrm{ss}}$, and the matrix of $\widetilde{I}_1^{\circ,\mathrm{ss}}$ on this subspace is

$$\begin{pmatrix} 1 & 0 \\ 1 & q+1 \end{pmatrix}.$$

This matrix is lower triangular, and the differences between the scalars on the diagonal are all invertible in $\mathbb{Q}[q]_{(q-1)}$. Therefore, it is diagonalizable over $\mathbb{Q}[q]_{(q-1)}$. Diagonalizing, we get the following eigenvectors modulo $K^{>1}(\mathfrak{Vect})_{(q-1)}$:

(i) $\overline{v}_{(1)} = [B\,\mathrm{GL}_2] - (1/q)[\widetilde{T}^*/N]$;

(ii) $\overline{v}_{(2)} = (1/q)[\widetilde{T}^*/N]$.

To get the actual eigenvectors, we project into $K^1(\mathfrak{Vect})_{(q-1)}$. We have:

(i) $\pi_1[B\,\mathrm{GL}_2] = [B\,\mathrm{GL}_2] - \frac{1}{2}[BT]$;

(ii) $\pi_1[\widetilde{T}^*/N] = [\widetilde{T}^*/N] - \frac{1}{2}(q-2)[BT]$

and hence

(i) $v_{(1)} = [B\,\mathrm{GL}_2] - (1/q)[\widetilde{T}^*/N] - (1/q)[BT]$;

(ii) $v_{(2)} = (1/q)[\widetilde{T}^*/N] - ((q-2)/2q)[BT]$.

If we add

(iii) $v_{(1,1)} = \frac{1}{2}[B\,\mathrm{GL}_2]$

we get the spectral decomposition $[B\,\mathrm{GL}_2] = v_{(1)} + v_{(2)} + v_{(1,1)}$ of $[B\,\mathrm{GL}_2]$, with respect to the operator $I^{\circ,\mathrm{ss}}$. This is, of course, the same as the spectral decomposition with respect to I^{ss}, which we computed in the introduction (after applying the bundle relations).

Remark 2.19. If we are willing to invert $(q-1)$, we can prove the diagonalizability of $I^{\circ,\mathrm{ss}} = I^{\mathrm{ss}}$ entirely within the context of strict algebroids. In fact, we can generalize the calculation in the introduction to accomplish this.

2.3 Graded structure of multiplication

We will now assume that $\mathfrak{M}$ admits all direct sums. Then we can define a commutative product on $K(\mathfrak{M})$ by

$$[X \to \mathfrak{M}] \cdot [Y \to \mathfrak{M}] = [X \times Y \to \mathfrak{M} \times \mathfrak{M} \xrightarrow{\oplus} \mathfrak{M}].$$

With this product, $K(\mathfrak{M})$ becomes a commutative $K(\mathrm{DM})$-algebra with unit $1 = [\mathrm{Spec}\, R \xrightarrow{0} \mathfrak{M}]$.

Proposition 2.20. *For $x, y \in K(\mathfrak{M})$, we have*

$$I^{\circ,\mathrm{ss}}(x \cdot y) = I^{\circ,\mathrm{ss}}(x) \cdot I^{\circ,\mathrm{ss}}(y).$$

Proof. This follows immediately from the fact that, for any two algebroids X, Y, we have $A_{X\times Y}^{\times,\mathrm{ss}} = A_X^{\times,\mathrm{ss}} \times A_Y^{\times,\mathrm{ss}}$ as algebroids over $X \times Y$. □

Denote the disjoint union of two partitions λ and μ by $\lambda + \mu$.

Corollary 2.21. *We have $K^{\lambda}(\mathfrak{M})_{(q-1)} \cdot K^{\mu}(\mathfrak{M})_{(q-1)} \subset K^{\lambda+\mu}(\mathfrak{M})_{(q-1)}$ and therefore also $K^{k}(\mathfrak{M})_{(q-1)} \cdot K^{\ell}(\mathfrak{M})_{(q-1)} \subset K^{k+\ell}(\mathfrak{M})_{(q-1)}$.*

So the $\mathbb{Q}[q]_{(q-1)}$-module

$$K(\mathfrak{M})_{(q-1)} = \bigoplus_{k\geqslant 0} K^{k}(\mathfrak{M})_{(q-1)}$$

is a graded $\mathbb{Q}[q]_{(q-1)}$-algebra with respect to the commutative product on $K(\mathfrak{M})_{(q-1)}$. We will prove next that this fact is true for $K(\mathfrak{M})$ itself.

Proposition 2.22. *For any $x, y \in K(\mathfrak{M})$ and any $p \geqslant 0$, we have*

$$E_p(x\cdot y) = \sum_{n,m} \begin{bmatrix} p \\ n, m \end{bmatrix} E_n(x)\cdot E_m(y).$$

Here $\left[\begin{smallmatrix} p \\ n,m \end{smallmatrix}\right]$ is the number of ways the set $\underline{p}$ can bewritten as the union of a subset of order n and a subset of order m.

Proof. Consider stack functions $X \to \mathfrak{M}$ and $Y \to \mathfrak{M}$. Then $E_p(X\times Y)$ is the stack of pairs (e, f), where $e = (e_\rho)_{\rho\in\underline{p}}$ is a complete set of orthogonal idempotents in A_X and $f = (f_\rho)_{\rho\in\underline{p}}$ is a complete set of orthogonal idempotents in A_Y such that for every $\rho = 1, \dots, p$, at least one of the two idempotents e_ρ, f_ρ is non-zero.

For every pair of strictly monotone maps $\underline{n} \hookrightarrow \underline{p}$, $\underline{m} \hookrightarrow \underline{p}$ whose images cover $\underline{p}$, we get a morphism of stack functions $E_n(X) \times E_m(Y) \to E_p(X \times Y)$ by mapping a pair of complete sets of orthogonal idempotents (e', f'), where $e' = (e'_\nu)_{\nu\in\underline{n}}$ and $f' = (f'_\mu)_{\mu\in\underline{m}}$, to the pair (e, f), defined by

$$e_\rho = \sum_{\nu\mapsto\rho} e'_\nu \quad \text{and} \quad f_\rho = \sum_{\mu\mapsto\rho} f'_\mu.$$

(As the maps $\underline{n} \to \underline{p}$ and $\underline{m} \to \underline{p}$ are injective, all these sums have either zero or one summand.)

Each of the morphisms $E_n(X)\times E_m(X) \to E_p(X\times Y)$ is an isomorphism onto a locally closed substack, because the locus of vanishing for an idempotent is closed. Moreover, the images of these morphisms are disjoint and from a cover. There are $\left[\begin{smallmatrix} p \\ n,m \end{smallmatrix}\right]$ of them. □

Corollary 2.23. *If $x \in K^{k}(\mathfrak{M})$ and $y \in K^{\ell}(\mathfrak{M})$, then $x \cdot y \in K^{k+\ell}(\mathfrak{M})$.*

Proof. We have

$$\begin{aligned}\pi_t(x\cdot y) &= \sum_p \binom{t}{p} E_p(x\cdot y)\\ &= \sum_p \binom{t}{p} \sum_{n,m} \begin{bmatrix} p \\ n, m \end{bmatrix} E_n(x)\cdot E_m(y)\end{aligned}$$

$$= \sum_{n,m} \left(\sum_p \begin{bmatrix} p \\ n, m \end{bmatrix} \binom{t}{p} \right) E_n(x) \cdot E_m(y)$$
$$= \sum_{n,m} \binom{t}{n} \binom{t}{m} E_n(x) \cdot E_m(x)$$
$$= \pi_t(x) \cdot \pi_t(y).$$

The step from line 3 to line 4 uses Proposition 2.24, below. □

A combinatorial lemma. Let $p \geqslant 0$.

For a non-negative integer n and a p-tuple of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_p)$, we define

$$\begin{bmatrix} n \\ \lambda \end{bmatrix} = \begin{bmatrix} n \\ \lambda_1, \dots, \lambda_p \end{bmatrix} \tag{18}$$

to be the number of indexed covers of $\underline{n}$ by subsets $S_1, \dots, S_p$ of cardinalities $\lambda_1, \dots, \lambda_p$. The non-negative integer $\left[\begin{smallmatrix} n \\ \lambda \end{smallmatrix}\right]$ vanishes, unless $\lambda_\rho \leqslant n$, for all$\rho = 1, \dots, p$ and $n \leqslant |\lambda|$, where $|\lambda| = \sum_\rho \lambda_\rho$.

We could not find the following combinatorial property in the literature.

Proposition 2.24. *For every p-tuple of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_p)$, we have*

$$\binom{t}{\lambda_1} \cdots \binom{t}{\lambda_p} = \sum_n \begin{bmatrix} n \\ \lambda \end{bmatrix} \binom{t}{n}.$$

Proof. Let $x_1, \dots, x_p$ be formal variables. We will prove that

$$\sum_\lambda \binom{t}{\lambda_1} \cdots \binom{t}{\lambda_p} x_1^{\lambda_1} \dots x_p^{\lambda_p} = \sum_\lambda \sum_n \begin{bmatrix} n \\ \lambda \end{bmatrix} \binom{t}{n} x_1^{\lambda_1} \dots x_p^{\lambda_p},$$

by proving that both sides of this equation are equal to

$$\prod_{i=1}^p (1 + x_i)^t.$$

On the one hand, we have

$$\prod_{i=1}^p (1 + x_i)^t = \prod_{i=1}^p \sum_n \binom{t}{n} x_i^n$$
$$= \sum_{\lambda_1, \dots, \lambda_p} \binom{t}{\lambda_1} \cdots \binom{t}{\lambda_p} x_1^{\lambda_1} \dots x_p^{\lambda_p}.$$

On the other hand, we have

$$\prod_{i=1}^p (1 + x_i)^t = \left(1 + \prod_{i=1}^p (1 + x_i) - 1 \right)^t$$
$$= \sum_n \binom{t}{n} \left(\prod_{i=1}^p (1 + x_i) - 1 \right)^n$$

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$$\begin{aligned}
&= \sum_n \binom{t}{n} \sum_j (-1)^j \binom{n}{j} \prod_{i=1}^p (1+x_i)^{n-j} \\
&= \sum_n \binom{t}{n} \sum_j (-1)^j \binom{n}{j} \prod_{i=1}^p \sum_\ell \binom{n-j}{\ell} x_i^\ell \\
&= \sum_n \binom{t}{n} \sum_j (-1)^j \binom{n}{j} \sum_{\lambda_1,\ldots,\lambda_p} \binom{n-j}{\lambda_i} x_1^{\lambda_1} \ldots x_p^{\lambda_p} \\
&= \sum_n \binom{t}{n} \sum_\lambda \Big(\sum_j (-1)^j \binom{n}{j} \binom{n-j}{\lambda_i} \Big) x^\lambda \\
&= \sum_n \binom{t}{n} \sum_\lambda \begin{bmatrix} n \\ \lambda_1, \ldots, \lambda_p \end{bmatrix} x^\lambda \\
&= \sum_\lambda \sum_n \begin{bmatrix} n \\ \lambda \end{bmatrix} \binom{t}{n} x^\lambda .
\end{aligned}$$

Here we have used the obvious inclusion–exclusion property satisfied by the covering numbers. □

3. The order filtration

The Hall algebra. Let $\mathfrak{M}$ be a linear algebraic stack admitting direct sums and direct summands, i.e., assume that $\mathfrak{M}$ is Karoubian (Remark 1.63). To define the Hall product, we need an additional structure on $\mathfrak{M}$. This is a linear algebraic substack $\mathfrak{M}^{(2)}$ of the stack of all sequences $M' \to M \to M''$ in $\mathfrak{M}$ such that for every R-scheme S, the fibre $\mathfrak{M}^{(2)}(S)$ defines the structure of an exact category on $\mathfrak{M}(S)$. The stack $\mathfrak{M}^{(2)}$ comes with a diagram of morphisms of linear algebraic stacks

$$\begin{array}{ccc}
\mathfrak{M}^{(2)} & \xrightarrow{\ b\ } & \mathfrak{M} \\
{\scriptstyle a_1 \times a_2}\big\downarrow & & \\
\mathfrak{M} \times \mathfrak{M} & &
\end{array}$$

where $a_1, a_2, b : \mathfrak{M}^{(2)} \to \mathfrak{M}$ are the projections of the sequence $M' \to M \to M''$ onto the objects M', M'', M, respectively. We require further that the morphism $a_1 \times a_2 : \mathfrak{M}^{(2)} \to \mathfrak{M} \times \mathfrak{M}$ is of finite type. Note that the morphism $b : \mathfrak{M}^{(2)} \to \mathfrak{M}$ is a representable morphism of algebroids, by the properties of exact categories.

We call such an $\mathfrak{M}$ an *exact linear algebraic stack.*

Example 3.1. The linear stacks $\mathfrak{Coh}_X$, $\mathfrak{Vect}$, and $\mathfrak{Rep}_Q$ of Examples 1.11, 1.12, and 1.13 satisfy these axioms. For $\mathfrak{Coh}_X$, see [Bri12, § 4.1].

In each case, the exact structure is given by all short exact sequences. Note that the categories $\mathfrak{M}(S)$ are not abelian, as the cokernel of a homomorphisms of flat sheaves is not necessarily flat.

Throughout the following discussion, we fix an exact linear algebraic stack $\mathfrak{M}$ and let $\mathfrak{A} \to \mathfrak{M}$ be its universal endomorphism algebra, as in § 2.

We have the following structures on $K(\mathfrak{M})$.

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(i) *Module structure.* The action of $K(\mathrm{DM})$ on $K(\mathfrak{M})$, given by $[Z]\cdot[X \to \mathfrak{M}] = [Z \times X \to X \to \mathfrak{M}]$, which turns $K(\mathfrak{M})$ into a $K(\mathrm{DM})$-module.

(ii) *Multiplication.* The commutative multiplication given by

$$[X \to \mathfrak{M}]\cdot[Y \to \mathfrak{M}] = [X \times Y \to \mathfrak{M} \times \mathfrak{M} \xrightarrow{\oplus} \mathfrak{M}].$$

(iii) *Hall product.* The Hall product of the stack functions $[X \to \mathfrak{M}]$ and $[Y \to \mathfrak{M}]$, which is defined by first constructing the fibred product

$$\begin{CD} X * Y @>>> \mathfrak{M}^{(2)} \\ @VVV @VV{a_1 \times a_2}V \\ X \times Y @>>> \mathfrak{M} \times \mathfrak{M} \end{CD}$$

and then setting

$$[X \to \mathfrak{M}] * [Y \to \mathfrak{M}] = [X * Y \longrightarrow \mathfrak{M}^{(2)} \xrightarrow{b} \mathfrak{M}].$$

The multiplication is associative and commutative; the Hall product is associative. The unit with respect to both multiplications is given by the 0-object of $\mathfrak{M}$:

$$1 = [\operatorname{Spec} R \xrightarrow{0} \mathfrak{M}].$$

We will refer to $K(\mathfrak{M})$ as the *Hall algebra* of $\mathfrak{M}$.

We define, inductively, $\mathfrak{M}^{(n)}$, for $n \geqslant 3$, by the following cartesian diagram.

$$\begin{CD} \mathfrak{M}^{(n)} @>{p_2}>> \mathfrak{M}^{(2)} \\ @V{p_1}VV @VV{a_1}V \\ \mathfrak{M}^{(n-1)} @>{b}>> \mathfrak{M} \end{CD}$$

The stack $\mathfrak{M}^{(n)}$ comes with structure maps $a_1, \ldots, a_n, b : \mathfrak{M}^{(n)} \to \mathfrak{M}$, defined recursively by $a_i = a_i \circ p_1$, for $i < n$, and $a_n = a_2 \circ p_2$, and $b = b \circ p_2$. The stack $\mathfrak{M}^{(n)}$ is the stack of objects in $\mathfrak{M}$ endowed with filtrations of length n. The subquotients are given by the maps $a_1, \ldots, a_n$, and the 'total object' by b.

It follows from the axioms for exact categories that there is also the following cartesian diagram.

$$\begin{CD} \mathfrak{M}^{(n)} @>>> \mathfrak{M}^{(n-1)} \\ @VVV @VV{b}V \\ \mathfrak{M}^{(2)} @>{a_2}>> \mathfrak{M} \end{CD}$$

The case $n = 3$ is known as *Noether's isomorphism* (see, for example, [Büh10, Lemma 3.5]) and is responsible for the associativity of the Hall product.

3.1 Filtered structure of the Hall algebra

Definition 3.2. For $n \geqslant 0$, we define

$$K^{\leqslant n}(\mathfrak{M}) = \ker E_{n+1} = \bigoplus_{k \leqslant n} K^k(\mathfrak{M}).$$

This is an ascending filtration on $K(\mathfrak{M})$, called the *filtration by the order of vanishing of inertia at $q = 1$*, or simply the *order filtration* of $K(\mathfrak{M})$.

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This is a slight abuse of language, because only the space obtained by extension of scalars $K^{\leqslant n}(\mathfrak{M})_{(q-1)}$ is the direct sum of all eigenspaces of $I^{\circ,\mathrm{ss}}$ whose corresponding eigenvalues $\mathfrak{Q} \in \mathbb{Q}[q]$ have order of vanishing at $q = 1$ less than or equal to n.

Theorem 3.3. *Suppose that $\xi \in K^{\leqslant n}(\mathfrak{M})$ and $\chi \in K^{\leqslant m}(\mathfrak{M})$. Then, $\xi * \chi \in K^{\leqslant n+m}(\mathfrak{M})$. Moreover, we have*

$$\xi * \chi \equiv \xi \cdot \chi \mod K^{<n+m}.$$

To prove this theorem we will prove the following lemma.

Lemma 3.4. *For any two stack functions $\xi, \chi \in K(\mathfrak{M})$ and for any integer $p \geqslant 0$, we have*

$$\frac{1}{p!} E_p(\pi_t(\xi) * \pi_t(\chi)) \equiv \sum_{i+j=p} \pi_i(\xi)\, \pi_j(\chi)\, t^p \mod t^{p+1} \tag{19}$$

as an equation in $K(\mathfrak{M})[t]$.

Before proving the lemma, let us indicate how the lemma implies the theorem. For this, suppose that $\xi \in K^{\leqslant k}(\mathfrak{M})$ and $\chi \in K^{\leqslant \ell}(\mathfrak{M})$. Then the degree of $\pi_t(\xi)$ in t is at most k and the degree of $\pi_t(\chi)$ is at most ℓ. So the degree of $E_p(\pi_t(\xi) * \pi_t(\chi))$ is at most $k+\ell$. So we see that if $p > k+\ell$, then $E_p(\pi_t(\xi) * \pi_t(\chi)) = 0$, which implies that $\xi * \chi \in K^{\leqslant k+\ell}(\mathfrak{M})$, by Corollary 2.8.

Now set $p = k + \ell$. The left-hand side of (19) has degree at most $k+\ell$, the right-hand side has degree exactly $k+\ell$, which implies that both sides are homogeneous of degree $k+\ell$, and we have

$$\frac{1}{(k+\ell)!} E_{k+\ell}(\pi_t(\xi) * \pi_t(\chi)) = \pi_k(\xi)\, \pi_\ell(\chi)\, t^{k+\ell}.$$

Now notice that if $x \in K^{\leqslant n}(\mathfrak{M})$, we have $\pi_n(x) = (1/n!)E_n(x)$. Hence, we can rewrite our equation as

$$\pi_{k+\ell}(\pi_t(\xi) * \pi_t(\chi)) = \pi_k(\xi) t^k\, \pi_\ell(\chi) t^\ell.$$

This proves the theorem.

Analysis of $E_p(E_n * E_m)$. Suppose $\xi = (X \to \mathfrak{M})$ and $\chi = (Y \to \mathfrak{M})$ are stack functions. The stack function $\xi * \chi$ is defined by the following cartesian diagram.

$$\begin{CD} X * Y @>>> \mathfrak{M}^{(2)} @>>> \mathfrak{M} \\ @VVV @VVV \\ X \times Y @>>> \mathfrak{M} \times \mathfrak{M} \end{CD}$$

Explicitly, $X * Y$ is the stack of triples (x, M, y)

$$\begin{CD} x @. @. y \\ @VVV @. @VVV \\ M' @>>> M @>>> M'' \end{CD} \tag{20}$$

where x and y are objects of X and Y, respectively, M is an object of $\mathfrak{M}^{(2)}$, i.e., a short exact sequence $M' \to M \to M''$ of objects in $\mathfrak{M}$, and $x \to M'$ and $y \to M''$ are isomorphisms from the images of x and y in $\mathfrak{M}$ to M' and M'', respectively. (We omit these isomorphisms from the triple to simplify the notation.)

The stack function $E_n(\xi) * E_m(\chi)$ is defined by the following enlarged diagram.

$$\begin{array}{ccccccc} E_n(X) * E_m(Y) & \longrightarrow & X * Y & \longrightarrow & \mathfrak{M}^{(2)} & \longrightarrow & \mathfrak{M} \\ \downarrow & & \downarrow & & \downarrow & & \\ E_n(X) \times E_m(Y) & \longrightarrow & X \times Y & \longrightarrow & \mathfrak{M} \times \mathfrak{M} & & \end{array}$$

Explicitly, $E_n(X) * E_m(Y)$ is the stack of 5-tuples $(x, (e_\nu), M, y, (f_\mu))$, where (x, M, y) represents a diagram (20), $(e_\nu) = (e_1, \dots, e_n)$ is a complete set of non-zero orthogonal idempotents in $A(x)$, and $(f_\mu) = (f_1, \dots, f_m)$ is a complete set of non-zero orthogonal idempotents in $A(y)$.

Finally, the stack $E_p(E_n(X) * E_m(Y))$ is the stack of objects of $E_n(X) * E_m(Y)$ endowed with a complete set of p non-zero labelled idempotents. Explicitly, it consists of 6-tuples

$$(x, (e_{\nu,\rho}), M, (g_\rho), y, (f_{\mu,\rho})), \tag{21}$$

where (x, M, y) is as in (20) and $(g_\rho)_{\rho \in \underline{p}}$ is a complete set of non-zero orthogonal idempotent endomorphisms of the short exact sequence $M' \to M \to M''$. Moreover, $(e_{\rho,\nu})_{\rho \in \underline{p}, \nu \in \underline{n}}$ is a pn-tuple of orthogonal idempotents in $A(x)$ and $(f_{\rho,\mu})_{\rho \in \underline{p}, \mu \in \underline{m}}$ is a pm-tuple of orthogonal idempotents in $A(y)$ such that for every $\rho = 1, \dots, p$ we have $\sum_{\nu=1}^{n} e_{\rho,\nu} = g_\rho|_{M'}$ and $\sum_{\mu=1}^{m} f_{\rho,\mu} = g_\rho|_{M''}$. Finally, we require for all $\nu = 1, \dots, n$ that $e_\nu = \sum_{\rho=1}^{p} e_{\rho,\nu} \neq 0$ and for all $\mu = 1, \dots, m$ that $f_\mu = \sum_{\rho=1}^{p} f_{\rho,\mu} \neq 0$.

Decomposing $E_p(E_n * E_m)$. Given p-tuples of non-negative integers $\varphi = (\varphi_1, \dots, \varphi_p)$ and $\psi = (\psi_1, \dots, \psi_p)$, we define a new stack function $(X * Y)_{\varphi,\psi} \to \mathfrak{M}$, denoted by $(\xi * \chi)_{\varphi,\psi}$, as follows.

Let $(X * Y)_{\varphi,\psi}$ be the algebraic stack of 6-tuples

$$(x, (e_\rho), M, (g_\rho), y, (f_\rho)), \tag{22}$$

where (x, M, y) is as in (20) and $(g_\rho)_{\rho=1,\dots,p}$ is a complete set of non-zero orthogonal idempotent endomorphisms of the short exact M. Moreover, for every $\rho = 1, \dots, p$, we require that $e_\rho = (e_1, \dots, e_{\varphi_\rho})$ and $f_\rho = (f_1, \dots, f_{\psi_\rho})$ are families of non-zero orthogonal idempotents for x and y, respectively, such that for all $\rho = 1, \dots, p$,

$$g_\rho|_{M'} = \sum_{\omega=1}^{\varphi_\rho} e_\omega \quad \text{and} \quad g_\rho|_{M''} = \sum_{\eta=1}^{\psi_\rho} f_\eta. \tag{23}$$

It follows that the union of $e_1, \dots, e_p$ is a complete set of orthogonal idempotents for x and the union of $f_1, \dots, f_p$ is a complete set of orthogonal idempotents for y.

There is a natural algebroid structure on $(X * Y)_{\varphi,\psi}$. The morphism to $\mathfrak{M}$ given by mapping the 6-tuple (22) to the middle object $b(M)$ of the short exact sequence M makes $(X * Y)_{\varphi,\psi}$ into a stack function.

Note that if for some $\rho = 1, \dots, p$, both integers φ_ρ and ψ_ρ vanish, then $(X * Y)_{\varphi,\psi} = \emptyset$, because all g_ρ are required to be non-zero.

Let us write $|\varphi| = \sum_\rho \varphi_\rho$ and $|\psi| = \sum_\rho \psi_\rho$. Let us assume that for every $\rho = 1, \dots, p$, at least one of the two integers φ_ρ, ψ_ρ is non-zero. Then we have a morphism

$$E_{|\varphi|}(X) \times E_{|\psi|}(Y) \longrightarrow (X * Y)_{\varphi,\psi} \tag{24}$$

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which maps a quadruple $(x,(e_\omega),y,(f_\eta))$ to the 6-tuple (22) where $M=M'\oplus M''$, with M' denoting the image of x in $\mathfrak{M}$ and M'' the image of y in $\mathfrak{M}$. To define (24), we break up the complete family of orthogonal idempotents $e_1,\dots,e_{|\varphi|}$ for x into p subfamilies, where the ρth subfamily has φ_ρ members. Similarly, we break up $f_1,\dots,f_{|\psi|}$ into p subfamilies whose sizes are $\psi_1,\dots,\psi_p$. Then the family of idempotents (g_ρ) on M is defined by formulas (23). Note that we need to make our assumption on the p-tuples φ, ψ, in order for every family member g_ρ to be non-zero.

Lemma 3.5. *If for every $\rho=1,\dots,p$, exactly one of the two integers φ_ρ, ψ_ρ is non-zero, (24) is an isomorphism. Hence, we have the equality*

$$(\xi*\chi)_{\varphi,\psi}=E_{|\varphi|}(\xi)E_{|\psi|}(\chi)$$

for stack functions.

Proof. Given an object (22) of $(X*Y)_{\varphi,\psi}$, the short exact sequence M is split into a direct sum of p short exact sequences. Each one of these sequences is canonically split, because either the subobject or the quotient object vanishes, by the assumption on φ and ψ. Therefore, the sequence M is split, canonically, too. □

Now suppose we are given strictly monotone maps

$$\Phi_\rho:\underline{\varphi_\rho}\longhookrightarrow\underline{n}\quad\text{and}\quad\Psi_\rho:\underline{\psi_\rho}\longhookrightarrow\underline{m},$$

for all $\rho=1,\dots,p$ such that the images of the Φ_ρ cover $\underline{n}$ and the images of the Ψ_ρ cover $\underline{m}$. The choice of these injections determines a morphism of algebraic stacks

$$(X*Y)_{\varphi,\psi}\longrightarrow E_p(E_n(X)*E_m(Y)) \tag{25}$$

by mapping the 6-tuple (22) to the 6-tuple (21) by defining

$$e_{\nu,\rho}=\sum_{\Phi_\rho(\omega)=\nu}(e_\rho)_\omega\quad\text{and}\quad f_{\mu,\rho}=\sum_{\Psi(\eta)=\mu}(f_\rho)_\eta.$$

By our assumptions, these sums are either empty or consist of a single summand, so the $e_{\nu,\rho}$ and the $f_{\mu,\rho}$ are obtained from the $(e_\rho)_\omega$ and the $(f_\rho)_\eta$ essentially by relabelling.

Note that the requirements $\bigcup_\rho\Phi_\rho(\underline{\varphi_\rho})=\underline{n}$ and $\bigcup_\rho\Psi_\rho(\underline{\psi_\rho})=\underline{m}$ are needed to assure that $\sum_\rho e_{\nu,\rho}$ and $\sum_\rho f_{\mu,\rho}$ are non-zero, for all $\nu=1,\dots,n$ and $\mu=1,\dots,m$.

Lemma 3.6. *The morphism (25) gives rise to a morphism of stack functions $(\xi*\chi)_{\varphi,\psi}\to E_p(E_n(\xi)*E_m(\chi))$, which is both an open and a closed immersion.*

If we change any of φ, ψ, or Φ, Ψ, we get a morphism with disjoint image. The images of all morphisms (25) cover $E_p(E_n(X)*E_m(Y))$.

Proof. This follows from the fact that the source and target of (25) differ only in the way the idempotents in A_x and A_y are indexed. □

Corollary 3.7. *Using the notation introduced in (18), we have the following equation in $K(\mathfrak{M})$:*

$$E_p(E_n(\xi)*E_m(\chi))=\sum_{\varphi,\psi}\sum_{\Phi,\Psi}(\xi*\eta)_{\varphi,\psi}=\sum_{\varphi,\psi}\begin{bmatrix}n\\ \varphi\end{bmatrix}\begin{bmatrix}m\\ \psi\end{bmatrix}(\xi*\eta)_{\varphi,\psi},$$

where φ and ψ run over all p-tuples of non-negative integers.

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For example, consider $\chi = 1$ and $m = 0$. If any of the ψ_ρ is non-zero, $(X * Y)_{\varphi,\psi}$ is empty. Hence,

$$E_p E_n(\xi) = \sum_{\varphi_1,\dots,\varphi_p>0} \begin{bmatrix} n \\ \varphi \end{bmatrix} E_{|\varphi|}(\xi),$$

where the sum is over all p-tuples of positive integers.

Proof of the main lemma. Using Proposition 2.24, we can now calculate as follows:

$$\begin{aligned} E_p(\pi_t(\xi) * \pi_t(\chi)) &= E_p\bigg(\sum_n \binom{t}{n} E_n(\xi) * \sum_m \binom{t}{m} E_m(\chi)\bigg) \\ &= \sum_{n,m} \binom{t}{n}\binom{t}{m} \sum_{\varphi,\psi} \begin{bmatrix} n \\ \varphi \end{bmatrix}\begin{bmatrix} m \\ \psi \end{bmatrix} (\xi * \chi)_{\varphi,\psi} \\ &= \sum_{\varphi,\psi} \bigg(\sum_n \binom{t}{n}\begin{bmatrix} n \\ \varphi \end{bmatrix}\bigg)\bigg(\sum_m \binom{t}{m}\begin{bmatrix} m \\ \psi \end{bmatrix}\bigg)(\xi * \chi)_{\varphi,\psi} \\ &= \sum_{\varphi,\psi} \binom{t}{\varphi_1}\cdots\binom{t}{\varphi_p}\binom{t}{\psi_1}\cdots\binom{t}{\psi_p}(\xi * \chi)_{\varphi,\psi}. \end{aligned} \tag{26}$$

For example, if $\chi = 1$, we get

$$E_p \pi_t(\xi) = \sum_{\varphi_1,\dots,\varphi_p>0} \binom{t}{\varphi_1}\cdots\binom{t}{\varphi_p} E_{|\varphi|}(\xi). \tag{27}$$

The lowest order term in (26) has degree p, since for $(\xi * \chi)_{\varphi,\psi}$ not to vanish, we need, for every $\rho = 1, \dots, p$, at least one of φ_ρ, ψ_ρ to be non-zero.

Modulo (t^{p+1}), only terms corresponding to pairs (φ, ψ), with the property that for every $\rho = 1, \dots, p$ exactly one of φ_ρ, ψ_ρ is non-zero, contribute to (26). These are exactly the terms to which Lemma 3.5 applies, and we deduce that, modulo t^{p+1}, we have:

$$E_p(\pi_t(\xi) * \pi_t(\chi)) \equiv \sum_{\varphi,\psi} \binom{t}{\varphi_1}\cdots\binom{t}{\varphi_p}\binom{t}{\psi_1}\cdots\binom{t}{\psi_p} E_{|\varphi|}(\xi)\, E_{|\psi|}(\chi),$$

where the sum is over all (φ, ψ), where the supports of φ and ψ form a partition of $\underline{p}$. By grouping terms corresponding to partitions of the same size together, we can rewrite this as

$$\sum_{i+j=p} \binom{p}{i} \sum_{\varphi_1,\dots,\varphi_i>0} \binom{t}{\varphi_1}\cdots\binom{t}{\varphi_i} E_{|\varphi|}(\xi) \sum_{\psi_1,\dots,\psi_j>0} \binom{t}{\psi_1}\cdots\binom{t}{\psi_j} E_{|\psi|}(\chi),$$

which is equal to

$$\sum_{i+j=p} \binom{p}{i} E_i \pi_t(\xi)\, E_j \pi_t(\chi),$$

by (27). Modulo t^{p+1}, this term is congruent to

$$\sum_{i+j=p} \binom{p}{i} E_i \pi_i(\xi)\, E_j \pi_j(\chi) = p! \sum_{i+j=p} \pi_i(\xi)\, \pi_j(\chi)\, t^p.$$

We conclude that

$$\frac{1}{p!} E_p(\pi_t(\xi) * \pi_t(\chi)) \equiv \sum_{i+j=p} \pi_i(\xi)\, \pi_j(\chi)\, t^p \mod t^{p+1},$$

which proves Lemma 3.4.

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3.2 The semi-classical Hall algebra

By Theorem 3.3, the submodule

$$\mathscr{K}(\mathfrak{M}) = \bigoplus_{n\geqslant 0} t^n K^{\leqslant n}(\mathfrak{M})$$

of $K(\mathfrak{M})[t]$ is a $K(\mathrm{DM})[t]$-subalgebra with respect to the Hall product. The algebra $\mathscr{K}(\mathfrak{M})$ is a one-parameter flat family of algebras. The special fibre at $t=0$ is canonically isomorphic to the graded algebra associated to the filtered algebra $(K(\mathfrak{M}),*)$. The quotient map $\mathscr{K}\to\mathscr{K}/t\mathscr{K}$ is identified with the map $\sum_n x_n t^n \mapsto \sum_n \pi_n(x_n)$.

The graded algebra associated to the filtered algebra $(K(\mathfrak{M}),*)$ is canonically isomorphic to the commutative graded algebra $(K(\mathfrak{M}),\,\cdot\,)$, by Theorem 3.3. The special fibre inherits therefore a Poisson bracket, which encodes the Hall product to second order. This Poisson bracket has degree -1 and is given by the formula

$$\{x,y\} = \pi_{k+\ell-1}(x*y - y*x) \quad \text{for } x\in K^k(\mathfrak{M}), y\in K^\ell(\mathfrak{M}). \tag{28}$$

Corollary 3.8. *The graded $K(\mathrm{DM})$-algebra $(K(\mathfrak{M}),\,\cdot\,)$ is endowed with a Poisson bracket of degree -1, given by (28).*

Corollary 3.9. *In particular, $K^1(\mathfrak{M})$ is a Lie algebra with respect to the Poisson bracket (28). In fact, for $x,y\in K^1(\mathfrak{M})$, we have that $x*y-y*x\in K^1(\mathfrak{M})$, so in this case, the Poisson bracket is equal to the Lie bracket. Thus, $K^1(\mathfrak{M})$ is a Lie algebra over the ring of scalars $K(\mathrm{DM})$.*

Proof. Equation (26) for $p=0$, together with Lemma 3.5, says

$$E_0(\pi_t(x)*\pi_t(y)) = E_0(x)E_0(y).$$

This proves that $E_0(x)=0$ or $E_0(y)=0$ implies that $E_0(x*y)=0$. □

Definition 3.10. We call $K^1(\mathfrak{M})$ the Lie algebra of *virtually indecomposable* stack functions. We will usually write $K^{\mathrm{vir}}(\mathfrak{M})$ for $K^1(\mathfrak{M})$.

This terminology is used in analogy with that of [Joy07a]. In Appendix A, we check that our notion of virtually indecomposable agrees with that of [Joy07a] in a special case.

3.3 Epsilon functions

We will prove that replacing direct sum decompositions by filtrations, in the formula

$$\pi_k = \sum_{n\geqslant k} \frac{s(n,k)}{n!} E_n,$$

will give rise to an operator mapping $K(\mathfrak{M})$ into $K^{\leqslant k}(\mathfrak{M})$. In particular, we will be able to construct virtually indecomposable stack functions as 'Hall algebra logarithms'.

Fix an algebraic substack $\mathfrak{N}\hookrightarrow\mathfrak{M}$ with the following properties:

(i) $\mathfrak{N}$ avoids the image of $\operatorname{Spec} R \xrightarrow{0} \mathfrak{M}$;
(ii) $\mathfrak{N}$ is closed under direct sums and direct summands, i.e., it is Karoubian (Remark 1.63) if we add $\operatorname{Spec} R \xrightarrow{0} \mathfrak{M}$ to it;

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(iii) for every positive integer n, the morphism $b|_{\mathfrak{N}^{(n)}} : \mathfrak{N}^{(n)} \to \mathfrak{M}$, illustrated in the diagram

$$\begin{array}{ccccc} \mathfrak{N}^{(n)} & \longrightarrow & \mathfrak{M}^{(n)} & \xrightarrow{b} & \mathfrak{M} \\ \downarrow & & \downarrow{\scriptstyle a_1\times\cdots\times a_n} & & \\ \mathfrak{N}^n & \longrightarrow & \mathfrak{M}^n, & & \end{array} \tag{29}$$

where the square is cartesian, is of finite type;

(iv) the disjoint union over all these morphisms $\coprod_{n>0} \mathfrak{N}^{(n)} \to \mathfrak{M}$ is still of finite type. This means that if $X \to \mathfrak{M}$ is a morphism with X of finite type, there exists an $N > 0$ such that for all $n \geqslant N$, the image of $b|_{\mathfrak{N}^{(n)}} : \mathfrak{N}^{(n)} \to \mathfrak{M}$ does not intersect the image of X in $\mathfrak{M}$.

Example 3.11. If $\mathfrak{M}$ is the stack of coherent sheaves on a projective curve, then the substack of non-zero semi-stable vector bundles of fixed slope is an example of a substack $\mathfrak{N}$ satisfying our conditions. More generally, we can take for $\mathfrak{N}$ the stack of all vector bundles whose Harder–Narasimhan slopes are contained in a fixed interval.

Example 3.12. If $\mathfrak{M}$ is the stack of representations of a quiver Q, then we can take $\mathfrak{N} = \mathfrak{M}_*$.

Consider an arbitrary stack function $X \to \mathfrak{M}$, and denote by F_nX, for $n \geqslant 1$, the stack

$$F_nX = \mathfrak{N}^{(n)} \times_{\mathfrak{M}} X.$$

It fits into the following cartesian diagram.

$$\begin{array}{ccccc} F_nX & \longrightarrow & X^{(n)} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle M} \\ \mathfrak{N}^{(n)} & \longrightarrow & \mathfrak{M}^{(n)} & \xrightarrow{b} & \mathfrak{M} \end{array}$$

Note that F_nX is of finite type, by our assumption on $\mathfrak{N}$, and also representable over $\mathfrak{M}$ (as an algebroid), because b is. Therefore, F_nX is another stack function.

The objects of F_nX are pairs (x, F), where x is an object of X and $F = (F_1 \to \cdots \to F_n)$ is a flag in $F_n = M$, where M is the image of x in $\mathfrak{M}$ such that all subquotients $F_\nu/F_{\nu-1}$, for $\nu = 1, \dots, n$, are in $\mathfrak{N}$.

We now consider $E_k(F_nX)$, for $k \geqslant 0$. This is the stack of triples

$$(x, (e_\kappa), F),$$

where the pair (x, F) is an object of F_nX and $(e_\kappa) = (e_1, \dots, e_k)$ is a complete set of non-zero orthogonal idempotents in $A(X)$ such that for every $\kappa = 1, \dots, k$ the endomorphism of M induced by e_κ respects the flag F. For every $\nu = 1, \dots, n$, we therefore get an induced idempotent operator

$$f_{\kappa,\nu} = e_\kappa|_{F_\nu/F_{\nu-1}}.$$

These idempotents have the properties:

(i) $\sum_\kappa f_{\kappa,\nu} = 1$, for all ν;

(ii) for every κ, at least one of the $f_{\kappa,\nu}$ does not vanish.

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The stack $E_k(F_nX)$ decomposes into a disjoint union of substacks according to which of the idempotents $(f_{\kappa,\nu})$ vanish.

To make this decomposition precise, consider a sequence of positive integers $(\alpha_1,\dots,\alpha_k)$. Define $F_\alpha X$ to be the stack of triples

$$(x,(e_\kappa),(F_\kappa)).$$

Here, x is an object of X, with image M in $\mathfrak{M}$, and (e_κ) is a complete set of orthogonal non-zero idempotents for x that decomposes M into a direct sum $M=\bigoplus_\kappa M_\kappa$. Moreover, F_κ is a flag of length α_κ on M_κ, with subquotients in $\mathfrak{N}$, for every $\kappa=1,\dots,k$.

For every k-tuple of strictly monotone maps $\Phi_\kappa:\underline{\alpha_\kappa}\hookrightarrow\underline{n}$, we define a morphism

$$F_\alpha X\longrightarrow E_k(F_nX) \tag{30}$$

by defining the flag F on M in terms of the k-tuple of flags (F_κ) by

$$F_\nu=\bigoplus_\kappa\sum_{\Phi_\kappa(\rho)\leqslant\nu}F_\rho.$$

Note that the sum for fixed κ is not really a sum, it is just the largest of the subobjects F_ρ of M_κ making up the flag $F_1\to\cdots\to F_{\alpha_\kappa}$ such that $\Phi_\kappa(\rho)\leqslant\nu$.

LEMMA 3.13. *The morphism (30) given by $(\Phi_\kappa)_{\kappa\in\underline{k}}$ is an isomorphism onto the locus in $E_k(F_nX)$, defined by $f_{\kappa,\nu}\neq0$ if and only if $\nu\in\Phi_\kappa(\underline{\alpha_\kappa})$, for all $\kappa=1,\dots,k$.*

COROLLARY 3.14. *If ξ denotes the element of $K(\mathfrak{M})$ defined by $X\to\mathfrak{M}$, we have*

$$E_k(F_n\xi)=\sum_\alpha\begin{bmatrix}n\\ \alpha\end{bmatrix}F_\alpha(\xi),$$

where the sum is taken over all k-tuples of positive integers.

If we set $F_0(\xi)=1$ and $F_\emptyset(\xi)=1$, this equality also holds for $n=0$.

DEFINITION 3.15. Define, for every $\xi\in K(\mathfrak{M})$,

$$\varepsilon_t(\xi)=\sum_{n\geqslant0}\binom{t}{n}F_n(\xi),$$

where for $n=0$, we set $F_0(\xi)=1$. This definition is justified, because by our assumptions on $\mathfrak{N}$, this sum is actually finite.

Expanding in powers of t defines the $\varepsilon_k(\xi)$ for $k\geqslant0$:

$$\varepsilon_t(\xi)=\sum_{k\geqslant0}\varepsilon_k(\xi)t^k.$$

For example, $\varepsilon_0=1$, and

$$\varepsilon_1(\xi)=\sum_{n>0}\frac{(-1)^{n+1}}{n}F_n(\xi).$$

In general,

$$\varepsilon_k(\xi)=\sum_{n\geqslant k}\frac{s(n,k)}{n!}F_n(\xi).$$

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Corollary 3.16. *For every* $k \geqslant 0$, *we have* $\varepsilon_k(\xi) \in K^{\leqslant k}(\mathfrak{M})$. *Hence,* $\varepsilon_t(\xi) \in \mathscr{K}(\mathfrak{M})$. *In particular,* $\varepsilon_1(\xi)$ *is virtually indecomposable for all* $\xi \in K(\mathfrak{M})$.

Proof. It suffices to prove that $E_k(\varepsilon_t(\xi)) \equiv 0 \bmod (t^k)$ for all k. In fact,

$$\begin{aligned} E_k(\varepsilon_t(\xi)) &= \sum_{n\geqslant 0} \binom{t}{n} E_k F_n(\xi) \\ &= \sum_{n\geqslant 0} \binom{t}{n} \sum_{\alpha_1,\ldots,\alpha_k>0} \begin{bmatrix} n \\ \alpha \end{bmatrix} F_\alpha(\xi) \\ &= \sum_{\alpha_1,\ldots,\alpha_k>0} \binom{t}{\alpha_1} \cdots \binom{t}{\alpha_k} F_\alpha(\xi) \end{aligned}$$

is, indeed, divisible by t^k, if all $\alpha_1, \ldots, \alpha_k$ are positive. □

Remark 3.17. The operator $F_n : K(\mathfrak{M}) \to K(\mathfrak{M})$ respects strict algebroids. The same is true for all ε_k.

Epsilons as logarithms. Suppose there exists an abelian group Γ and a decomposition of $\mathfrak{M}$ (as an algebroid, not a linear stack) into a disjoint union

$$\mathfrak{M} = \coprod_{\gamma\in\Gamma} \mathfrak{M}_\gamma.$$

We require that if $\mathfrak{E}_{\gamma,\beta}$ is defined by the cartesian diagram

$$\begin{array}{ccc} \mathfrak{E}_{\gamma,\beta} & \longrightarrow & \mathfrak{M}^{(2)} \\ \big\downarrow & & \big\downarrow{\scriptstyle a_1\times a_2} \\ \mathfrak{M}_\gamma \times \mathfrak{M}_\beta & \longrightarrow & \mathfrak{M}\times\mathfrak{M} \end{array}$$

then the composition

$$\mathfrak{E}_{\gamma,\beta} \longrightarrow \mathfrak{M}^{(2)} \xrightarrow{b} \mathfrak{M}$$

factors through $\mathfrak{M}_{\gamma+\beta} \subset \mathfrak{M}$.

We call such Γ a *grading group* for $\mathfrak{M}$.

The grading group Γ decomposes $K(\mathfrak{M})$ into a direct sum

$$K(\mathfrak{M}) = \bigoplus_{\gamma\in\Gamma} K(\mathfrak{M})_\gamma, \tag{31}$$

where $K(\mathfrak{M})_\gamma$ is the submodule of $K(\mathfrak{M})$ generated by stack functions $X \to \mathfrak{M}$ that factor through $\mathfrak{M}_\gamma$. The Hall product, as well as the commutative product, are graded with respect to (31). For $x \in K(\mathfrak{M})$, we denote the projection of x into the component $K(\mathfrak{M})_\gamma$ by x_γ.

Let $\mathfrak{N} \subset \mathfrak{M}$ be a linear algebraic substack avoiding $\operatorname{Spec} R \xrightarrow{0} \mathfrak{M}$, with the properties:

(i) every intersection $\mathfrak{N}_\gamma = \mathfrak{N} \cap \mathfrak{M}_\gamma$ is of finite type;

(ii) $\mathfrak{N}$ is closed under direct summands and extensions in $\mathfrak{M}$, the latter meaning that if $\mathfrak{N}^{(2)}$ is defined as in (29), then the composition $\mathfrak{N}^{(2)} \to \mathfrak{M}^{(2)} \xrightarrow{b} \mathfrak{M}$ factors through $\mathfrak{N} \subset \mathfrak{M}$;

(iii) there is a submonoid $\Gamma_+ \subset \Gamma$ such that $\mathfrak{N}_\gamma \neq \emptyset$ implies that $\gamma \in \Gamma_+ \backslash \{0\}$. The monoid Γ_+ is required to have the property that every $\gamma \in \Gamma_+$ admits only finitely many decompositions $\gamma = \alpha + \beta$ such that both $\alpha, \beta \in \Gamma_+$. We will further assume that Γ_+ has the property that the intersection of all cofinite ideals is empty.

If these axioms hold, $\mathfrak{N}$ satisfies the finiteness conditions above, so that the $\varepsilon_k(\xi)$ are defined for all $\xi \in K(\mathfrak{M})$.

Remark 3.18. If $\mathfrak{M}$ is the stack of coherent sheaves on a projective curve, then we can take $\Gamma = \mathbb{Z}^2$ and define $\mathfrak{M}_{(d,r)}$, for $(d, r) \in \mathbb{Z}^2$ to be the stack of sheaves of rank r and degree d. Suppose $\mathfrak{N}$ is the stack of bundles whose Harder–Narasimhan slopes are contained in the interval $(a, b) \subset \mathbb{R}$. Then we can take

$$\Gamma_+ = \{(0,0)\} \cup \left\{ (d,r) \in \mathbb{Z}^2 \;\middle|\; r > 0 \text{ and } a < \frac{d}{r} < b \right\},$$

and the above requirements will be satisfied.

Remark 3.19. If $\mathfrak{M}$ is the stack of representations of a quiver Q, we can take $\Gamma = \mathbb{Z}^{Q_0}$, where Q_0 is the set of vertices of Q, and then set $\mathfrak{M}_\gamma$, for $\gamma \in \Gamma$, equal to the stack of representations with dimension vector γ. If we take $\mathfrak{N} = \mathfrak{M}_*$, we can take $\Gamma_+ = \mathbb{Z}^{Q_0}_{\geqslant 0}$.

Let us also define

$$K(\mathfrak{M})_S = \bigoplus_{\gamma \in S} K(\mathfrak{M})_\gamma \subset K(\mathfrak{M})$$

for any cofinite ideal $S \subset \Gamma_+$. For every such S, the group $K(\mathfrak{M})_S$ is an ideal (with respect to both multiplications) in the ring $K(\mathfrak{M})_+ = K(\mathfrak{M})_{\Gamma_+}$, and we may complete $K(\mathfrak{M})_+$ with respect to this collection of ideals to obtain $\hat{K}(\mathfrak{M})_+$. The morphism $K(\mathfrak{M})_+ \to \hat{K}(\mathfrak{M})_+$ is injective and both multiplications extend to $\hat{K}(\mathfrak{M})_+$.

In $\hat{K}(\mathfrak{M})_+$, the sum

$$[\mathfrak{N}] = \sum_{\gamma \in \Gamma_+} [\mathfrak{N}_\gamma \to \mathfrak{M}]$$

converges.

The idempotent operators commute with the Γ-grading, and so everything defined in terms of them does too. In particular, $\hat{K}(\mathfrak{M})_+$ is a graded Poisson algebra, and the Lie algebra of degree $+1$ elements in $\hat{K}(\mathfrak{M})_+$, which we will denote $\hat{K}^{\mathrm{vir}}(\mathfrak{M})_+$, is equal to the completion of $K^{\mathrm{vir}}(\mathfrak{M})_+ = \bigoplus_{\gamma \in \Gamma_+} K^{\mathrm{vir}}(\mathfrak{M})_\gamma$.

PROPOSITION 3.20. *In $\hat{K}(\mathfrak{M})_+[[t]]$, we have*

$$\varepsilon_t[\mathfrak{N}] = \sum_{n \geqslant 0} \binom{t}{n} [\mathfrak{N}]^{*n}.$$

Hence, we can write

$$\varepsilon_t[\mathfrak{N}] = (1 + [\mathfrak{N}])^{*t} = \exp_*(t \log_*(1 + [\mathfrak{N}])),$$

where exponential and logarithm are defined by their power series using the Hall product. In particular,

$$\varepsilon_1[\mathfrak{N}] = \log_*(1 + [\mathfrak{N}]) \in \hat{K}^{\mathrm{vir}}(\mathfrak{M})_+$$

and

$$\varepsilon_k[\mathfrak{N}] = \frac{1}{k!} \log(1 + [\mathfrak{N}])^{*k} = \frac{1}{k!} (\varepsilon_1[\mathfrak{N}])^{*k}.$$

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Proof. Using the formula

$$F_n[\mathfrak{N}] = [\mathfrak{N}^{(n)} \xrightarrow{b} \mathfrak{M}] = \underbrace{[\mathfrak{N}] * \cdots * [\mathfrak{N}]}_{n \text{ times}},$$

the result follows. □

Remark 3.21. Compare the two formulas

$$\begin{aligned} \varepsilon_t[\mathfrak{N}] &= (1 + [\mathfrak{N}])^{*t}, \\ \pi_t[\mathfrak{N}] &= (1 + [\mathfrak{N}])^{t}. \end{aligned}$$

Remark 3.22. Let us write $\hat{\mathscr{K}}(\mathfrak{M})_+$ for the subspace of $\hat{K}(\mathfrak{M})_+[[t]]$ defined by requiring the coefficient of t^k to be contained in $\hat{K}^{\leqslant k}(\mathfrak{M})_+$, for all k. Then

$$\varepsilon_t[\mathfrak{N}] = (1 + [\mathfrak{N}])^{*t} \in \hat{\mathscr{K}}(\mathfrak{M})_+.$$

Remark 3.23. Setting $t = 1$, we also get that $1 + [\mathfrak{N}] = \exp_*(\varepsilon_1[\mathfrak{N}])$. One should think of $1 + [\mathfrak{N}]$ as *group-like*.

Hopf algebra. We make a brief remark, without striving for generality.

Let us fix $\mathfrak{N} \subset \mathfrak{M}$ and $\Gamma_+ \subset \Gamma$ as before. Assume for simplicity that Γ is free. In particular, $\alpha + \beta = 0$, for $\alpha, \beta \in \Gamma_+$, implies $\alpha = \beta = 0$.

For $0 \neq \gamma \in \Gamma_+$, abbreviate the element $[\mathfrak{N}_\gamma \to \mathfrak{M}] \in K(\mathfrak{M})$ by $[\gamma]$.

For a finite sequence $(\gamma) = \gamma_1, \ldots, \gamma_n$ of non-zero elements of Γ_+, write

$$[(\gamma)] = [\gamma_1, \ldots, \gamma_n] = [\gamma_1] * \cdots * [\gamma_n].$$

In particular, for $n = 0$, we have $[\emptyset] = 1$.

In many cases of interest, the Hall algebra elements $[(\gamma)]$, as (γ) runs over all finite sequences of non-zero elements of Γ_+, are linearly independent over $\mathbb{Q}$. Let us assume that this is the case. Then the $\mathbb{Q}$-span of all $[(\gamma)]$ is a $\mathbb{Q}$-subalgebra of $K(\mathfrak{M})$ that is free on the generators $[\gamma]$, for $\gamma \in \Gamma_+ \backslash \{0\}$, as a unitary $\mathbb{Q}$-algebra. Let us denote this algebra by U. Let us further assume that the morphism induced by the commutative product $U \otimes U \to K(\mathfrak{M})$ is injective. (Again, this will hold in many cases of interest.)

We will now define a comultiplication Δ on U, making a U a cocommutative Hopf algebra over $\mathbb{Q}$.

To define Δ, it is convenient to extend the notation $[(\gamma)]$ to finite sequences of elements of Γ_+, which may be zero. This is done by setting $[0] = 1$. Thus, $[(\gamma)]$ is unchanged by 'crossing off its zeros'. We then define

$$\begin{aligned} \Delta : U &\longrightarrow U \otimes U \\ [(\gamma)] &\longmapsto \sum_{(\alpha)+(\beta)=(\gamma)} [(\alpha)] \otimes [(\beta)], \end{aligned}$$

where the sum is over all pairs of sequences of the same length as γ, but allowing zeros.

This defines on U the structure of a cocommutative Hopf algebra.

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LEMMA 3.24. *The diagram*

$$\begin{array}{ccc} U^* & \xrightarrow{x \longmapsto \Delta(x) - 1\otimes x - x \otimes 1} & U \otimes U \\ & \searrow^{E_2} & \downarrow{\scriptstyle \text{comm. mult.}} \\ & & K(\mathfrak{M}) \end{array}$$

is commutative, where $U^* \subset U$ is the augmentation ideal.

It follows that the virtual indecomposables in U are equal to the primitives with respect to the Hopf algebra structure:

$$U^{\mathrm{prim}} = U^{\mathrm{vir}}.$$

As U is a cocommutative Hopf algebra, it is isomorphic to the universal enveloping algebra of U^{prim}, by the Cartier–Gabriel theorem [Car07, Theorem 3.8.2].

The Lie algebra U^{vir} is free, as a Lie algebra over $\mathbb{Q}$, on the elements

$$\varepsilon[\gamma] = \sum_{n>0} \frac{(-1)^{n+1}}{n} \sum_{\substack{\gamma=\gamma_1+\cdots+\gamma_n \\ \gamma_1,\ldots,\gamma_n>0}} [\gamma_1] * \cdots * [\gamma_n],$$

for $\gamma \in \Gamma_+ \backslash \{0\}$.

Remark 3.25. Suppose $\mathfrak{N} = \mathfrak{M} = \mathfrak{Vect}$ is the stack of vector bundles. We take $\Gamma = \mathbb{Z}$ and $\Gamma_+ = \mathbb{Z}_{\geqslant 0}$. Then the $[(\gamma)]$ are, indeed, linearly independent over $\mathbb{Q}$, at least if our ground ring R is a field. Moreover, $U \otimes U \to K(\mathfrak{M})$ is injective. The Hopf algebra we obtain is the Hopf algebra of *non-commutative symmetric functions*, see [Car07, Example 4.1 (F)].

Remark 3.26. It is doubtful that it is possible to extend the coproduct to all of $K(\mathfrak{M})$ in such a way that $K^{\mathrm{vir}}(\mathfrak{M}) = K(\mathfrak{M})^{\mathrm{prim}}$. By the above considerations, we consider the family of operators (E_n) as a substitute, which allows us to prove at least some of the result one would expect in a cocommutative Hopf algebra. In particular, we find it unlikely that, in general, $K(\mathfrak{M})$ would be isomorphic to the universal enveloping algebra of $K^{\mathrm{vir}}(\mathfrak{M})$.

4. Integration

The integral of a stack function $(X, A) \to (\mathfrak{M}, \mathfrak{A})$ does three things: it forgets the structure map to $(\mathfrak{M}, \mathfrak{A})$, it forgets the algebroid structure, mapping (X, A) to X, and it introduces the bundle relations in $K(\mathrm{St})$, for non-inert bundles of algebraic stacks.

The vector bundle relations. Let $K(\mathrm{St})$ be the Grothendieck $K(\mathrm{DM})$-algebra of algebraic stacks (finite type, with affine diagonal), modulo the scissor and the bundle relations. A *bundle relation* is any equation of the form

$$[Y] = [F \times X]$$

for a fibre bundle $Y \to X$ of algebraic stacks with special structure group and fibre F.

It is well known that

$$K(\mathrm{St}) = K(\mathrm{Var})\left[\frac{1}{q}\right]\left[\frac{1}{q^n - 1}\right]_{n \geqslant 1}.$$

A proof can be found, for example in [Bri12]. Along the same lines, one proves that

$$K(\mathrm{St}) = K(\mathrm{DM})\left[\frac{1}{q}\right]\left[\frac{1}{q^n-1}\right]_{n\geqslant 1}.$$

Note that the (connected, semi-simple) inertia operator does not preserve non-inert bundle relations. Therefore, in $K(\mathrm{St})$, we cannot talk about $I^{\circ,\mathrm{ss}}[X]$, only about $[I_X^{\circ,\mathrm{ss}}]$.

Regular motivic weights.

Definition 4.1. We say an element of $K(\mathrm{St})$ is *regular* if it can be written with a denominator which does not vanish at $q = 1$. Thus, the subalgebra of regular motivic weights $K(\mathrm{St})_{\mathrm{reg}} \subset K(\mathrm{St})$ is by definition the image of the morphism of $K(\mathrm{Var})$-algebras:

$$K(\mathrm{Var})\left[\frac{1}{q}\right]\left[\frac{1}{q^n+\cdots+1}\right]_{n\geqslant 1} \longrightarrow K(\mathrm{Var})\left[\frac{1}{q}\right]\left[\frac{1}{q^n-1}\right]_{n\geqslant 1} = K(\mathrm{St}).$$

The image of $K(\mathrm{DM})$ in $K(\mathrm{St})$ is contained in $K(\mathrm{St})_{\mathrm{reg}}$. Hence, we can also think of $K(\mathrm{St})_{\mathrm{reg}}$ as a $K(\mathrm{DM})$-algebra. This follows from the following lemma.

Lemma 4.2. *Every finite type stack with quasi-finite stabilizer has regular motivic weight in* $K(\mathrm{St})$.

Proof. Let Z be a stack with quasi-finite stabilizer (meaning that its inertia I_Z is quasi-finite over Z). By stratifying Z, if necessary, we may assume that the inertia stack of Z is in fact finite. By [Kre99, Proposition 3.5.7], Z is stratified by global quotient stacks, so we may assume that $Z = Y/\mathrm{GL}_m$, for an algebraic space Y, such that GL_m acts on Y with finite stabilizer. The maximal torus $T \subset \mathrm{GL}_m$ then also acts with finite stabilizer on Y. The flattening stratification $\coprod Y_i \to Y$ of the stabilizer $\mathrm{Stab}_T Y$ is then T-equivariant, so that T acts on each Y_i, and by passing to open and closed subspaces of the Y_i, we may assume that the action of T on Y_i has constant stabilizer (see [Oes14, I, 5.4]). Then T acts on Y_i freely through a quotient T_i by a finite subgroup. We conclude

$$[Z] = \frac{1}{[\mathrm{GL}_m]}\sum_i [Y_i] = \frac{1}{[\mathrm{GL}_m]}\sum_i [T_i][Y_i/T_i] = \frac{[T]}{[\mathrm{GL}_m]}\sum_i [Y_i/T_i], \tag{32}$$

because each quotient T_i of T is isomorphic to T. The last term in (32) is regular, because each Y_i/T_i is an algebraic space, and

$$\frac{[T]}{[\mathrm{GL}_m]} = q^{-(1/2)m(m-1)}\prod_{i=1}^{m}\frac{1}{q^i+\cdots+1}$$

does not vanish at $q = 1$. □

The integral. Mapping a stack function $(X, A) \to (\mathfrak{M}, \mathfrak{A})$ to the class $[X] \in K(\mathrm{St})$ gives rise to a well-defined homomorphism $K(\mathfrak{M}) \to K(\mathrm{St})$ of $K(\mathrm{DM})$-modules. We denote this homomorphism by

$$\int : K(\mathfrak{M}) \to K(\mathrm{St}).$$

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4.1 The 'no poles' theorem

Theorem 4.3. *The composition $\int\circ I^{\circ,\mathrm{ss}}$ factors through the algebra of regular motivic weights.*

$$\begin{array}{ccc} K(\mathfrak{M}) & \xrightarrow{I^{\circ,\mathrm{ss}}} & K(\mathfrak{M}) \\ {\scriptstyle \int\circ I^{\circ,\mathrm{ss}}}\downarrow & & \downarrow{\scriptstyle \int} \\ K(\mathrm{St})_{\mathrm{reg}} & \hookrightarrow & K(\mathrm{St}) \end{array} \tag{33}$$

Proof. It suffices to prove that $[A^{\times,\mathrm{ss}}]\in K(\mathrm{St})$ is regular for every clear algebroid (X,A) admitting a faithful representation. This will suffice, by Proposition 1.66. If (X,A) is such an algebroid, there exists a Deligne–Mumford stack Y with a left GL_n-action together with a strict GL_n-equivariant algebra subbundle $B\hookrightarrow M_{n\times n}|_Y$ such that $B^\times=\mathrm{Stab}_{\mathrm{GL}_n}Y$ and $(X,A)=(\mathrm{GL}_n\backslash Y,\mathrm{GL}_n\backslash B)$.

Let $D_n\subset M_{n\times n}$ be the diagonal subalgebra and $T=D_n^\times$ the standard maximal torus of GL_n. As GL_n acts on $B\subset M_{n\times n}|_Y$, so does the torus T. We will now stratify B by the stabilizer with respect to the action of T. For this stratification to be canonical, we need $\mathrm{Stab}_T B$ to be the units in a finite type algebra over B.

In fact, such an algebra $C\subset D_n|_B$ is given as the intersection of $D_n|_B$ with $(B|_B)^{\mathrm{fix}}$ inside $M_{n\times n}|_B$. Here, $(B|_B)^{\mathrm{fix}}$ is the centralizer of the tautological section of $B|_B$, or, under the identification $B|_B=B\times_Y B$, the stack of commuting pairs. Thus, a section $(u,b,y)\in D_n|_B$ is in C if and only if $u\in Z_{B(y)}(b)$.

We have, indeed, an equality

$$C^\times=\mathrm{Stab}_T B$$

of relative group schemes over B, because for $t\in T$ and $(b,y)\in B\subset M_{n\times n}|_Y$,

$$\begin{aligned} t\in\mathrm{Stab}_T(b,y) &\iff {}^t b=b \text{ and } ty=y\\ &\iff tb=bt \text{ and } t\in\mathrm{Stab}_{\mathrm{GL}_n(y)}\\ &\iff tb=bt \text{ and } t\in B^\times(y)\\ &\iff t\in Z_{B^\times(y)}(b)\\ &\iff t\in C^\times(b,y). \end{aligned}$$

The subalgebras of D_n are in one-to-one correspondence with partitions $I=\{I_1,\dots,I_r\}$ of the set $\underline{n}=\{1,\dots,n\}$. The partition $\underline{n}=I_1\sqcup\dots\sqcup I_r$ corresponds to the subalgebra D_I whose primitive idempotents are the $e_{I_\rho}=\sum_{i\in I_\rho}e_i$, for $\rho=1,\dots,r$. Let us write $T_I=D_I^\times$ for the torus of units in D_I.

Now there is a unique stratification

$$\coprod_I B_I\longrightarrow B, \tag{34}$$

such that a section (b,y) of B factors through B_I if and only if the pullback of $C\subset D_n|_B$ via (b,y) is equal to D_I. The existence of this stratification is proved by passing to the rank stratification (see Definition 1.23) of C and observing that a subalgebra bundle $C\subset D_n|_S$, for any stack S, decomposes S into a disjoint union of open and closed substacks such that C is constant over these components. We may reformulate the defining property of B_I by saying that $(b,y)\in B_I$ if and only if $Z_{B(y)}(b)\cap D_n=D_I$. We also have, for $(b,y)\in B_I$, that $\mathrm{Stab}_T(b,y)=C^\times(b,y)=D_I^\times=T_I$.

The stratification (34) is T-equivariant, because for $t \in T$ and $(b, y) \in B$, we have

$$\begin{aligned}(b,y) \in B_I &\iff Z_{B(y)}(b) \cap D_n = D_I \\ &\iff Z_{B(ty)}({}^t b) \cap {}^t D_n = {}^t D_I \\ &\iff Z_{B(ty)}({}^t b) \cap D_n = D_I \\ &\iff ({}^t b, ty) \in B_I,\end{aligned}$$

as $T \subset D_n$ and D_n is commutative.

(Let us remark that we were not able to prove that for a general action of T on a Deligne–Mumford stack Y, the stabilizer stratifies Y equivariantly. The fact that the stabilizer is equal to the units in an algebra helps. Note also that we did not prove a defining property for (34) in terms of stabilizers in T.)

So, for every partition I of $\underline{n}$, the torus T acts on $B_I \subset B$, with stabilizer T_I. We therefore get an induced action of T/T_I on B_I. Matrix conjugation preserves units, so we get an induced action of T/T_I on

$$B_I^{\times} = B_I \cap B^{\times}.$$

In fact, this action even respects $B_I^{\times,\mathrm{ss}} = B_I \cap B^{\times,\mathrm{ss}}$, but the following modification does not.

Consider the action of $T_I \subset T$ on B_I by left multiplication:

$$t(b, y) = (tb, y).$$

This is a well-defined action, because $t \in T_I$ and $(b, y) \in B_I$ implies that $t \in Z_{B(y)}(b)$. In particular, $t \in B(y)$, so that $(t, y) \in B$, and the product $(t, y)(b, y) = (tb, y)$ in B exists. Moreover, $(tb, y) \in B_I$, because $Z_{B(y)}(tb) \cap D_n = Z_{B(y)}(b) \cap D_n$. This action of T_I on B_I preserves $B_I^{\times}$. Over *fields*, it also preserves sections which are semi-simple, because the product of two semi-simple commuting matrices is again a semi-simple matrix. (Note that this does not imply that T_I acts on B_I^{ss}, because even if (b, y) is a strict section of B, the product (tb, y) may not be strict.)

We finally consider the action of $T(I) = T_I \times T/T_I$ on $B_I^{\times}$, defined by

$$(t', t) * (b, y) = (t'\, {}^t b, ty). \tag{35}$$

The quotient stack $Z_I = B_I^{\times}/T(I)$ is a finite type scheme over R, so its Zariski topological space $|Z_I|$ is a Zariski space (see [LMB00, ch. 5]). By Chevalley's theorem (see [LMB00]), the image of $|B_I^{\times,\mathrm{ss}}|$ in $|Z_I|$ is constructible, so we can find disjoint, locally closed (reduced) algebraic substacks $Z_1, \dots, Z_n \subset Z_I$ such that this image is equal to $|Z_1| \sqcup \dots \sqcup |Z_n| \subset |Z_I|$. Let

$$\widetilde{Z}_I = Z_1 \amalg \cdots \amalg Z_n.$$

This is a finite type algebraic stack endowed with a representable monomorphism $\widetilde{Z}_I \to Z_I$.

We claim that $\widetilde{Z}_I$ is an algebraic stack with quasi-finite stabilizer. This will follow from the fact that, for field-valued points, the action of $T(I)$ on $B_I^{\times,\mathrm{ss}}$ has finite stabilizers. To see this, assume that

$$(t'\, {}^t b, ty) = (b, y),$$

for $(t', t) \in T_I \times T/T_I$ and $(b, y) \in B_I^{\times,\mathrm{ss}}$. This implies that $ty = y$, and hence conjugation by t preserves the fibre $B(y)$ of B over y. We have

$$t'\, {}^t b = b,$$

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where t' commutes with b. Rewriting as ${}^t b = t'^{-1} b$, we see that ${}^t b$ commutes with b. Changing basis, if necessary, we can diagonalize the three matrices b, ${}^t b$, and t' simultaneously. Since b and ${}^t b$ have the same eigenvalues, we see that the entries of the diagonal matrix ${}^t b$ are obtained from those of b by a permutation. Hence, there are at most $n!$ possible values for ${}^t b$, and hence also for $t' = b\,{}^t b^{-1}$. For every one of these possible values of t', there is at most one $t \in T/T_I$ such that ${}^t b = t'^{-1} b$. Thus, the action (35) has finite stabilizers, at least on field-valued points of $B_I^{\times,\mathrm{ss}}$, as required.

Consider the cartesian diagram

$$\begin{array}{ccccc} \widetilde{B}_I^{\times,\mathrm{ss}} & \xrightarrow{\widetilde{\beta}} & \widetilde{B}_I^{\times} & \xrightarrow{\widetilde{\pi}} & \widetilde{Z}_I \\ {\scriptstyle \alpha'}\downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} \\ B_I^{\times,\mathrm{ss}} & \xrightarrow{\beta} & B_I^{\times} & \xrightarrow{\pi} & Z_I \end{array}$$

obtained by pulling back $B_I^{\times,\mathrm{ss}} \to B_I^{\times} \to Z_I$ via $\widetilde{Z}_I \to Z_I$. The morphisms α and β are disjoint unions of isomorphisms onto locally closed substacks, so the same is true for α' and $\widetilde{\beta}$. But both α' and $\widetilde{\beta}$ are surjective on underlying Zariski topological spaces, so by the scissor relations, we have

$$[B_I^{\times,\mathrm{ss}}] = [\widetilde{B}_I^{\times,\mathrm{ss}}] = [\widetilde{B}_I^{\times}]$$

in $K(\mathrm{St})$. The morphism π is a principal $T(I)$-bundle, so the same is true for $\widetilde{\pi}$, and so by the bundle relations, we have

$$[B_I^{\times,\mathrm{ss}}] = [\widetilde{B}_I^{\times}] = [T(I)][\widetilde{Z}_I] = (q-1)^n[\widetilde{Z}_I]$$

in $K(\mathrm{St})$. It follows that we have

$$\begin{aligned} [A^{\times,\mathrm{ss}}] = [\mathrm{GL}_n \backslash B^{\times,\mathrm{ss}}] = \frac{[B^{\times,\mathrm{ss}}]}{[\mathrm{GL}_n]} &= \frac{1}{[\mathrm{GL}_n]} \sum_I [B_I^{\times,\mathrm{ss}}] \\ &= \frac{1}{[\mathrm{GL}_n]} \sum_I (q-1)^n [\widetilde{Z}_I] = q^{-(1/2)n(n-1)} \prod_{i=1}^{n-1} \frac{1}{q^i + \cdots + 1} \sum_I [\widetilde{Z}_I]. \end{aligned}$$

The claim now follows from Lemma 4.2. □

COROLLARY 4.4. *The multiple $(q-1)^k \int$ of the integral takes regular values on $K^{\leqslant k}(\mathfrak{M})$, for every $k \geqslant 0$.*

$$\begin{array}{ccc} K^{\leqslant k}(\mathfrak{M}) & & \\ \downarrow & \searrow{\scriptstyle (q-1)^k \int} & \\ K(\mathrm{St})_{\mathrm{reg}} & \hookrightarrow & K(\mathrm{St}) \end{array}$$

Proof. Consider the following diagram.

$$\begin{array}{ccccc} K^k(\mathfrak{M}) & \overset{(q-1)^k \int}{\dashrightarrow} & & & \\ \downarrow & & & & \\ \bigoplus_{\ell(\lambda)=k} K^{\lambda}(\mathfrak{M})_{(q-1)} & & K(\mathrm{St})_{\mathrm{reg}} & \hookrightarrow & K(\mathrm{St}) \\ & \searrow{\scriptstyle \bigoplus_{\ell(\lambda)=k} (k!/\widetilde{Q}_\lambda(q)) \int \circ I^{\mathrm{ss}}} & \downarrow & & \downarrow \\ & & K(\mathrm{Var})_{(q-1)} & \hookrightarrow & K(\mathrm{St})(q) \end{array}$$

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The dotted arrow exists because the square in the lower right of this diagram is cartesian, and the outer part of the diagram commutes. (Here we have identified the localization $K(\mathrm{Var})_{(q-1)}$ with its image in $K(\mathrm{St})(q) = K(\mathrm{St}) \otimes_{\mathbb{Q}[q]} \mathbb{Q}(q)$.) □

Corollary 4.5. *Defining* $\int t = q - 1$ *extends the integral to a* $K(\mathrm{DM})$*-linear homomorphism*

$$\int : \mathscr{K}(\mathfrak{M}) \longrightarrow K(\mathrm{St})_{\mathrm{reg}}.$$

4.2 The integral versus the Hall product

The Γ*-indexed integral.* Let Γ be a grading group for $\mathfrak{M}$, as in § 3.3. We assume, in addition, that Γ is endowed with a $\mathbb{Z}$-valued bilinear form χ.

Definition 4.6. We say $\mathfrak{M}$ is *hereditary* if for every $\gamma, \beta \in \Gamma$, the morphism $\mathfrak{E}_{\gamma,\beta} \to \mathfrak{M}_\gamma \times \mathfrak{M}_\beta$ is a vector bundle stack [BF97, Definition 1.9] of rank $-\chi(\beta, \gamma)$.

Let us assume henceforth that $\mathfrak{M}$ is hereditary.

We define $K(\mathrm{St})[\Gamma]$ to be the free $K(\mathrm{St})$-module on the symbols u^γ, for $\gamma \in \Gamma$, and introduce an associative product on $K(\mathrm{St})[\Gamma]$ by the formula

$$u^\gamma * u^\beta = q^{-\chi(\beta,\gamma)} u^{\gamma+\beta}$$

and extend it linearly to make $K(\mathrm{St})[\Gamma]$ a $K(\mathrm{St})$-algebra. Regular coefficients form a subalgebra $K(\mathrm{St})_{\mathrm{reg}}[\Gamma]$.

We define the Γ-indexed integral

$$\int : \mathscr{K}(\mathfrak{M}) \longrightarrow K(\mathrm{St})_{\mathrm{reg}}[\Gamma]$$
$$\sum_{\gamma\in\Gamma} x_\gamma \longmapsto \sum_{\gamma\in\Gamma} u^\gamma \int x_\gamma. \tag{36}$$

Proposition 4.7. *If* $\mathfrak{M}$ *is hereditary, the* Γ*-indexed integral preserves the star product. In fact, for* $x, y \in \mathscr{K}(\mathfrak{M})$ *we have*

$$\int x * y = \int x * \int y \in K(\mathrm{St})_{\mathrm{reg}}[\Gamma].$$

Proof. This is a straightforward calculation. One uses the fact that for $X \to \mathfrak{M}_\gamma$ and $Y \to \mathfrak{M}_\beta$, the morphism $X * Y \to X \times Y$ is a vector bundle stack of rank $-\chi(\gamma, \beta)$, and hence, in $K(\mathrm{St})$, we have $[X * Y] = q^{-\chi(\gamma,\beta)}[X][Y]$. □

Semi-classical limit. We will pass to the semi-classical limit of the integral $\int : \mathscr{K}(\mathfrak{M}) \to K(\mathrm{St})_{\mathrm{reg}}[\Gamma]$ by setting $t = 0$ (in the source) and hence $q = 1$ (in the target). As $\int$ respects the $*$-product, the semi-classical limit will be a morphism of Poisson algebras.

Modulo $(q - 1)$, the star product on $K(\mathrm{St})_{\mathrm{reg}}[\Gamma]$ is commutative; in fact, modulo $(q - 1)$ it is given by the commutative product $u^\gamma \cdot u^\beta = u^{\gamma+\beta}$. Hence, the quotient $K(\mathrm{St})_{\mathrm{reg}}[\Gamma]/(q - 1)$ inherits a Poisson bracket, defined by

$$x * y - y * x \equiv \{x, y\}(q - 1) \mod (q - 1)^2.$$

Explicitly, it is given by

$$\{u^\gamma, u^\beta\} = -\widetilde{\chi}(\beta, \gamma)\, u^{\gamma+\beta}, \tag{37}$$

where $\widetilde{\chi}$ is (twice) the anti-symmetrization of χ:

$$\widetilde{\chi}(\beta, \gamma) = \chi(\beta, \gamma) - \chi(\gamma, \beta).$$

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We conclude the following result.

Theorem 4.8. *If $\mathfrak{M}$ is hereditary, we have a morphism of Poisson algebras*

$$\int_{q=1} : K(\mathfrak{M}) \longrightarrow K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma]. \tag{38}$$

The Poisson structure on $K(\mathfrak{M})$ is described in § 3.2; the one on $K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma]$, above, see (37). The u^γ coefficient of the integral $\int_{q=1}$ may be expressed as

$$\int_{q=1} x = \left(\int \pi_{q-1}(x)\right)\bigg|_{q=1} = \sum_{n=0}^{\infty} \binom{q-1}{n} \int E_n(x)\bigg|_{q=1}$$

for $x \in K(\mathfrak{M})_\gamma$. Here we have used the operator π_t of Definition 2.11 and substituted $t = q-1$.

Proof. The homomorphism (38) is obtained by setting $t = 0$ in (36). Note that the deformation parameter t is mapped to the deformation parameter $(q-1)$, so that the Poisson bracket (which depends on the choice of the deformation parameter) is preserved.

To calculate $\int_{q=1}$, note that $x \mapsto \pi_t(x)$ is a section of the quotient map $\mathscr{K}(\mathfrak{M}) \to K(\mathfrak{M})$, obtained by setting $t = 0$. This gives rise to the displayed formula. □

Remark 4.9. Note that the diagram

$$\begin{array}{ccccc} K(\mathfrak{M}) \otimes \mathbb{Q}[t,1/t] & \xleftarrow{\text{invert } t} & \mathscr{K}(\mathfrak{M}) & \xrightarrow{t \mapsto 0} & K(\mathfrak{M}) \\ \Big\downarrow{\scriptstyle\int} & & \Big\downarrow{\scriptstyle\int} & & \Big\downarrow{\scriptstyle\int_{q=1}} \\ K(\mathrm{St})[\Gamma] & \longleftarrow & K(\mathrm{St})_{\mathrm{reg}}[\Gamma] & \xrightarrow{q \mapsto 1} & K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma] \end{array}$$

commutes. The central column is a morphism of one-parameter families of non-commutative algebras. The left-hand column is the general fibre and a morphism of non-commutative $K(\mathrm{DM}) \otimes \mathbb{Q}[t,1/t]$-algebras; the right-hand column is the semi-classical limit and hence a morphism of Poisson algebras.

Restricting the theorem to the virtually indecomposable elements, we obtain the following result.

Corollary 4.10. *The semi-classical limit of the integral defines a morphism of Lie algebras over $K(\mathrm{DM})$:*

$$\int_{q=1} : K^{\mathrm{vir}}(\mathfrak{M}) \longrightarrow K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma].$$

The bracket in $K^{\mathrm{vir}}(\mathfrak{M})$ is the commutator bracket of the Hall product, and the bracket in $K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma]$ is given in (37). The integral $\int_{q=1}$ is given by the formula

$$\int_{q=1} x = \sum_{\gamma \in \Gamma} u^\gamma \left((q-1)\int x_\gamma\right)\bigg|_{q=1}$$

for a virtually indecomposable Hall algebra element $x \in K^{\mathrm{vir}}(\mathfrak{M})$.

Remark 4.11. We have a surjective morphism of $K(\mathrm{Var})$-algebras

$$K(\mathrm{Var})/(q-1) \longrightarrow K(\mathrm{St})_{\mathrm{reg}}/(q-1)$$

and an isomorphism

$$K(\mathrm{Var})/((q-1)+\mathrm{Ann}(q-1)) \xrightarrow{\sim} K(\mathrm{St})_{\mathrm{reg}}/(q-1).$$

The former morphism is (most likely) not injective, because there is no (obvious) reason why elements in $\mathrm{Ann}(q-1) \subset K(\mathrm{Var})$ should map to zero in $K(\mathrm{Var})/(q-1)$, although they certainly map to zero in $K(\mathrm{St})_{\mathrm{reg}}/(q-1)$.

Without too much more effort, it is possible to prove that the semi-classical limit of the integral lifts to a $K(\mathrm{Var})$-linear homomorphism

$$\int_{q=1} : \mathscr{K}(\mathfrak{M}) \longrightarrow K(\mathrm{Var})/(q-1)[\Gamma].$$

Unfortunately, we cannot, at the moment, prove that this lift is a morphism of Poisson algebras.

Remark 4.12. Let $\Gamma_+ \subset \Gamma$ be a submonoid as above, i.e., having the property that every $\gamma \in \Gamma_+$ admits only finitely many decompositions $\gamma = \alpha + \beta$, with both $\alpha, \beta \in \Gamma_+$, and such that the intersection of all cofinite ideals is empty.

For every ideal $S \subset \Gamma_+$, the group $K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma_S]$ is an ideal in $K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma_+]$, and we let $K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma_+]^\wedge$ be the completion of $K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma_+]$ with respect to this collection of ideals.

The restriction of the semi-classical integral

$$\int_{q=1} : K(\mathfrak{M})_+ \longrightarrow K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma_+]$$

is continuous with respect to the filtrations induced by the cofinite ideals in Γ_+ and hence extends to the completions, giving rise to a morphism of Poisson-algebras

$$\int_{q=1} : \hat{K}(\mathfrak{M})_+ \longrightarrow K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma_+]^\wedge,$$

and a morphism of Lie algebras

$$\int_{q=1} : \hat{K}^{\mathrm{vir}}(\mathfrak{M})_+ \longrightarrow K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma_+]^\wedge.$$

If we now have a substack $\mathfrak{N} \subset \mathfrak{M}$ as above, i.e., every $\mathfrak{N}_\gamma = \mathfrak{N} \cap \mathfrak{M}_\gamma$ is of finite type, $\mathfrak{N}$ is closed under direct sums and extensions in $\mathfrak{M}$, and $\mathfrak{N}_\gamma \neq \emptyset$ only if $\gamma \in \Gamma_+$, we can form $\varepsilon_1[\mathfrak{N}] \in \hat{K}^{\mathrm{vir}}(\mathfrak{M})_+$ and apply to it the semi-classical integral $\int_{q=1}$ to obtain the *Joyce-type invariant*

$$\begin{aligned}\int_{q=1} \varepsilon_1[\mathfrak{N}] &= \int_{q=1} \log_*(1+[\mathfrak{N}]) \\ &= \left((q-1) \sum_{\substack{n \geqslant 1 \\ \gamma_1 \dots, \gamma_n \in \Gamma_+}} \frac{(-1)^{n+1}}{n} [\mathfrak{N}_{\gamma_1}] * \cdots * [\mathfrak{N}_{\gamma_n}]\right)\Bigg|_{q=1} \end{aligned} \tag{39}$$

in $K(\mathrm{St})_{\mathrm{reg}}/(q-1)[\Gamma_+]^\wedge$.

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Appendix. Comparison with Joyce's virtual projections in an example

Let us write $[n] = [B\,\mathrm{GL}_n \to \mathfrak{Vect}] \in K(\mathfrak{Vect})$. We have (cf. Remark 3.21)

$$E_r[n] = \sum_{\substack{\ell_1+\cdots+\ell_r=n\\ \ell_1\ldots\ell_r>0}} [\ell_1]\ldots[\ell_r].$$

This gives us

$$\begin{aligned}\pi_k[n] &= \sum_r \frac{s(r,k)}{r!} E_r[n]\\ &= \sum_r \frac{s(r,k)}{r!} \sum_{\substack{\ell_1+\cdots+\ell_r=n\\ \ell_1\ldots\ell_r>0}} [\ell_1]\cdots[\ell_r]\\ &= \sum_{\lambda\vdash n} \frac{s(\ell(\lambda),k)}{|\mathrm{Aut}\,\lambda|} \prod_i [\lambda_i].\end{aligned}$$

We remark also that the formula of Remark 2.12 gives us

$$\pi_t\bigg(\sum_{n\geqslant 0}[n]\,u^n\bigg) = \bigg(\sum_{n\geqslant 0}[n]\,u^n\bigg)^t,$$

which contains the above formulas for $\pi_k[n]$.

In [Joy07b, § 5.2], Joyce defines projection operators $\Pi_n^{\mathrm{vi}} : K(\mathfrak{M}) \to K(\mathfrak{M})$, which pairwise commute and add up to the identity (although he works with bare algebraic stacks, not algebroids). We expect that modulo this difference, we have

$$\pi_k = \Pi_k^{\mathrm{vi}}.$$

We will prove that these operators take the same values on the elements $[n] \in K(\mathfrak{Vect})$.

Proposition A.1. *We have*

$$\pi_k[n] = \Pi_k^{\mathrm{vi}}[n],$$

for all k and n.

Proof. Let T_n be the n-dimensional torus of diagonal matrices inside GL_n.

Joyce's $\mathcal{P}$ set [Joy07b, Definition 5.3] is trivial in this case because as a quotient stack $B\,\mathrm{GL}_n = */\mathrm{GL}_n$ where $*$ is a point, so $\mathcal{P}(*, T_n) = \{T_n\}$. The $\mathcal{Q}$ set $\mathcal{Q}(\mathrm{GL}_n, T_n)$ is computed in [Joy07b, Example 5.7] to be the set of all tori

$$T_\varphi := \{\mathrm{diag}(z_1,\ldots,z_n) : z_i \in \mathbb{G}_m, z_i = z_j \text{ if } \varphi(i) = \varphi(j), \forall i,j\},$$

where φ ranges over all surjection maps $\varphi : \underline{n} \to \underline{r}$. Finally, the $\mathcal{R}$ set coincides with $\mathcal{Q}$. Joyce's definition then needs computation of $M_G^X(P,Q,R)$, where P, Q, and R are selected respectively

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from $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$. In our case, this is

$$M^*_{\mathrm{GL}_n}(T_n,Q,R)=\left|\frac{N_{\mathrm{GL}_n}(T_n)}{C_{\mathrm{GL}_n}(Q)\cap N_{\mathrm{GL}_n}(T_n)}\right|^{-1} n^{\mathrm{GL}_n}_{T_n}(R,Q)$$

for all choices of $R,Q\in\mathcal{Q}(\mathrm{GL}_n,T_n)$ such that $R\subseteq Q$. Now we unwind the definition of $n^{\mathrm{GL}_n}_{T_n}(R,Q)$.

$$n(R,Q)=\sum_{\substack{B\subseteq\{\hat Q\in\mathcal{Q}:\hat Q\subseteq Q\}\\ Q\in B,\bigcap_{\hat Q\in B}\hat Q=R}}(-1)^{|B|-1}.$$

We can finally define the virtual projections of $B\,\mathrm{GL}_n$ as

$$\Pi^{vi}_k(B\,\mathrm{GL}_n)=\sum_{R:\dim R=k}\sum_{Q:R\subseteq Q}M^*_{\mathrm{GL}_n}(T_n,Q,R)[BC_G(Q)].$$

We say $Q\in\mathcal{Q}(\mathrm{GL}_n,T_n)$ is of 'type σ' if the corresponding surjection $\varphi:\{1,\dots,n\}\to\{1,\dots,r\}$ induces the partition $\sigma\vdash n$. Note that there are $n!/\sigma_1!\cdots\sigma_n!(1!)^{\sigma_1}\cdots(n!)^{\sigma_n}$ of them. Also, $C_{\mathrm{GL}_n}Q$ depends only on the type of Q and is isomorphic to $\prod_{i=1}^n[\mathrm{GL}_i]^{\sigma_i}$. The normalizer of T_n is $S_n\ltimes T^n$, and therefore

$$C_{\mathrm{GL}_n}(T_\varphi)\cap N_{\mathrm{GL}_n}(T_n)=\prod_{i=1}^n[S_i\ltimes T_i]^{\sigma_i}$$

and

$$\left|\frac{N_{\mathrm{GL}_n}(T^n)}{C_{\mathrm{GL}_n}(Q)\cap N_{\mathrm{GL}_n}(T_n)}\right|^{-1}=\frac{(1!)^{\sigma_1}\cdots(n!)^{\sigma_n}}{n!}.$$

We have

$$\begin{aligned}\Pi^{vi}_k(B\,\mathrm{GL}_n)&=\sum_{R:\dim R=k}\sum_{Q:R\subseteq Q}M^*_{\mathrm{GL}_n}(T_n,Q,R)[BC_G(Q)]\\&=\sum_Q\left|\frac{N_{\mathrm{GL}_n}(T_n)}{C_{\mathrm{GL}_n}(Q)\cap N_{\mathrm{GL}_n}(T_n)}\right|^{-1}\Bigg(\sum_{\substack{R\subseteq Q\\ \dim R=k}}n^{\mathrm{GL}_n}_{T_n}(R,Q)\Bigg)[BC_G(Q)]\\&=\sum_\sigma(\#Q\text{ of type }\sigma).\frac{(1!)^{\sigma_1}\cdots(n!)^{\sigma_n}}{n!}s(\ell(\sigma),k)\prod_{i=1}^n[B\,\mathrm{GL}_i]^{\sigma_i}\\&=\sum_\sigma\frac{1}{\sigma_1!\cdots\sigma_n!}s(\ell(\sigma),k)\prod_{i=1}^n[B\,\mathrm{GL}_i]^{\sigma_i},\end{aligned}$$

where the third line follows from the lemma below. We conclude that Joyce's virtual projections of $B\,\mathrm{GL}_n$ are identical to our eigenprojections. □

Lemma A.2. *For a Q of type σ, we have*

$$\sum_{R:\dim R=k}n(R,Q)=s(\ell(\sigma),k).$$

Proof. We let $m=\dim Q=\ell(\sigma)$ in this proof. Obviously, if $\ell(\sigma)<k$, there is no possible choice of

$$B\subseteq\{\hat Q:\hat Q\subseteq Q\}:Q\in B,\quad\bigcap_{\hat Q\in B}\hat Q=R,$$

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therefore proving

$$\sum_{R:\dim R=k} n(R,Q) = s(\ell(\sigma),k) \quad \text{if } \ell(\sigma) < k.$$

In the case that $\ell(\sigma) = k$, the only choice of R is Q itself and the only choice of B is the set $B = \{Q\}$. This proves

$$\sum_{R:\dim R=k} n(R,Q) = s(\ell(\sigma),k) \quad \text{if } \ell(\sigma) = k.$$

All other values of $s(\ell(\sigma),k)$ are defined recursively by

$$s(m,k) = s(m-1,k-1) - (m-1)s(m-1,k).$$

So it suffices to show that $\sum_{R:\dim R=k} n(R,Q)$ satisfies the same recursive relation.

For any choice of R,

$$n(R,Q) = \sum_{\substack{B\subseteq\{\hat{Q}\in\mathcal{Q}:\hat{Q}\subseteq Q\}\\ Q\in B,\bigcap_{\hat{Q}\in B}\hat{Q}=R}} (-1)^{|B|-1}$$

can also be computed by choosing only those $\hat{Q}$ that are codimension 1 inside Q. This is because for every $\hat{Q}$ of codimension > 2, the number t of intermediate subtori Q'

$$\hat{Q} \subset Q' \subseteq Q$$

is positive, and therefore B containing $\hat{Q}$ is included in 2^t possible choices of B with cancelling size parities.

Let us write the points of Q as m-tuples $(x_1,\ldots,x_m)$ with $x_i \in \mathbb{G}_m$. Let W be the $(m-1)$-dimensional torus consisting of points $(x_1,\ldots,x_{m-1})$. Any R with $\dim R = k$ is given by a set of defining equations

$$x_{i_1} = \cdots = x_{i_{k_i}}, \quad i = 1,2,\ldots.$$

In the defining equation of R with $\dim R = k$, either x_m does not appear, in which case $R|_W$ is $(k-1)$-dimensional, or x_m does appear, in which case $R|_W$ is k-dimensional and any choice of B consisting of only codimension 1 elements satisfying $\bigcap_{\hat{Q}\in B}\hat{Q} = R$ loses one of its elements after restriction to $R|_W$. This shows that

$$\sum_{\substack{B\subseteq\{\hat{Q}\in\mathcal{Q}:\hat{Q}\subseteq Q\}\\ Q\in B,\dim\bigcap_{\hat{Q}\in B}\hat{Q}=k}} (-1)^{|B|-1} = \sum_{\substack{B\subseteq\{\hat{Q}\in\mathcal{Q}:\hat{Q}\subseteq W\}\\ W\in B,\dim\bigcap_{\hat{Q}\in B}\hat{Q}=k-1}} (-1)^{|B|-1} - (m-1)\sum_{\substack{B\subseteq\{\hat{Q}\in\mathcal{Q}:\hat{Q}\subseteq W\}\\ W\in B,\dim\bigcap_{\hat{Q}\in B}\hat{Q}=k}} (-1)^{|B|-1}$$

which completes the proof. □

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