BULL. AUSTRAL. MATH. SOC. 46M05
VOL. 8 (1973), 211-214. (46J05)

A constructive proof of Gelbaum's theorem on tensor products

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A constructive proof is given of Gelbaum's result that the maximal ideal space of the tensor product of commutative Banach algebras is homeomorphic to the cartesian product of the maximal ideal spaces.

It is a well-known result of Gelbaum [1] that if A and B are commutative Banach algebras and $\Delta(A)$ and $\Delta(B)$ their respective maximal ideal spaces, then the maximal ideal space $\Delta(A \otimes_\gamma B)$ of $A \otimes_\gamma B$ is naturally homeomorphic to $\Delta(A) \times \Delta(B)$, where γ is the greatest cross-norm. We are concerned with giving a much shorter, constructive proof of this result, using only the universal mapping property of tensor products for the cross-norm γ, and the known representations of elements of $A \otimes_\gamma B$.

Recall that any element $z \in A \otimes_\gamma B$ may be written $z = \sum_{i=1}^{\infty} a_i \otimes b_i$, where $\sum_{i=1} \|a_i\|\|b_i\| < \infty$, and that, if S and T are continuous endomorphisms of A and B, respectively, then

$$\|(S \otimes T)(z)\| = \left\|\sum_{i=1}^{\infty} (Sa_i) \otimes (Tb_i)\right\| \le \|S\|\|T\|\|z\| .$$

THEOREM (Gelbaum). *Let A and B be commutative Banach algebras. Then $\Delta(A \otimes_\gamma B)$ is weak-* homeomorphic to $\Delta(A) \times \Delta(B)$.*

Received 2 November 1972.

Proof. Let $\varphi : \Delta(A) \times \Delta(B) \to \left(A \otimes_\gamma B\right)^*$ be the map such that $\varphi(f, g) = f \otimes g$. It is easily checked that $f \otimes g \in \Delta\left(A \otimes_\gamma B\right)$ and that φ is injective.

We claim φ is surjective: let $h \in \Delta\left(A \otimes_\gamma B\right)$, and let $z \in A \otimes_\gamma B$, written $z = \sum_{i=1}^{\infty} a_i \otimes b_i$, be an identity mod h. Define $f_z \in A^*$ by $f_z(a) = h\left(\sum_i aa_i \otimes b_i\right)$, for $a \in A$. (We note that $\sum_i aa_i \otimes b_i$ is well-defined, by the universal mapping property.) It is clear that f_z is linear on A, and, if $a, a' \in A$, we have

$$\begin{aligned} f_z(aa') &= h\left(\sum_i aa'a_i \otimes b_i\right) = h\left(\left(\sum_i aa'a_i \otimes b_i\right)\left(\sum_i a_i \otimes b_i\right)\right) \\ &= h\left(\sum_{i,j} aa'a_ia_j \otimes b_ib_j\right) \\ &= h\left(\left(\sum_i aa_i \otimes b_i\right)\left(\sum_i a'a_i \otimes b_i\right)\right) \\ &= f_z(a)f_z(a') , \end{aligned}$$

so that f_z is multiplicative on A, and hence $f_z \in \Delta(A)$. Finally, f_z is independent of our choice of identity mod h. For, if $z' = \sum_i a_i' \otimes b_i'$ is another identity mod h, and if $a \in A$, we have

$$\begin{aligned} f_z(a) &= h\left(\sum_i aa_i \otimes b_i\right) = h\left(\left(\sum_i aa_i \otimes b_i\right)\left(\sum_i a_i' \otimes b_i'\right)\right) \\ &= h\left(\sum_{i,j} aa_ia_j' \otimes b_ib_j'\right) \\ &= h\left(\left(\sum_i aa_i' \otimes b_i'\right)\left(\sum_i a_i \otimes b_i\right)\right) = f_{z'}(a) . \end{aligned}$$

We now define $g_z \in \Delta(B)$ in a similar manner, and claim $h = f_z \otimes g_z$. Let $x \in A \otimes_\gamma B$ be of form $x = a \otimes b$. Then

https://doi.org/10.1017/S0004972700042453 Published online by Cambridge University Press

$$\left(f_z \otimes g_z\right)(a \otimes b) = f_z(a)g_z(b) = h\left(\sum_i aa_i \otimes b_i\right)h\left(\sum_i a_i \otimes bb_i\right)$$
$$= h\left(\sum_{i,j} aa_ia_j \otimes bb_ib_j\right)$$
$$= h\left((a \otimes b)\left(\sum_i a_i \otimes b_i\right)\left(\sum_i a_i \otimes b_i\right)\right)$$
$$= h(a \otimes b) .$$

Since elements of form $a \otimes b$ span a dense subspace of $A \otimes_\gamma B$, we see that $h = f_z \otimes g_z$, and hence φ is surjective.

If $h \in \Delta(A \otimes_\gamma B)$, we now write $h = f \otimes g$, where $f \in \Delta(A)$ and $g \in \Delta(B)$.

It is easily shown that φ is continuous. Suppose that $(f_p, g_p) \xrightarrow{p} (f, g)$ in $\Delta(A) \times \Delta(B)$. If $z \in A \otimes_\gamma B$ is of form $z = \sum_{i=1}^{n} a_i \otimes b_i$ (that is, if $z \in A \otimes B$), it is clear that $(f_p \otimes g_p)(z) \xrightarrow{p} (f \otimes g)(z)$. Now, let $z \in A \otimes_\gamma B$, and let $\varepsilon > 0$ be given. Choose $z' \in A \otimes B$ such that $\|z-z'\| < \varepsilon/3$. Then, if p is sufficiently large, we have

$$\left|(f_p \otimes g_p)(z)-(f \otimes g)(z)\right| \le \left|(f_p \otimes g_p)(z)-(f_p \otimes g_p)(z')\right|$$
$$+ \left|(f_p \otimes g_p)(z')-(f \otimes g)(z')\right| + \left|(f \otimes g)(z')-(f \otimes g)(z)\right| < \varepsilon ,$$

since $\|f_p \otimes g_p\| = \|f_p\|\|g_p\| \le 1$ and $\|f \otimes g\| = \|f\|\|g\| \le 1$, so that φ is continuous.

Finally, we claim φ^{-1} is continuous. Let $h_p = f_p \otimes g_p \xrightarrow{p} h = f \otimes g$ be a convergent net in $\Delta(A \otimes_\gamma B)$. We claim $f_p \xrightarrow{p} f$ and $g_p \xrightarrow{p} g$ in $\Delta(A)$ and $\Delta(B)$, respectively. To show $f_p \xrightarrow{p} f$, it will suffice to show that every subnet $\{f_q\}$ of $\{f_p\}$ has a subsubnet $\{f_s\}$ such that $f_s \xrightarrow{s} f$. Let $\{f_q\}$ be a subnet of $\{f_p\}$. Since $\overline{\Delta(A)}$ is compact, there is a subnet $\{f_r\}$ of $\{f_q\}$ which converges

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to some f' in $\overline{\Delta(A)}$. By similar reasoning, there exists a subnet $\{g_s\}$ of $\{g_r\}$ such that $g_s \xrightarrow{s} g'$ for some g' in $\overline{\Delta(B)}$. We then have $f_s \otimes g_s \xrightarrow{s} f' \otimes g'$, and, since $\{f_s \otimes g_s\}$ is a subnet of $\{f_p \otimes g_p\}$, we have $f_s \otimes g_s \xrightarrow{s} f \otimes g$. Thus, $f' \otimes g' = f \otimes g$, so that $f' \in \Delta(A)$ and $g' \in \Delta(B)$. Then, since φ is injective, $f' = f$. Similarly, we may show that $g_p \xrightarrow{p} g$, so that φ^{-1} is continuous, and the homeomorphism is established. //

With suitable modifications, our proof will serve to show the same result for any tensor product $A \otimes_\alpha B$ of commutative Banach algebras A and B , where $\lambda \le \alpha \le \gamma$ is any uniform cross-norm such that $A \otimes_\alpha B$ is a Banach algebra and where λ is the "least" cross-norm.

Reference

[1] Bernard R. Gelbaum, "Tensor products of Banach algebras", *Canad. J. Math.* 11 (1959), 297-310.

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