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# ORBITAL DECOMPOSITIONS OF REPRESENTATIONS OF NON-SIMPLY CONNECTED NILPOTENT GROUPS

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An orbital integral formula is proven for the direct integral decomposition of an induced representation of a connected nilpotent Lie group. Previous work required simple connectivity. An explicit description of the spectral measure and spectral multiplicity function is derived in terms of orbital parameters. It is also proven that connected (but not necessarily simply connected) exponential solvable symmetric spaces are multiplicity free. Finally, the qualitative properties of the spectral multiplicity function are examined via several illuminating examples.

## 1. INTRODUCTION

There is a growing body of literature [**3, 7, 8, 5**] devoted to orbital parametrisations of induced representations $\mathrm{Ind}_H^G \nu$ in the case that both $G$ and $H$ are connected Lie groups. Some of this work involves general groups [**7, 8**], but much of it is concerned with $G$ nilpotent. Moreover, in all of the latter, $G$ is assumed to be simply connected. Very beautiful formulas have been derived in that case [**3, 7**] — for the spectrum, multiplicity and spectral measure — as well as some interesting qualitative results regarding the multiplicity function [**4**]. In this paper I shall retain the nilpotent assumption on $G$, but drop the simple connectivity condition. Because the fundamental group is *infinite* cyclic, it turns out that — although orbital parametrisations of induced representations are still valid — they cannot be derived from the simply connected case (see below for a more detailed explanation). The proof must be accomplished by an induction argument that generalises that of [**7**]. In addition, regarding the qualitative results on the multiplicity, it turns out that one is still true, but the others are false (again see below). Finally, we shall prove that all connected (but not necessarily simply connected) exponential solvable symmetric spaces are multiplicity free.

Here is a more detailed explanation of the contents and import of the paper. Let $G$ be a connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. We do *not* assume $G$ is simply connected. Let $\widetilde{G}$ be a simply connected Lie group having $\mathfrak{g}$ as its Lie algebra. Then $\widetilde{Z} = \mathrm{Cent}\, \widetilde{G}$ is a vector group and there is a discrete subgroup $\Gamma \subset \widetilde{Z}$ such that

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$G \cong \widetilde{G}/\Gamma$. We write $P\colon \widetilde{G} \to G$ for the canonical projection. The irreducible unitary representations of $\widetilde{G}$ are parametrised orbitally by the co-adjoint orbits $\mathfrak{g}^*/\widetilde{G}$. (We assume the reader is familiar with the orbit method — see [6, 2].) Now suppose $\widetilde{H}$ is a connected (therefore simply connected) closed subgroup of $\widetilde{G}$, $\mathfrak{h}$ its Lie algebra. The orbital integral formula for the description of an induced representation is as follows. Let $\nu \in \widehat{\widetilde{H}}$, $\mathcal{O}_\nu \in \mathfrak{h}^*/\widetilde{H}$ the corresponding orbit. Then

$$(0) \qquad \operatorname{Ind}_{\widetilde{H}}^{\widetilde{G}} \nu = \int_{p^{-1}(\mathcal{O}_\nu)/H}^{\oplus} \pi_\varphi d\mu(\varphi) = \int_{\widetilde{G}\cdot p^{-1}(\mathcal{O}_\nu)/\widetilde{G}}^{\oplus} n_\varphi^\nu \pi_\varphi d\mu'(\varphi),$$

where $p\colon \mathfrak{g}^* \to \mathfrak{h}^*$ is the canonical projection, $n_\varphi^\nu = \#[\widetilde{G}\cdot\varphi \cap p^{-1}(\mathcal{O}_\nu)]/\widetilde{H}$, and $\mu$, $\mu'$ are push-forwards of the canonical measure on $p^{-1}(\mathcal{O}_\nu)$ (see [7, 8]).

Now suppose $H \subset G$ is a closed connected subgroup. For $\nu \in \widehat{H}$ we wish to describe the direct integral decomposition of $\operatorname{Ind}_H^G \nu$. It does *not* follow as a simple consequence of formula (0) applied to the induced representation $\operatorname{Ind}_{\widetilde{H}}^{\widetilde{G}} \nu \circ P$. Here is the explanation. Let $\widetilde{H}$ be the analytic subgroup of $\widetilde{G}$ whose Lie algebra is $\mathfrak{h}$. Then $P\left(\widetilde{H}\right) = H$, but $P^{-1}(H) = \widetilde{H}\Gamma$ will be disconnected and so bigger than $\widetilde{H}$ in general. It is obvious that understanding $\operatorname{Ind}_H^G \nu$ is exactly the same as understanding $\operatorname{Ind}_{\widetilde{H}\Gamma}^{\widetilde{G}} \nu \circ P$. By the simply connected theory, we understand $\operatorname{Ind}_{\widetilde{H}}^{\widetilde{G}} \nu \circ P$. Moreover,

$$\operatorname{Ind}_{\widetilde{H}}^{\widetilde{G}} \nu \circ P = \operatorname{Ind}_{\widetilde{H}\Gamma}^{\widetilde{G}} \operatorname{Ind}_{\widetilde{H}}^{\widetilde{H}\Gamma} \nu \circ P = \int_{(\Gamma/\Gamma\cap H)\widehat{}}^{\oplus} \operatorname{Ind}_{\widetilde{H}\Gamma}^{\widetilde{G}} (\nu \circ P) \times \omega d\omega.$$

Were $\Gamma$ a finite group, the latter would be a direct sum and we could read off information on the finitely many constituents. But in general $(\Gamma/\Gamma \cap H)\widehat{}$ is a torus and each of the constituents is infinitesimal (of measure zero). Thus we can deduce nothing (for $\omega = 1$) about the representation $\operatorname{Ind}_{\widetilde{H}\Gamma}^{G} \nu \circ P$. To illustrate, let $\mathfrak{g}$ be the three-dimensional Heisenberg Lie algebra with generators $X$, $Y$, $Z$ satisfying the bracket relation $[X, Y] = Z$. Let $\widetilde{G}$ be the corresponding simply connected Lie group, $\Gamma = \exp \mathbf{Z}Z$ and take $\widetilde{H} = \exp \mathbf{R}X$. For $\xi \in \mathbf{R}$, define $\chi_\xi \in \left(\widetilde{H}\right)\widehat{}$ by $\chi_\xi(\exp rX) = e^{2\pi i r\xi}$, $r \in \mathbf{R}$. Then we have

$$\operatorname{Ind}_{\widetilde{H}}^{\widetilde{G}} \chi_\xi = \int_{\mathbf{R}}^{\oplus} \pi_\zeta d\zeta,$$

where for $\zeta \neq 0$, $\pi_\zeta = \operatorname{Ind}_{\widetilde{H}\widetilde{Z}}^{\widetilde{G}} \chi_\xi \chi_\zeta$ is the unique irreducible of $\widetilde{G}$ having central character $\chi_\zeta(\exp tZ) = e^{2\pi i t\zeta}$. On $G = \widetilde{G}/\Gamma$, $H = \widetilde{H}/\widetilde{H} \cap \Gamma \cong \widetilde{H}$, we have

$$\operatorname{Ind}_H^G \chi_\xi = \int_{\mathbf{R}}^{\oplus} \chi_{\xi,\eta} d\eta \oplus \sum_{\substack{\zeta \in \mathbf{Z},\\ \zeta \neq 0}}^{\oplus} \pi_\zeta,$$

where $\chi_{\xi,\eta}$ is the character $\chi_{\xi,\eta}(\exp rX \exp sY \exp tZ) = e^{2\pi i(r\xi+s\eta)}$ The latter, when lifted to $\widetilde{G}$, is *not* a subrepresentation of the former.

Nevertheless, we are able to give a precise orbital interpretation and parametrisation of the induced representation $\mathrm{Ind}_H^G \nu$ for any connected nilpotent groups $G$, $H$. The formula is developed in Section 2. As explained above, we cannot derive the decomposition from those of [**3**, **7**]. The method of proof is induction on $\dim G/H$. Since $G$ is nilpotent, we can utilise the existence of a co-dimension 1 connected normal subgroup $N$ between $H$ and $G$. But we must be careful to take into account its relationship to $\Gamma$. The precise orbital integral formula is found in Theorem 1. We merely remark here that the multiplicity function is unchanged from (0), the spectrum is modified only in an obvious way to account for $\Gamma$, but the spectral measure is considerably more complicated than in the simply connected case.

In Section 3 we consider homogeneous spaces $G/H$ which are symmetric — that is, $G$ is a connected Lie group with an involution $\tau$ and $H$ is an open subgroup of the stability group $G^\tau$. In case $G$ is simply connected nilpotent, one knows [**1**] that $L^2(G/H)$ — that is, $\mathrm{Ind}_H^G 1$ — is multiplicity free. If $H = G^\tau$, this can be read off from (0) since it is easy to show that $\#[G\cdot\varphi \cap \mathfrak{h}^\perp]/H = 1$ for $\varphi \in \mathfrak{h}^\perp$. Now for virtually the same reasons as above, it is impossible to deduce from these facts anything about non-simply connected nilpotent symmetric spaces. Nevertheless we prove (Theorem 3) that any connected nilpotent symmetric space is multiplicity free. In fact we are able to prove Theorem 3 when $G$ is exponential solvable. This generalises Benoist's theorem [**1**] to arbitrary connected exponential solvable symmetric spaces.

Finally in Section 4 we take up the qualitative results on the multiplicity function $n_\varphi^\nu$ obtained by Corwin-Greenleaf [**4**] in the simply connected nilpotent case. In brief they: show that the multiplicity is always either uniformly infinite or finite almost everywhere; give a criterion for the latter; and show that in the finite multiplicity case, the multiplicity is bounded and of constant parity. We prove here that with the exception of the boundedness result, all of the above are false in the non-simply connected case. (The boundedness result says that even if infinite multiplicity occurs, there is an upper bound on the finite multiplicities that appear.) Several examples are given to illustrate Theorems 1 and 3, and other interesting aspects of the behaviour of induced representations for non-simply connected nilpotent groups.

## 2. The non-simply connected orbital integral formula

We continue with the notation of Section 1. We assume $\Gamma$ is not trivial. Let $r = \mathrm{rank}\,\Gamma \leqslant \dim \widetilde{Z}$, $r > 0$. We set $\Lambda = \log \Gamma$, a discrete subspace of $\mathfrak{z} = \mathrm{Lie}\left(\widetilde{Z}\right) = \mathrm{Aut}\,\mathfrak{g}$. We also set $Z = P\left(\widetilde{Z}\right) = \mathrm{Cent}\,G$. Now define

$$\mathfrak{g}^*_\Lambda = \{\varphi \in \mathfrak{g}^* : \varphi(\Lambda) \subset \mathbb{Z}\}.$$

If $\kappa : \mathfrak{g}^* \to \widehat{\left(\widetilde{G}\right)}$ denotes the ($\widetilde{G}$-equivariant) Kirillov map, then the image of $\mathfrak{g}^*_\Lambda$ is identified with $\widehat{G}$ because $\widehat{G} = \widehat{\left(\widetilde{G}\right)}_\Gamma = \{\pi \in \widehat{\left(\widetilde{G}\right)} : \pi|_\Gamma = \mathrm{Id}\}$ and obviously $\kappa(\varphi) \in \widehat{\left(\widetilde{G}\right)}_\Gamma \Leftrightarrow \varphi \in \mathfrak{g}^*_\Lambda$. It is also clear that $\mathfrak{g}^*_\Lambda$ is a $\widetilde{G}$-space and

$$\mathfrak{g}^*_\Lambda/G = \mathfrak{g}^*_\Lambda/\widetilde{G} \underset{\kappa}{\overset{\cong}{\leftrightarrow}} \widehat{\left(\widetilde{G}\right)}_\Gamma = \widehat{G}.$$

Now suppose $H \subset G$ is a connected closed subgroup $\mathfrak{h} = \mathrm{Lie}(H)$. As in Section 1, we take $\widetilde{H} = \exp \mathfrak{h}$ in $\widetilde{G}$ so that $P\left(\widetilde{H}\right) = H$, $P^{-1}(H) = \widetilde{H}\Gamma$. Obviously $P^{-1}(H) = \widetilde{H} \Leftrightarrow \Gamma \subset \widetilde{H}$. We set $\Delta = \Gamma \cap \widetilde{H}$ and $\Omega = \log \Delta = \Lambda \cap \mathfrak{h}$. We fix $\nu \in \widehat{H}$. Then there is a unique $H$-orbit $\mathcal{O}_\nu \subset \mathfrak{h}^*_\Omega$ such that $\kappa(\mathcal{O}_\nu) = \nu$. We select any $\psi \in \mathcal{O}_\nu$ so that $\mathcal{O}_\nu = H \cdot \psi$. If we restrict the projection $p : \mathfrak{g}^* \to \mathfrak{h}^*$ to $\mathfrak{g}^*_\Lambda$ we obtain $p(\mathfrak{g}^*_\Lambda) = \mathfrak{h}^*_\Omega$. However, $p^{-1}(\mathfrak{h}^*_\Omega)$ may not be contained in $\mathfrak{g}^*_\Lambda$ — for example, if $\Omega = \{0\}$ but $\Lambda \neq \{0\}$, then $p^{-1}(\mathfrak{h}^*_\Omega) = p^{-1}(\mathfrak{h}^*) = \mathfrak{g}^* \neq \mathfrak{g}^*_\Lambda$. Thus we set $p_\Lambda = p|_{\mathfrak{g}^*_\Lambda}$, $p_\Lambda : \mathfrak{g}^*_\Lambda \to \mathfrak{h}^*_\Omega$, and pay attention to $p_\Lambda^{-1}(\mathcal{O}_\nu)$ in what follows. Here is the orbital integral formula for the direct integral decomposition of an induced representation.

**Theorem 1.**

$$\mathrm{Ind}_H^G \nu = \int^\oplus_{p_\Lambda^{-1}(\mathcal{O}_\nu)/H} \pi_\varphi d\mu^\nu_{G,H}(\varphi) = \int^\oplus_{G \cdot p_\Lambda^{-1}(\mathcal{O}_\nu)/G} n^\nu_\varphi \pi_\varphi d\widetilde{\mu}^\nu_{G,H}(\varphi),$$

*where* $n^\nu_\varphi = \#[G\cdot\varphi \cap p_\Lambda^{-1}(\mathcal{O}_\nu)]/H$ *and* $\mu^\nu_{G,H}$, $\widetilde{\mu}^\nu_{G,H}$ *are push-forwards of the canonical measure on* $p_\Lambda^{-1}(\mathcal{O}_\nu)$.

Before beginning the proof we describe the canonical measure on $p_\Lambda^{-1}(\mathcal{O}_\nu)$. As in the simply connected case [7, Section 3], it is explained most readily in terms of a fiber space picture. Indeed, $p_\Lambda^{-1}(\mathcal{O}_\nu)$ is a fiber space

$$\text{“}(\mathfrak{g}/\mathfrak{h} + \Lambda)^*\text{”} \longrightarrow p_\Lambda^{-1}(\mathcal{O}_\nu)$$
$$\downarrow$$
$$\mathcal{O}_\nu.$$

The base is the $H$-orbit $\mathcal{O}_\nu = H \cdot \psi$ which carries its canonical ($H$-invariant) measure. The fiber is described as follows. First note that $\mathfrak{g}^*_\Lambda$ is a discrete disjoint union (paramterised by $\Lambda^* = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$) of affine spaces of dimension $\dim_{\mathbb{R}} \mathfrak{g} - rk_{\mathbb{Z}}(\Lambda)$. Let us write $\mathbb{R}(\mathfrak{h} + \Lambda)$ to denote the $\mathbb{R}$-span of the $\mathbb{Z}$-space $\mathfrak{h} + \Lambda$. Then the fiber $(\mathfrak{g}/(\mathfrak{h} + \Lambda))^*$ is the discrete disjoint union (parametrised by $(\Lambda/\Omega)^* = \mathrm{Hom}_{\mathbb{Z}}(\Lambda/\Omega, \mathbb{Z})$)

of affine spaces of dimension $\dim_{\mathbb{R}} \mathfrak{g}/\mathsf{R}(\mathfrak{h}+\Lambda)$. The canonical measure on the fiber is Lebesgue measure on each of its components. It is clear that $p_\Lambda^{-1}(\mathcal{O}_\nu)$ has components which are manifolds, and the canonical measure $p_\Lambda^{-1}(\mathcal{O}_\nu)$ agrees on each component with the canonical measure class of the manifold.

PROOF OF THEOREM 1: The result is self-evident if $H = G$, so we may assume $\dim \mathfrak{h} < \dim \mathfrak{g}$. We first show that it is also no loss of generality to assume $\dim \mathsf{R}(\mathfrak{h}+\Lambda) < \dim \mathfrak{g}$. So suppose $\mathsf{R}(\mathfrak{h}+\Lambda) = \mathfrak{g}$. Then certainly $\mathfrak{g} = \mathfrak{h} + \mathfrak{z}$. This forces $\mathfrak{h}$ to be an ideal in $\mathfrak{g}$ since $[\mathfrak{g}, \mathfrak{h}] = [\mathfrak{h}+\mathfrak{z}, \mathfrak{h}] = [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. We also have $G = HZ$. Set $Z_1 = H \cap Z$ a central subgroup of $H$. The assumption $\mathsf{R}(\mathfrak{h}+\Lambda) = \mathfrak{g}$ forces $G/H \cong Z/Z_1$ to be compact. Let $\chi$ be the central character of $\nu$ restricted to $Z_1$, $\nu|_{Z_1} = \chi \,\mathrm{Id}$. Write $\mathrm{Ind}_{Z_1}^{Z} \chi = \sum^{\oplus}_{(Z_\chi)\widehat{}} \lambda$, $(Z_\chi)\widehat{} = \{\lambda \in \widehat{Z} : \lambda|_{Z_1} = \chi\}$. Then clearly

$$\mathrm{Ind}_H^G \nu = \sum^{\oplus}_{(Z_\chi)\widehat{}} \nu_\lambda, \tag{1}$$

where $\nu_\lambda$ is the unique irreducible representation of $G = HZ$ satisfying $\nu_\lambda|_H = \nu$, $\nu_\lambda|_Z = \lambda\,\mathrm{Id}$. We show that the direct sum decomposition (1) agrees with the direct integral decomposition in the statement of Theorem 1. First, since $p_\Lambda^{-1}(\mathcal{O}_\nu) = \{\varphi \in \mathfrak{g}_\Lambda^* : \varphi|_{\mathfrak{h}} \in \mathcal{O}_\nu\}$, $p^{-1}(\mathcal{O}_\nu)/H$ is parametrised by $(Z_\chi)\widehat{}$. In fact, since $\mathfrak{g} = \mathfrak{h} + \mathfrak{z}$, the parameter is $d\lambda = (1/i)\varphi|_{\mathfrak{z}}$. Next the multiplicity function in Theorem 1 is one. This is because $G \cdot \varphi = HZ \cdot \varphi = H \cdot \varphi$ for any $\varphi \in p_\Lambda^{-1}(\mathcal{O}_\nu)$ so that

$$G \cdot \varphi \cap p_\Lambda^{-1}(\mathcal{O}_\nu) = H \cdot \varphi.$$

Finally, we have agreement of the measures also. This is because the affine spaces are trivial and the components of $p_\Lambda^{-1}(\mathcal{O}_\nu)$ are parametrised precisely by $(\Lambda/\Omega)^* \sim (Z_\chi)\widehat{}$.

We now proceed to the main argument, an induction on $\dim \mathfrak{g}/\mathfrak{h}$. So suppose that $\dim \mathfrak{g}/\mathfrak{h} = 1$. Since by the previous paragraph, we may (henceforth) assume $\dim \mathsf{R}(\mathfrak{h}+\Lambda) < \dim \mathfrak{g}$, it must be the case that $\Lambda \subset \mathfrak{h}$. Any co-dimension 1 subalgebra is an ideal, so $\mathfrak{h}$ is an ideal. Then $\Gamma \subset \widetilde{H} \triangleleft \widetilde{G}$ and $G/H \cong \left(\widetilde{G}/\Gamma\right)/\left(\widetilde{H}/\Gamma\right) \cong \widetilde{G}/\widetilde{H} \cong \mathsf{R}$. Set $\tilde{\nu} = \nu \circ P$. We know that $\mathrm{Ind}_{\widetilde{H}}^{\widetilde{G}} \tilde{\nu}$ is either irreducible or a one-parameter direct integral of irreducibles [6]. In either case, the orbital integral formula is true [7]

$$\mathrm{Ind}_{\widetilde{H}}^{\widetilde{G}} \tilde{\nu} = \int^{\oplus}_{p^{-1}(\mathcal{O}_\nu)/\widetilde{H}} \pi_\varphi \, d\mu^{\tilde{\nu}}_{\widetilde{G},\widetilde{H}}(\varphi).$$

But clearly $\mathrm{Ind}_H^G \nu \circ P = \mathrm{Ind}_{\widetilde{H}}^{\widetilde{G}} \tilde{\nu}$. Everything above factors through $P$ (for example, $p^{-1}(\mathcal{O}_\nu) = p_\Lambda^{-1}(\mathcal{O}_\nu)$, since $\Gamma \subset \widetilde{H}$). The conclusion is therefore completely clear.

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Now we come to the main induction step: $\dim \mathfrak{g}/\mathfrak{h} > 1$ and we assume the result is proven for lesser co-dimension. Since $\dim \mathsf{R}(\mathfrak{h}+\Lambda) < \dim \mathfrak{g}$ we know (by nilpotence) that we can find an intermediate ideal of codimension 1, that is, $\mathfrak{n} \lhd \mathfrak{g}$, $\dim \mathfrak{g}/\mathfrak{n} = 1$ and

$$\mathsf{R}(\mathfrak{h}+\Lambda) \subset \mathfrak{n} \lhd \mathfrak{g}.$$

We employ in succession induction in stages, the induction hypothesis, and the fact that direct integrals commute with induced representations. These account for the first three of the following equivalences

$$\begin{aligned} \operatorname{Ind}_H^G \nu = \operatorname{Ind}_N^G \operatorname{Ind}_H^N \nu &= \operatorname{Ind}_N^G \int_{p_{\mathfrak{n},\Lambda}^{-1}(\mathcal{O}_\nu)/H}^{\oplus} \gamma_\theta d\mu_{N,H}^\nu(\theta) \\ \text{(A)} \qquad &= \int_{p_{\mathfrak{n},\Lambda}^{-1}(\mathcal{O}_\nu)/H} \operatorname{Ind}_N^G \gamma_\theta \, d\mu_{N,H}^\nu(\theta) = \int_{p_{\mathfrak{g},\Lambda}^{-1}(\mathcal{O})/H}^{\oplus} \pi_\varphi d\mu_{G,H}^\nu(\varphi). \end{aligned}$$

(Notational clarification: $p_{\mathfrak{n},\Lambda} \colon \mathfrak{n}_\Lambda^* \to \mathfrak{h}_\Omega^*$, $p_{\mathfrak{g},\Lambda} \colon \mathfrak{g}_\Lambda^* \to \mathfrak{h}_\Omega^*$.)

It remains to prove the last equation (A). The equality of spectrum is almost obvious. The orbital spectrum of $\operatorname{Ind}_N^G \gamma_\theta$ consists of $p_{\mathfrak{g},\mathfrak{n}}^{-1}(N \cdot \theta)$ regardless of whether the co-dimension 1 induced representation is irreducible or not — see [7, Section 0]. Thus the spectrum of the left side of (A) is $p_{\mathfrak{g},\mathfrak{n}}^{-1}\left(p_{\mathfrak{n},\Lambda}^{-1}(\mathcal{O}_\nu)\right)$. That clearly equals $p_{\mathfrak{g},\Lambda}^{-1}(\mathcal{O}_\nu)$, which is the spectrum of the right side. To demonstrate equality of mulitplicity, we require a generalisation of [7, Proposition 1.7]. The proper generalisation takes into account the disconnected nature of $p_{\mathfrak{g},\Lambda}^{-1}(\mathcal{O}_\nu)$. We formulate the result quite generally.

**Proposition 2.** *Let $H \subset N \lhd G$ be connected nilpotent Lie groups. Fix $\nu \in \widehat{H}$. Then generically on $p_{\mathfrak{n},\Lambda}^{-1}(\mathcal{O}_\nu)$ we have*

$$\dim G \cdot \theta \cap p_{\mathfrak{n},\Lambda}^{-1}(\mathcal{O}_\nu) = \dim \mathfrak{g} \cdot \theta \cap p_{\mathfrak{n}A,\Lambda}^{-1}(\mathfrak{h} \cdot \psi).$$

Note. It is crucial to observe that $p_{\mathfrak{n},\Lambda}^{-1}(\mathcal{O}_\nu)$ is a countable disjoint union of (simply) connected spaces and that on each component we have generically the equality of dimension — however that generic dimension may vary from component to component.

Proof: The proof is identical to that of [7] as soon as we make the proper modification. Namely, the Zariski open set $\mathcal{U} \subset \mathfrak{n}^*$ selected in [7] is only servicable one component at a time. Thus the correct order of selection is: fix a component, then choose a corresponding $Z$-open set, and then the reasoning proceeds exactly as in [7].

Now we are ready for the proof of equal multiplicity. We fix a component in $p_{\mathfrak{n},\Lambda}^{-1}(\mathcal{O}_\nu)$. Then not only do the orbit intersections have a generic dimension, but we also know that generically $\operatorname{Ind}_N^G \gamma_\theta$ is either irreducible or a 1-parameter direct integral

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of irreducibles. Our argument splits according to this dichotomy. This is reminiscent of [7], but I have simplified the argument in two ways. Firstly (as in [9]), I have removed the intermediate step of inducing through a character. Secondly, for the 1-parameter case, I am able to give a single argument which applies to both the finite and infinite multiplicity situations. Thus, if one compares the following with [7, Section 2], there are three subcases instead of four.

(i) $\mathrm{Ind}_N^G \gamma_\theta$ not irreducible (generically). Then, on the component in question, I claim that the map $\varphi \to \theta = \varphi|_{\mathfrak{n}}$ when restricted to

$$G \cdot \varphi \cap p_{\mathfrak{g},\Lambda}^{-1}(\mathcal{O}_\nu) \to G \cdot \theta \cap p_{\mathfrak{n},\Lambda}^{-1}(\mathcal{O}_\nu),$$

is a bijection — in particular, generates a bijection of $H$-orbits. This follows easily from the equality $G_\varphi = G_\theta$ which must hold when $\mathrm{Ind}_N^G \gamma_\theta$ is not irreducible [7, Section 2]. In fact, if for $\varphi$, $g \cdot \varphi \in G \cdot \varphi \cap p_{\mathfrak{g},\Lambda}^{-1}(\mathcal{O}_\nu)$ we have $\theta = \varphi|_n = g \cdot \varphi|_{\mathfrak{n}} = g \cdot \theta$, then $g \in G_\theta = G_\varphi$ and the claim is established.

(ii) $\mathrm{Ind}_N^G \gamma_\theta$ is irreducible (generically on the component in question). In this case we must split the argument according to whether the multiplicity is finite or infinite. These are characterised by

(a) $\dim \mathfrak{g} \cdot \varphi \cap p^{-1}(\mathfrak{h} \cdot \psi) > \dim \mathfrak{h} \cdot \varphi$

(b) $\dim \mathfrak{g} \cdot \varphi \cap p^{-1}(\mathfrak{h} \cdot \psi) = \dim \mathfrak{h} \cdot \varphi$.

In the former case the multiplicity on the right side of (A) is $+\infty$; in the latter it is finite and equal to $n_\varphi^\nu = \#[G \cdot \varphi \cap p_{\mathfrak{g},\Lambda}^{-1}(\mathcal{O}_\nu)]/H$.

(iia) It is enough in this instance to prove uniform infinite multiplicity on the left side of (A). Hence, again invoking Proposition 2, it is clearly enough to show

$$\dim \mathfrak{g} \cdot \theta \cap p_{\mathfrak{n}}^{-1}(\mathfrak{h} \cdot \psi) > \dim \mathfrak{h} \cdot \theta.$$

In fact, we have (using [7, p.443]) that

$$\begin{aligned}
\dim \mathfrak{g} \cdot \theta \cap p^{-1}(\mathfrak{h} \cdot \psi) &= \dim \mathfrak{g} \cdot \theta \cap p_{\mathfrak{n}}^{-1}\left(\mathfrak{h}_\psi^\perp(\mathfrak{h})\right) \\
&= \dim \mathfrak{n}_\varphi^\perp \cap \mathfrak{h}_\psi^\perp \\
&= \dim \left(\mathfrak{n}_\varphi + \mathfrak{h}_\psi\right)^\perp \\
&= \dim \mathfrak{n}/\left(\mathfrak{n}_\varphi + \mathfrak{h}_\psi\right) \\
&= \dim \mathfrak{n} - \dim \mathfrak{n}_\varphi - \dim \mathfrak{h}_\psi + \dim \mathfrak{h}_\varphi \\
&= \dim \mathfrak{n}/\mathfrak{n}_\varphi + \dim \mathfrak{h}/\mathfrak{h}_\psi - \dim \mathfrak{h}/\mathfrak{h}_\varphi \\
&= \dim \mathfrak{g} \cdot \theta + \dim \mathfrak{h} \cdot \psi - \dim \mathfrak{h} \cdot \varphi.
\end{aligned}$$

(In the above, all perps but the first are taken in $\mathfrak{n}^*$.) Now we know (from [7, Section 1]) that condition (a) is equivalent to $\dim \mathfrak{g} \cdot \varphi > 2 \dim \mathfrak{h} \cdot \varphi - \dim \mathfrak{h} \cdot \psi$. Both sides of

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this inequality are even. That, together with the observation $\dim \mathfrak{g}\cdot\theta \geqslant \dim \mathfrak{g}\cdot\varphi - 1$, yields

$$\dim \mathfrak{g}\cdot\theta > 2\dim \mathfrak{h}\cdot\varphi - \dim \mathfrak{h}\cdot\psi.$$

Combining these, we obtain

$$\begin{aligned}\dim G\cdot\theta \cap p_{\mathfrak{n},\Lambda}^{-1}(\mathcal{O}_\nu) &= \dim \mathfrak{g}\cdot\theta \cap p^{-1}(\mathfrak{h}\cdot\psi)\\ &= \dim \mathfrak{g}\cdot\theta + \dim \mathfrak{h}\cdot\psi - \dim \mathfrak{h}\cdot\varphi\\ &> \dim \mathfrak{h}\cdot\varphi\\ &\geqslant \dim \mathfrak{h}\cdot\theta.\end{aligned}$$

This completes the argument in case (a).

(iib) In this case the multiplicities in (A) are finite and we show they are equal by demonstrating that generically on $p_{\mathfrak{g},\Lambda}^{-1}(\mathcal{O}_\nu)$ the restriction map

$$G\cdot\varphi \cap p_{\mathfrak{g},\Lambda}^{-1}(\mathcal{O}_\nu) \to G\cdot\theta \cap p_{\mathfrak{n},\Lambda}^{-1}(\mathcal{O}_\nu),$$

although not a pointwise bijection, is a bijection of $H$-orbits. In fact, it is clearly enough to show that if for $\varphi$, $g\cdot\varphi \in G\cdot\varphi \cap p_{\mathfrak{g},\Lambda}^{-1}(\mathcal{O}_\nu)$, $\theta = \varphi|_{\mathfrak{n}}$, it is true that $g\cdot\theta = h\cdot\theta$ for some $h \in H$, then $g\cdot\varphi = h'\cdot\varphi$ for some $h' \in H$. Now $h^{-1}g \in G_\theta$. But in case (b), $G_\theta = N_\theta$ (see [**7**, Section 2]). Then it suffices to prove $N_\theta\cdot\varphi = H_\theta\cdot\varphi$. For if so, then $h^{-1}g\cdot\varphi = h'\cdot\varphi$ for some $h' \Rightarrow g\cdot\varphi = hh'\cdot\varphi$. Now we know that when $\mathrm{Ind}_N^G\gamma_\theta$ is irreducible that $N_\theta\cdot\varphi = \varphi + \mathbf{R}\alpha$, $\alpha \in \mathfrak{g}^*$, $\alpha \neq 0$, $\alpha|_{\mathfrak{n}} = 0$. The only possibilities for $H_\theta\cdot\varphi$ are $\varphi$ or $\varphi + \mathbf{R}\alpha$. We can rule out the former by the following computations. First

$$2\dim \mathfrak{h}\cdot\theta - \dim \mathfrak{h}\cdot\psi \leqslant \dim \mathfrak{n}\cdot\theta = \dim \mathfrak{g}\cdot\varphi - 2 = 2\dim \mathfrak{h}\cdot\varphi - \dim \mathfrak{h}\cdot\psi - 2.$$

Hence

$$\dim \mathfrak{h}\cdot\theta \leqslant \dim \mathfrak{h}\cdot\varphi - 1.$$

Strict inequality is impossible, so $\dim \mathfrak{h}\cdot\theta = \dim \mathfrak{h}\cdot\varphi - 1$. Therefore $\dim \mathfrak{h}_\theta = \dim \mathfrak{h}_\varphi + 1$, and so $H_\theta\cdot\varphi = \varphi$ is not possible.

Our task is reduced to proving equivalence of the spectral measures in equation (A). We take our motivation partly from [**8**, Section 4 Spectral Measures]. It is clear that the $H$-equivariant projection $p_{\mathfrak{g},\Lambda}^{-1}(\mathcal{O}_\nu) \to p_{\mathfrak{n},\Lambda}^{-1}(\mathcal{O}_\nu)$ takes components to components — since $\Lambda \subset \mathfrak{n}$. Hence it preserves canonical measure classes. Therefore in case (ii) (where the induced representation is irreducible), the map factors through $H$ and it is totally clear that it takes $\mu_{G,H}^\nu$ to $\mu_{N,H}^\nu$. Case (i) requires somewhat more subtle reasoning. In this case we have

$$\mathrm{Ind}_N^G\gamma_\theta = \int_{\mathbf{R}}^{\oplus} \pi_{\varphi+s\alpha}\,ds.$$

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It is again appropriate to work within a single component. Let $\mathcal{C}$ be a Borel cross-section for a component in $p_{\mathfrak{n},\Lambda}^{-1}(\mathcal{O}_\nu)$. Fix $Y \in g \setminus \mathfrak{n}$. For any $\theta \in \mathfrak{n}^*$, define $\varphi_\theta$ by setting it equal to $\theta$ on $\mathfrak{n}$ and $0$ on $Y$. Set $\mathcal{S} = \{\varphi_\theta + s\alpha : \theta \in \mathcal{C}, s \in \mathbf{R}\}$. Then I claim $\mathcal{S}$ is a cross-section for the corresponding component in $p_{\mathfrak{g},\Lambda}^{-1}(\mathcal{O}_\nu)$. Accepting that momentarily, we see that the Borel surjection $\varphi_\theta + s\alpha \to \theta$, $\mathcal{S} \to \mathcal{C}$ takes the class of $\mu_{G,H}^\nu$ to that of $\mu_{N,H}^\nu$.

Now we prove that $\mathcal{S}$ is a cross-section. First, for any $h \in H$, $\theta \in \mathfrak{n}^*$, there exists $s_{h,\theta} \in \mathbf{R}$ such that

$$h \cdot \varphi_\theta = \varphi_{h\cdot\theta} + s_{h,\theta}\alpha.$$

Then if $\varphi \in p_{\mathfrak{g},\Lambda}^{-1}(\mathcal{O}_\nu)$ (actually in our prescribed component), $\theta = \varphi|_{\mathfrak{n}}$ and $h \cdot \theta \in \mathcal{C}$, we have

$$h \cdot \varphi = h \cdot (\varphi_\theta + \varphi(Y)\alpha) = \varphi_{h\cdot\theta} + (s_{h,\theta} + \varphi(Y))\alpha \in \mathcal{S},$$

where we used the easily-verified equality $h\alpha = \alpha$. On the other hand, if $\varphi$, $h \cdot \varphi \in \mathcal{S}$, then since $\theta$ and $h \cdot \theta \in \mathcal{C}$, we must have $h \in H_\theta$. Hence

$$h \cdot \varphi = h \cdot (\varphi_\theta + \varphi(Y)\alpha) = \varphi_{h\cdot\theta} + (s_{h,\theta} + \varphi(Y))\alpha = \varphi + s_{h,\theta}\alpha.$$

But in case (a) the functionals $\varphi$ and $\varphi + s\alpha$ lie in distinct $G$-orbits if $s \neq 0$. Thus it must be that $s_{h,\theta} = 0 \Rightarrow h \cdot \varphi = \varphi$. $\square$

## 3. Exponential solvable symmetric spaces are multiplicity free

We now generalise Benoist's theorem [1] to arbitrary exponential solvable symmetric spaces. Let $G$ be a connected exponential solvable Lie group. That means $\mathfrak{g}$ is solvable and has no purely imaginary eigenvalues. We do not assume $G$ is simply connected. Still we have

**Theorem 3.** *Let $G$ be connected exponential solvable, $\tau$ an involutive automorphism of $G$. Let $H$ be an open subgroup of $G^\tau$. Then $\mathrm{Ind}_H^G 1$ is multiplicity free.*

Proof: Exactly as with nilpotent groups, we let $\widetilde{G}$ be a simply connected Lie group having $\mathfrak{g} = \mathrm{Lie}(G)$ as its Lie algebra. Then there is a discrete subgroup $\Gamma \subset \widetilde{Z} = \mathrm{Cent}\,\widetilde{G}$ such that $G \cong \widetilde{G}/\Gamma$. $P\colon \widetilde{G} \to G$ denotes the canonical projection. We consider first the case that $H = (G^\tau)_0$, the identity component of the stability group. Let $\widetilde{H}$ be the (*a fortiori* simply connected) analytic subgroup of $\widetilde{G}$ with $\mathfrak{h} = \mathrm{Lie}(H)$ as its Lie algebra. Then clearly

$$\left(\mathrm{Ind}_H^G 1\right) \circ P \cong \mathrm{Ind}_{\widetilde{H}\Gamma}^{\widetilde{G}} 1.$$

Therefore, for this case, it is enough to prove

**Theorem 3′.** *Let $G$ be simply connected exponential solvable, $\tau$ an involutive automorphism of $G$, $H = G^\tau$, $\Gamma \subset \operatorname{Cent} G$ a discrete subgroup. Then $\operatorname{Ind}_{H\Gamma}^G 1$ is multiplicity free.*

Proof: We let $Z$ be the connected subgroup of $\operatorname{Cent} G$ generated by $\Gamma$. Then

$$\operatorname{Ind}_{H\Gamma}^G 1 = \sum_{\lambda \in (Z/Z\cap H)\widehat{}} \operatorname{Ind}_{HZ}^G \lambda.$$

Since any constituent of $\operatorname{Ind}_{HZ}^G \lambda$ has $Z$-central character $\lambda$, it is absolutely clear that to prove $\operatorname{Ind}_{H\Gamma}^G 1$ is multiplicity free, it suffices to prove that each of the representations $\operatorname{Ind}_{HZ}^G \lambda$ is multiplicity free. But we have an orbital integral expression for these, namely

$$\operatorname{Ind}_{HZ}^G \lambda = \int_{(\mathfrak{h}+\mathfrak{z})_\omega^\perp/H}^\oplus \pi_\varphi d\dot{\varphi}$$

where $\omega = (1/i)\, d\lambda$, $(h + \mathfrak{z})_\omega^\perp = \{\varphi \in \mathfrak{g}^* : \varphi|_\mathfrak{h} = 0, \varphi|_\mathfrak{z} = \omega\}$. Also $G/H$ symmetric implies that for $\varphi \in \mathfrak{h}^\perp$, we have $G\cdot\varphi \cap \mathfrak{h}^\perp = H\cdot\varphi$ (see [1, 7]). Hence for $\varphi \in (\mathfrak{h} + \mathfrak{z})_\omega^\perp$, we have $g \cdot \varphi|_\mathfrak{z} = \phi|_\mathfrak{z}$ and therefore

$$G \cdot \varphi \cap (\mathfrak{h} + \mathfrak{z})_\omega^\perp = H \cdot \varphi.$$

∎

Summarising, we have proven Theorem 3 in case $H$ is connected — that is, if $\tau$ is an involutive automorphism of a connected exponential solvable group $G$ and $H = (G^\tau)_0$, then $\operatorname{Ind}_H^G 1$ is multiplicity free. We now complete the proof by demonstrating;

**Lemma 4.** *Let $G$ be connected exponential solvable, $\tau$ an involutive automorphism. Then $G^\tau/(G^\tau)_0$ is finite.*

If so, then for any $H$, $(G^\tau)_0 \subset H \subset G^\tau$, we have

$$\operatorname{Ind}_{(G^\tau)_0}^G 1 = \sum_{\chi \in [H/(G^\tau)_0]\widehat{}}^\oplus \operatorname{Ind}_H^G \chi,$$

a finite direct sum. Clearly the whole sum can be multiplicity free only if each constituent is. This completes the proof of Theorem 3 once we have the

Proof of Lemma 4: If $G$ is simply connected, then $G^\tau = (G^\tau)_0$. Otherwise we employ the usual structure: $G = \widetilde{G}/\Gamma$, $P: \widetilde{G} \to G$. Given $\tau$, there exists a unique involutive automorphism $\tilde{\tau}$ of $\widetilde{G}$ such that $d\tilde{\tau} = d\tau$. Then clearly

$$\tau(\Gamma) = \Gamma. \tag{2}$$

In fact, the involutions of $G$ are in $1-1$ correspondence with the involutions of $\widetilde{G}$ that satisfy (2). Now

$$G^\tau = \{\widetilde{g}\Gamma : \widetilde{\tau}(\widetilde{g})\Gamma = \widetilde{g}\Gamma\}$$
$$P^{-1}(G^\tau) = \{\widetilde{g} \in \widetilde{G} : \widetilde{g}^{-1}\widetilde{\tau}(\widetilde{g}) \in \Gamma\} \stackrel{\text{def}}{=} \widetilde{G}^\tau_\Gamma$$
$$P^{-1}((G^\tau)_0) = \widetilde{G}^\tau\Gamma.$$

Thus to prove the lemma, it suffices to prove ▯

**Lemma 4′.** *Let $G$ be simply connected exponential solvable, $\Gamma \subset \operatorname{Cent} G$ a discrete subgroup, $\tau$ an involutive automorphism of $G$ satisfying $\tau(\Gamma) = \Gamma$. Then if $G^\tau_\Gamma = \{g \in G : g^{-1}\tau(g) \in \Gamma\}$, we have $G^\tau_\Gamma / G^\tau\Gamma$ is finite.*

Proof: Set $H = G^\tau$. Then $G = H \exp \mathfrak{q}$ where $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$, $\mathfrak{h} = \operatorname{Lie}(H)$, $\mathfrak{q} =$ the $-1$ eigenspace of $d\tau$. For $g \in G$, write $g = h \exp X$, $h \in H$, $X \in \mathfrak{q}$. Then $\tau(g) = h \exp -X$ and

$$g^{-1}\tau(g) = \exp -2X \in \Gamma \Leftrightarrow -2X \in \Lambda = \log \Gamma.$$

Thus it is clear that $\# G^\tau_\Gamma / G^\tau\Gamma \leqslant 2^{\operatorname{rank}\Lambda} < \infty$. ▯

## 4. Qualitative results and examples

The orbital formula (0) has been further elucidated by several qualitative results of Corwin-Greenleaf on the multiplicity function $n^\nu_\varphi$. Those results are:

- CG(i) The multiplicity is either uniformly $+\infty$ or finite almost everywhere;
- (ii) the finite multiplicity case is characterised by the equality $\dim \mathfrak{g} \cdot \varphi = 2 \dim \mathfrak{h} \cdot \varphi - \dim \mathfrak{h} \cdot \psi$, generically in $\varphi \in p^{-1}(\mathcal{O}_\nu)$;
- (iii) if finite, the multiplicity is bounded;
- (iv) if finite, the multiplicity has constant parity.

The attention that has been paid to generalisation of these results has been directed towards the possibilities that might occur if the nilpotent assumption is relaxed to exponential solvable or even completely solvable. (The author has a forthcoming paper along these lines.) Here I want to focus attention in a different direction — retain the nilpotent assumption, but drop simple connectivity.

**Proposition 5.** *For induced representations $\operatorname{Ind}_H^G \nu$ wherein $G$, $H$ are only connected — but not necessarily simply connected — nilpotent Lie groups, properties CG(i), (ii), (iv) are false, while CG(iii) is true.*

Proof: For the counterexamples to (i) and (iv), see Examples (1) and (2) below. Once (i) is false, property (ii) has no real meaning. The truth of CG(iii) is actually a

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relatively simple consequence of the proof of [7, Proposition 5.1]. It is evident from [7, p.457, especially formula (1)] that there is a single fixed upper bound $M$ which depends only on the structure constants for $\widetilde{G}$, such that for *any* discrete group $\Gamma \subset \operatorname{Cent}\widetilde{G}$, for *any* connected subgroup $H \subset G \cong \widetilde{G}/\Gamma$ and for *any* $\nu \in \widehat{H}$, the *finite* multiplicity that occurs in $\operatorname{Ind}_H^G \nu$ cannot exceed $M$. ∎

We conclude the paper with a brief presentation of several illustrative examples.

(1) Let $G = \widetilde{G}/\Gamma$ be the example from the introduction, but take $H = \{1\}$ and $\nu = 1$. Then the regular representation of $G$

$$\operatorname{Ind}_{\{1\}}^G 1 = \int_{\mathbf{R}^2}^{\oplus} \chi_{\xi,\eta} d\xi d\eta \oplus \infty \sum_{\substack{\zeta \in \mathbf{Z} \\ \zeta \neq 0}}^{\oplus} \pi_\zeta$$

has mixed finite and infinite multiplicity.

(2) Let $\mathfrak{g}$ be the four-dimensional nilpotent Lie algebra with generators $X$, $Y$, $Z$, $W$ satisfying bracket relations $[X, Y] = Z$, $[X, Z] = W$; $\widetilde{G}$ is the corresponding simply connected nilpotent Lie group. We write $\varphi \in \mathfrak{g}^*$ as $\varphi = \xi X^* + \eta Y^* + \zeta Z^* + \omega W^*$. The representations of $\widetilde{G}$ are parametrised by the following orbital cross-section:

$$\begin{aligned} &\text{generic} && \mathcal{C}_1 = \{\varphi = (0, \eta, 0, \omega) : \eta \in \mathbf{R}, \omega \neq 0\} \\ &\text{degenerate, but inf-dim} && \mathcal{C}_2 = \{\varphi = (0, 0, \zeta, 0) : \zeta \neq 0\} \\ &\text{characters} && \mathcal{C}_3 = \{\varphi = (\xi, \eta, 0, 0) : \xi, \eta \in \mathbf{R}\}. \end{aligned}$$

Now take $\Gamma = \exp \mathbf{Z}W$, $G = \widetilde{G}/\Gamma$ and $H = \exp \mathbf{R}Y$. The representations that pass to $G$ require $\omega \in \mathbf{Z}$ in $\mathcal{C}_1$, but no conditions on $\mathcal{C}_2$ or $\mathcal{C}_3$. Then

$$\operatorname{Ind}_H^G 1 = 2 \sum_{\substack{\omega \in \mathbf{Z} \\ \omega \neq 0}}^{\oplus} \int_{\operatorname{sgn}(\omega)\eta < 0}^{\oplus} \pi_{\varphi(0,\eta,0,\omega)} d\eta \oplus \int_{\mathbf{R}}^{\oplus} \pi_{\varphi(0,0,\zeta,0)} d\zeta,$$

an example of mixed parity.

(3) Taking an outer tensor product of the last two examples, we clearly get an example in which the multiplicities $1$, $2$, $\infty$ each occur on sets of positive measure.

(4) Both Examples (1) and (2) provide instances of mixed type — that is, both generic and degenerate representations occurring on sets of positive measure. In Example 1 the degenerate representations have finite multiplicity and the non-degenerate representations have infinite multiplicity. Here is an example where the roles are reversed. Take $G$ as in Example (2), but let $H = \exp \mathbf{R}Z$. Then for $\zeta \neq 0$

$$\operatorname{Ind}_H^G \chi_\zeta = \infty \pi_{\varphi(0,0,\zeta,0)} \oplus \sum_{\substack{\omega \in \mathbf{Z} \\ \omega \neq 0}}^{\oplus} \int^{\oplus} \pi_{\varphi(0,\eta,0,\omega)} d\eta.$$

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(5) In the simply connected case any induced representation has spectrum concentrated in one "layer" of orbits — a property which we have seen is violated in the non-simply connected case. Another feature of the simply connected case is that an induced representation is either irreducible, a continuous one-parameter direct integral of irreducibles, or an infinite multiple of one of the previous two. This configuration also fails in the non-simply connected case — in fact, we can have an infinite direct *sum* of inequivalent irreducibles. To see that, once again let $G$ be as in Example 2, but set $H = \exp \mathbf{R}Y + \mathbf{R}Z$. Then for $\eta \in \mathbf{R}$, $\zeta \neq 0$, we have

$$\mathrm{Ind}_H^G \chi_{\eta,\zeta} = \sum_{\substack{\omega \in \mathbf{Z} \\ \omega \neq 0}}^{\oplus} \pi_{\varphi(0,\eta-\zeta^2/2\omega,0,\omega)} \oplus \pi_{\varphi(0,0,\zeta,0)}.$$

(6) Finally we illustrate the results of Section 3, that is, Theorem 3 and Lemma 4. We take the five-dimensional Lie algebra $\mathfrak{g}$ spanned by $X$, $Y$, $Z$, $U$, $V$ satisfying bracket relations $[X, Y] = Z$, $[X, Z] = U$, $[Y, Z] = V$. We write $\varphi \in \mathfrak{g}^*$ as $\varphi = \xi X^* + \eta Y^* + \zeta Z^* + \mu U^* + \nu V^*$. The representations of $\widetilde{G}$ are parametrised by the following orbital cross-sections:

| | |
|---|---|
| generic | $\varphi(\xi, 0, 0, \mu, \nu)$, $\xi, \mu \in \mathbf{R}$, $\nu \neq 0$ |
| slightly degenerate | $\varphi(0, \eta, 0, \mu, 0)$, $\eta \in \mathbf{R}$, $\mu \neq 0$ |
| quite degenerate | $\varphi(0, 0, \zeta, 0, 0)$, $\zeta \neq 0$ |
| characters | $\varphi(\xi, \eta, 0, 0, 0)$, $\xi, \eta \in \mathbf{R}$. |

We take $\Gamma = \exp \mathbf{Z}U + \mathbf{Z}V$, $G = \widetilde{G}/\Gamma$ and $H = \exp \mathbf{R}Z$. Then

$$\mathrm{Ind}_H^G 1 = \int_{\mathbf{R}^2}^{\oplus} \chi_{\xi,\eta} d\xi d\eta \oplus \int^{\oplus} \sum_{\substack{\mu \in \mathbf{Z} \\ \mu \neq 0}}^{\oplus} \pi_{\varphi(0,\eta,0,\mu,0)} d\eta \oplus \int^{\oplus} \sum_{\substack{\mu,\nu \in \mathbf{Z} \\ \nu \neq 0}}^{\oplus} \pi_{\varphi(\xi,0,0,\mu,\nu)} d\xi.$$

The involution here is given, at the Lie algebra level, by $\tau : X \to -X$, $Y \to -Y$, $Z \to Z$, $U \to -U$, $V \to -V$. One computes readily that

$$\widetilde{G}_\Gamma^\tau = \{\exp zZ \exp uU \exp vV : z \in \mathbf{R},\, u, v \in \tfrac{1}{2}\mathbf{Z}\},$$

an extension of order 4 over $\widetilde{G}^\tau \Gamma$.

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