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SOME PROPERTIES OF CLASS $A(k)$ OPERATORS AND THEIR HYPONORMAL TRANSFORMS

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Abstract. In this paper we shall first show that if T is a class $A(k)$ operator then its operator transform $\hat{T}$ is hyponormal. Secondly we prove some spectral properties of T via $\hat{T}$. Finally we show that T has property (β).

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Let **H** be a complex Hilbert space and **L(H)** the algebra of all bounded linear operators on **H**. An operator $T \in$ **L(H)** has a unique polar decomposition $T = U|T|$ where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the partial isometry satisfying $N(U) = N(T) = N(|T|)$ and $N(U^*) = N(T^*)$.

An operator $T \in$ **L(H)** is said to be hyponormal if $T^*T \geq TT^*$ where T^* is the adjoint of T. As a generalisation of hyponormal operators, p-hyponormal and log-hyponormal operators are defined in **[2]** and **[9]** respectively. An operator T is said to be *p-hyponormal* if and only if $(T^*T)^p \geq (TT^*)^p$ for a positive number p and *log-hyponormal* if and only if T is invertible and $\log(T^*T) \geq \log(TT^*)$. An operator T is said to be of *class A* if and only if $|T^2| \geq |T|^2$. See **[9]**. As a generalisation of class A, class $A(k)$ and class $A(s, t)$ are defined in **[9]** and **[8]** respectively. T belongs to class $A(k)$, if and only if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$ where $k > 0$. For positive numbers s and t, T belongs to class $A(s, t)$ if and only if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$. In particular a class $A(k, 1)$ operator is a class $A(k)$ operator **[18]**. It is well known that inequalities $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$ and $(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2$ are equivalent **[18]**.

The following inclusion relations hold among these classes:

$$
\begin{aligned}
\{\text{hyponormal}\} &\subset \{p\text{-hyponormal}, 0 < p < 1\} \ [\mathbf{12}] \\
&\subset \{\text{class } A(s, t), s, t \in (0, 1]\} \ [\mathbf{8}] \\
&\subset \{\text{class } A\} \ [\mathbf{12}] \\
&\subset \{\text{class } A(k), k \geq 1\} \ [\mathbf{9}].
\end{aligned}
$$

The Aluthge transform $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ was introduced in **[1]**. An operator is *w-hyponormal* if $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$ **[3]**. The Aluthge transforms are useful in the study of these new classes of operators. “The Aluthge transform is an operator transform from the class of w-hyponormal and semi-hyponormal operators to the class of semi-hyponormal and hyponormal respectively. By using Aluthge transforms we can obtain spectral properties of these new classes of operators from those of hyponormal operators" [7]. But so far we have not obtained any property of a class $A(k)$ operator and it becomes difficult to study its properties. In this paper a new operator transform $\hat{T}$ of T from the class $A(k)$ to the class of hyponormal operators is given by

$$|\hat{T}| = ||T|^k T|^{\frac{1}{k+1}}.$$

We denote the spectrum, the point spectrum, the approximate point spectrum and the residual spectrum of an operator T by $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$ and $\sigma_r(T)$ respectively. A complex number λ is in the *normal approximate point spectrum* $\sigma_{na}(T)$ if there exists a sequence $\{y_n\}$ of unit vectors such that $(T-\lambda)y_n \to 0$ and $(T-\lambda)^* y_n \to 0$ as $n \to \infty$. For a hyponormal operator T, $\sigma_a(T) = \sigma_{na}(T)$ because the inequality $\|(T-\lambda)^* y\| \leq \|(T-\lambda)y\|$ always hold for all $\lambda \in \mathbf{C}$ and all $y \in \mathbf{H}$ [7].

In the following theorem we shall show that the operator transform $\hat{T}$ is hyponormal when T is a class $A(k)$ operator, where $k > 1$. Throughout this paper we assume that $k > 1$.

THEOREM 1. *If $T = U|T|$ is the polar decomposition of a class $A(k)$ operator, then $\hat{T} = WU||T|^k T|^{\frac{1}{k+1}}$ is hyponormal, where $|T||T^*| = W\,|\,|T||T^*|\,|$ is the polar decomposition.*

The following theorems play an important role in the proof of Theorem 1.

Theorem R_1 [12]. *Let A and B be positive operators. Then for each $p \geq 0$ and $r \geq 0$ the following assertions hold:*

(a) *If $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, then $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$.*

(b) *If $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$ and $N(A) \subset N(B)$, then $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.*

Theorem R_2 (Löwner-Heinz inequality [12]). *$A \geq B \geq 0$ ensures that $A^\alpha \geq B^\alpha$ for any $\alpha \in (0, 1]$.*

Theorem R_3 [13]. *Let $T = U|T|$ and $S = V|S|$ and $|T||S^*| = W\,|\,|T||S^*|\,|$ be the polar decompositions. Then $TS = UWV|TS|$ is also the polar decomposition.*

Proof of Theorem 1. By assumption T is a class $A(k)$ operator. The following inequalities hold.

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} = (|T|U^*|T|^{2k}U|T|)^{\frac{1}{k+1}} \geq |T|^2 \Longleftrightarrow (|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2. \quad (1)$$

Applying Theorem R_1 we obtain

$$|T|^{2k} \geq (|T|^k |T^*|^2 |T|^k)^{\frac{k}{k+1}}. \quad (2)$$

Since $\frac{1}{k} < 1$, by Theorem R_2 we have

$$|T|^2 \geq (|T|^k |T^*|^2 |T|^k)^{\frac{1}{k+1}} = (|T|^k U|T|^2 U^* |T|^k)^{\frac{1}{k+1}}. \quad (3)$$

From (1) and (3) we get

$$(|T|U^*|T|^{2k}U|T|)^{\frac{1}{k+1}} \geq |T|^2 \geq (|T|^k U|T|^2 U^* |T|^k)^{\frac{1}{k+1}}. \quad (4)$$

Let $S = |T|^k U|T| = |T|^k T$. Then (4) becomes, $(S^*S)^{\frac{1}{k+1}} \geq (SS^*)^{\frac{1}{k+1}}$. This shows that $S = |T|^k T$ is $\frac{1}{k+1}$ hyponormal. Besides, since $T = U|T|$ and $|T|^k = U^*U|T|^k$ are the

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polar decompositions, by Theorem R_3, $|T|^kT$ has the following polar decomposition

$$|T|^kT = U^*UWU||T|^kT|, \tag{5}$$

where $|T|^k|T^*| = W||T|^k|T^*||$ is the polar decomposition. Accordingly we have $N(U) \subseteq N(|T^*||T|^k) = N(W^*)$ and $W^*U^*U = W^*$ on $\mathbf{H} = N(U) \oplus R(U^*)$.

Hence (5) can be written as $|T|^kU|T| = U^*UWU||T|^kU|T|| = WU||T|^kT|$ which is $\frac{1}{k+1}$ hyponormal. It follows that $\hat{T} = WU||T|^kT|^{\frac{1}{k+1}}$ is hyponormal. □

We note that $\hat{T}||T|^kT|^{\frac{k}{k+1}} = \hat{T}|\hat{T}|^k = WU||T|^kT| = |T|^kT$.

THEOREM 2. *Let T be a class $A(k)$ operator and $\{y_n\}$ be a sequence of unit vectors in $\boldsymbol{H}$ such that $\lim_{n\to\infty}(\hat{T}-\lambda)y_n = 0$. If $\lim_{n\to\infty}|\hat{T}|^ky_n$ and $\lim_{n\to\infty}|T|^ky_n$ exist, then $\lim_{n\to\infty}(T-\lambda)y_n = 0$ and $\lim_{n\to\infty}(T-\lambda)^*y_n = 0$ where $\lambda \in \boldsymbol{C}$.*

Proof. Since $\hat{T}$ is hyponormal, $\lim_{n\to\infty}(\hat{T}-\lambda)y_n = 0$ implies that $\lim_{n\to\infty}(\hat{T}-\lambda)^*y_n = 0$. When $\lambda = 0$, $\lim_{n\to\infty}\hat{T}y_n = 0$ and hence $\lim_{n\to\infty}\|\hat{T}y_n\| = 0$. Since T is a class $A(k)$ operator we have

$$\begin{aligned}
\|Ty_n\|^2 &= (|T|^2y_n, y_n)\\
&\le ((T^*|T|^{2k}T)^{1/k+1}y_n, y_n)\\
&= (||T|^kT|^{2/k+1}y_n, y_n)\\
&= \|\hat{T}y_n\|^2 \quad \text{since} \quad \hat{T} = WU||T|^kT|^{1/k+1}.
\end{aligned}$$

It follows that $\lim_{n\to\infty}\|Ty_n\| \le \lim_{n\to\infty}\|\hat{T}y_n\| = 0$ and hence $\lim_{n\to\infty}Ty_n = 0$.

Also, since $\|T^*y_n\| \le \|Ty_n\|$, we have $\lim_{n\to\infty}\|T^*y_n\| \le \lim_{n\to\infty}\|Ty_n\| = 0$ and hence $\lim_{n\to\infty}T^*y_n = 0$.

On the other hand, when $\lambda \ne 0$ we have $\lim_{n\to\infty}(\hat{T}-\lambda)y_n = 0$ and $\lim_{n\to\infty}(\hat{T}-\lambda)^*y_n = 0$ so that

$$\lim_{n\to\infty}(|\hat{T}|^2 - |\lambda|^2)y_n = 0 \text{ and } \lim_{n\to\infty}|(\hat{T})^*|^2 - |\lambda|^2)y_n = 0. \tag{6}$$

Since $|\hat{T}|^2 = ||T|^kT|^{\frac{2}{k+1}} = (T^*|T|^{2k}T)^{\frac{1}{k+1}}$ and $|(\hat{T})^*|^2 = |T^*|T|^k|^{\frac{2}{k+1}} = (|T|^k|T^*|^2|T|^k)^{\frac{1}{k+1}}$, we obtain from (6) that

$$\lim_{n\to\infty}((T^*|T|^{2k}T)^{\frac{1}{k+1}} - |\lambda|^2)y_n = 0 \text{ and } \lim_{n\to\infty}((|T|^k|T^*|^2|T|^k)^{\frac{1}{k+1}} - |\lambda|^2)y_n = 0. \tag{7}$$

Since T belongs to class $A(k)$,

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \ge |T|^2 \ge (|T|^k|T^*|^2|T|^k)^{\frac{1}{k+1}},$$

and hence by (7) we have

$$\lim_{n\to\infty}\big((|T|^2 - |\lambda|^2)y_n, y_n\big) = 0. \tag{8}$$

Also,

$$\big\|\big[(T^*|T|^{2k}T)^{\frac{1}{k+1}} - |T|^2\big]^{\frac{1}{2}}y_n\big\|^2 = ([(T^*|T|^{2k}T)^{\frac{1}{k+1}} - |\lambda|^2]y_n, y_n) - ([|T|^2 - |\lambda|^2]y_n, y_n).$$

It follows from (7) and (8) that $\lim_{n\to\infty}\|[(T^*|T|^{2k}T)^{\frac{1}{k+1}} - |T|^2]^{\frac{1}{2}}y_n\|^2 = 0$.

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Consequently we obtain

$$\lim_{n\to\infty}(|T|^2-|\lambda|^2)y_n=\lim_{n\to\infty}[|T|^2-(T^*|T|^{2k}T)^{1/k+1}]y_n + \lim_{n\to\infty}[(T^*|T|^{2k}T)^{1/k+1}-|\lambda|^2]y_n=0.$$

Hence $\lim_{n\to\infty}(|T|-|\lambda|)y_n=0$. By hypothesis $\lim_{n\to\infty}|T|^k y_n$ and $\lim_{n\to\infty}|\hat{T}|^k y_n$ exist, so that we get

$$\lim_{n\to\infty}(|T|^k-|\lambda|^k)y_n=0, \tag{9}$$

$$\lim_{n\to\infty}(|\hat{T}|^k-|\lambda|^k)y_n=0. \tag{10}$$

Now $|T|^kT=WU||T|^kT|$ and $\hat{T}=WU||T|^kT|^{1/k+1}=WU|\hat{T}|$ implies $|T|^kT=\hat{T}|\hat{T}|^k$. Hence $T^*|T|^k=|\hat{T}|^k(\hat{T})^*$ and so by (9) and (10)

$$(T^*-\bar{\lambda})y_n=\frac{T^*}{|\lambda|^k}(|\lambda|^k-|T|^k)y_n+\frac{|\hat{T}|^k}{|\lambda|^k}((\hat{T})^*-\bar{\lambda})y_n+\frac{\bar{\lambda}}{|\lambda|^k}(|\hat{T}|^k-|\lambda|^k)y_n\longrightarrow 0$$

as $n\to\infty$. That is $\lim_{n\to\infty}(T-\lambda)^*y_n=0$. Since $|\,\|T^*y_n\|-|\lambda|\,|\leq\|(T-\lambda)^*y_n\|$, we have

$$\lim_{n\to\infty}\|T^*y_n\|=|\lambda|. \tag{11}$$

Also

$$\begin{aligned}\|(TT^*-|\lambda|^2)^{1/2}y_n\|^2 &= ((TT^*-|\lambda|^2)y_n,y_n)\\ &= (TT^*y_n,y_n)-|\lambda|^2\\ &= \|T^*y_n\|^2-|\lambda|^2,\end{aligned}$$

and by (11)

$$\lim_{n\to\infty}((TT^*-|\lambda|^2)y_n,y_n)=0. \tag{12}$$

Hence by (12) and (8),

$$\begin{aligned}\lim_{n\to\infty}\|(|T^*|^2-|T|^2)^{1/2}y_n\|^2 &= \lim_{n\to\infty}((|T^*|^2-|T|^2)y_n,y_n)\\ &= \lim_{n\to\infty}[((|T^*|^2-|\lambda|^2)y_n,y_n)-((|T|^2-|\lambda|^2)y_n,y_n)]\\ &= 0.\end{aligned}$$

It follows that

$$\lim_{n\to\infty}(|T^*|^2-|T|^2)y_n=0. \tag{13}$$

By (13) $\lim_{n\to\infty}(|T^*|^2-|\lambda|^2)y_n=\lim_{n\to\infty}[(|T^*|^2-|T|^2)y_n+(|T|^2-|\lambda|^2)]y_n=0$. Finally, $\lim_{n\to\infty}(T-\lambda)y_n=\lim_{n\to\infty}(\bar{\lambda})^{-1}[(|T^*|^2-|\lambda|^2)y_n-T(T^*-\bar{\lambda})y_n]=0$. □

COROLLARY 3. *Let T be a class $A(k)$ operator. Suppose that $\lambda\in\sigma_{na}(\hat{T})$ and $\{y_n\}$ is a corresponding sequence of unit vectors such that $(\hat{T}-\lambda)y_n\to 0$ and $(\hat{T}-\lambda)^*y_n\to 0$ as $n\to\infty$. If $\lim_{n\to\infty}|\hat{T}|^k y_n$ and $\lim_{n\to\infty}|T|^k y_n$ exist, then $\sigma_{na}(\hat{T})\subseteq\sigma_{na}(T)$.*

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Proof. By hypothesis, $\lambda \in \sigma_{na}(\hat{T}) \Longrightarrow \lim_{n\to\infty}(\hat{T}-\lambda)y_n = 0$ and by Theorem 2 $\lim_{n\to\infty}(T-\lambda)y_n = 0$ and $\lim_{n\to\infty}(T-\lambda)^* y_n = 0$.

That is $\lambda \in \sigma_{na}(T)$. Hence $\sigma_{na}(\hat{T}) \subseteq \sigma_{na}(T)$. □

THEOREM 4. *Let T be a class $A(k)$ operator and $\{y_n\}$ be a sequence of unit vectors in $\boldsymbol{H}$ such that $\lim_{n\to\infty}|\hat{T}|^k y_n$ and $\lim_{n\to\infty}|T|^k y_n$ exist then $\lim_{n\to\infty}(T-\lambda)y_n = 0$ and $\lim_{n\to\infty}(T-\lambda)^* y_n = 0 \Longrightarrow \lim_{n\to\infty}(\hat{T}-\lambda)y_n = 0$, where $\lambda \in \boldsymbol{C}$.*

Proof. When $\lambda = 0$ we have

$$\begin{aligned}\|\hat{T}y_n\|^2 &= \|WU||T|^kT|^{\frac{1}{k+1}}y_n\|^2\\ &= (||T|^kT|^{\frac{2}{k+1}}y_n, y_n)\\ &\le (T^*|T|^{2k}Ty_n, y_n)^{\frac{1}{k+1}}.\end{aligned} \tag{14}$$

Since $\lim_{n\to\infty} Ty_n = 0$ we have $\lim_{n\to\infty}(T^*|T|^{2k}Ty_n, y_n) = 0$. Also from (14) we have $\lim_{n\to\infty}\|\hat{T}y_n\| = 0$ and $\lim_{n\to\infty}\hat{T}y_n = 0$. When $\lambda \neq 0$, by hypothesis $\lim_{n\to\infty}(T-\lambda)y_n = 0$ and $\lim_{n\to\infty}(T-\lambda)^* y_n = 0$. It follows that

$$\lim_{n\to\infty}(|T|^2 - |\lambda|^2)y_n = 0 \quad \text{and} \quad \lim_{n\to\infty}(|T| - |\lambda|)y_n = 0.$$

By the continuity of operators we have the following equations

$$\begin{aligned}&\lim_{n\to\infty}(|T|^k - |\lambda|^k)y_n = 0,\\ &\lim_{n\to\infty}(|T|^kT - |\lambda|^k\lambda)y_n = 0,\\ &\lim_{n\to\infty}(T^*|T|^k - |\lambda|^k\overline{\lambda})y_n = 0,\\ &\lim_{n\to\infty}((T^*|T|^k|T|^kT) - |\lambda|^{2(k+1)})y_n = 0,\\ &\lim_{n\to\infty}(\left||T|^kT\right|^2 - |\lambda|^{2(k+1)})y_n = 0,\\ &\lim_{n\to\infty}(\left||T|^kT\right|^{\frac{2}{k+1}} - |\lambda|^2)y_n = 0.\end{aligned} \tag{15}$$

That is $\lim_{n\to\infty}(|\hat{T}|^2 - |\lambda|^2)y_n = 0$ and $\lim_{n\to\infty}(|\hat{T}| - |\lambda|)y_n = 0$. By hypothesis, $\lim_{n\to\infty}|\hat{T}|^k y_n$ exists and hence,

$$\lim_{n\to\infty}(|\hat{T}|^k - |\lambda|^k)y_n = 0. \tag{16}$$

Since $\hat{T}|\hat{T}|^k = |T|^kT$ we have $(\hat{T}-\lambda)y_n = (-)\frac{\hat{T}}{|\lambda|^k}(|\hat{T}|^k - |\lambda|^k)y_n + \frac{|T|^k}{|\lambda|^k}(T-\lambda)y_n + \frac{\lambda}{|\lambda|^k}(|T|^k - |\lambda|^k)y_n$. Consequently by (15) and (16) we get $\lim_{n\to\infty}(\hat{T}-\lambda)y_n = 0$. □

COROLLARY 5. *Let T be a class $A(k)$ operator. Suppose $\lambda \in \sigma_{na}(T)$ and $\{y_n\}$ is a corresponding sequence of unit vectors such that $(T-\lambda)y_n \to 0$ and $(T-\lambda)^* y_n \to 0$ as $n \to \infty$. If $\lim_{n\to\infty}|\hat{T}|^k y_n$ and $\lim_{n\to\infty}|T|^k y_n$ exist, then $\sigma_{na}(T) \subseteq \sigma_{na}(\hat{T})$.*

Proof. By hypothesis, $\lambda \in \sigma_{na}(T) \Longrightarrow \lim_{n\to\infty}(T-\lambda)y_n = 0$ and $\lim_{n\to\infty}(T-\lambda)^* y_n = 0$. By Theorem 4 $\lim_{n\to\infty}(\hat{T}-\lambda)y_n = 0$. That is $\lambda \in \sigma_{na}(\hat{T})$. Hence $\sigma_{na}(T) \subseteq \sigma_{na}(\hat{T})$ □

In the following theorem we shall show that for a class $A(k)$ operator T, $\sigma_a(T) = \sigma_{na}(T)$.

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THEOREM 6. *Let T be a class $A(k)$ operator. Suppose $\{y_n\}$ is a sequence of unit vectors in* $\boldsymbol{H}$ *such that $(T-\lambda)y_n \to 0$ and $\||\hat{T}|^2 y_n\| - |\lambda|^2 \to 0$ as $n \to \infty$, then* $\lim_{n\to\infty}(T-\lambda)^* y_n = 0$.

Proof. By assumption $\lim_{n\to\infty}(T-\lambda)y_n = 0$. Since $|\, \|Ty_n\| - |\lambda|\, | \leq \|(T-\lambda)y_n\|$ we obtain $\lim_{n\to\infty} \|Ty_n\| = |\lambda|$. Also T is a class $A(k)$ operator implies that

$$\begin{aligned}
\|Ty_n\|^2 &= (|T|^2 y_n, y_n) \\
&\leq \left((T^*|T|^{2k}T)^{\frac{1}{k+1}} y_n, y_n\right) \\
&= \left(||T|^k T|^{\frac{2}{k+1}} y_n, y_n\right) \\
&\leq \|||T|^k T|^{\frac{2}{k+1}} y_n\| \text{ (Cauchy-Schwarz inequality)} \\
&= \||\hat{T}|^2 y_n\|.
\end{aligned}$$

That is $\lim_{n\to\infty} \|Ty_n\|^2 \leq \lim_{n\to\infty} \||\hat{T}|^2 y_n\|$ and so $|\lambda|^2 \leq \lim_{n\to\infty} \||\hat{T}|^2 y_n\|$.

By hypothesis $\lim_{n\to\infty} \||\hat{T}|^2 y_n\| = |\lambda|^2$ and hence we obtain

$$\lim_{n\to\infty} (||T|^k T|^{\frac{2}{k+1}} y_n, y_n) = |\lambda|^2. \tag{17}$$

Now by (17)

$$\lim_{n\to\infty} \left\| \left(||T|^k T|^{\frac{2}{k+1}} - |\lambda|^2\right)^{\frac{1}{2}} y_n \right\|^2 = \lim_{n\to\infty} (||T|^k T|^{\frac{2}{k+1}} y_n, y_n) - |\lambda|^2 = 0.$$

It follows that

$$\lim_{n\to\infty} (||T|^k T|^{\frac{2}{k+1}} - |\lambda|^2) y_n = 0. \tag{18}$$

Also

$$\lim_{n\to\infty} \left\| \left(||T|^k T|^{\frac{2}{k+1}} - |T|^2\right)^{\frac{1}{2}} y_n \right\|^2 = \lim_{n\to\infty} \left[\left(||T|^k T|^{\frac{2}{k+1}} y_n, y_n\right) - (|T|^2 y_n, y_n) \right] = 0,$$

and hence we obtain

$$\lim_{n\to\infty} (||T|^k T|^{\frac{2}{k+1}} - |T|^2) y_n = 0. \tag{19}$$

From (18) and (19) we get

$$\lim_{n\to\infty} \left(|T|^2 - |\lambda|^2\right) y_n = \lim_{n\to\infty} (|T|^2 - ||T|^k T|^{2/k+1}) y_n + \lim_{n\to\infty} (||T|^k T|^{\frac{2}{k+1}} - |\lambda|^2) y_n = 0.$$

As a consequence,

$$\lim_{n\to\infty} (T-\lambda)^* y_n = \frac{1}{\lambda} \lim_{n\to\infty} [(|T|^2 - |\lambda|^2) y_n - T^*(T-\lambda) y_n] = 0.$$

Hence $\lambda \in \sigma_{na}(T)$. □

THEOREM 7. *Let T be a class $A(k)$ operator. Suppose $\lambda \in \sigma_a(T)$ and $\{y_n\}$ is a corresponding sequence of unit vectors sucht that $(T-\lambda)y_n \to 0$ and $\||\hat{T}|^2 y_n\| - |\lambda|^2 \to 0$ as $n \to \infty$ then $\sigma(T) = \sigma(\hat{T})$*

To prove Theorem 7 we need the following theorems.

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Theorem R_4 [11]

1. *If A is normal, then for any $B \in L(H)$, $\sigma(AB) = \sigma(BA)$.*
2. *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator$(p > 0)$. Then for any $t > 0$, $\sigma(U|T|^t) = \{e^{i\theta}\rho^t : e^{i\theta}\rho \in \sigma(T)\}$.*

Theorem R_5 [17] *Let R be a subset of the complex plane C, $T(t)$ an operator-valued function of $t \in [0, 1]$ that is continuous in the norm topology, $\tau_t, t \in [0, 1]$, a family of bijective mappings from R onto $\tau_t(R) \subset C$ and, for any fixed $z \in R$, $\tau_t(z)$ is a continuous function of $t \in [0, 1]$ such that τ_0 is the identity function. Suppose that*

$$\sigma_a(T(t)) \cap \tau_t(R) = \tau_t(\sigma_a(T(0)) \cap R)$$

for all $t \in [0, 1]$. Then for all $t \in [0, 1]$,

$$\sigma_r(T(t)) \cap \tau_t(R) = \tau_t(\sigma_r(T(0)) \cap R),$$
$$\sigma(T(t)) \cap \tau_t(R) = \tau_t(\sigma(T(0)) \cap R).$$

Let F be the set of all strictly monotone increasing continuous nonnegative functions on $R^+ = [0, \infty)$. Let $F_0 = \{\Psi \in F : \Psi(0) = 0\}$ and $T = U|T|$. For $\Psi \in F_0$, the mapping $\widetilde{\Psi}$ is defined by $\widetilde{\Psi}(\rho e^{i\theta}) = e^{i\theta}\Psi(\rho)$ and $\widetilde{\Psi}(T) = U\Psi(|T|)$.

Theorem R_6 [6] *Let $T = U|T|$ and $\Psi \in F_0$. Then $\sigma_{na}(\widetilde{\Psi}(T)) = \widetilde{\Psi}(\sigma_{na}(T))$.*

Proof. Let $T = U|T|$ be the polar decomposition of T. We shall prove that if T is class $A(k)$, then $\sigma(U|T|^{k+1}) = \{\rho^{k+1}e^{i\theta} : \rho e^{i\theta} \in \sigma(T)\}$. Let $T(t) = U|T|^{k+t}$ and $\tau_t(\rho e^{i\theta}) = e^{i\theta}\rho^{k+t}$. Since $|T(t)| = |T|^{k+t}$ and $|T(t)^*| = |T^*|^{k+t}$ we have the following implications.

$$\begin{aligned}
&T \text{ belongs to class } A(k), \Leftrightarrow (|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2 \\
&\Leftrightarrow (|T(t)^*|^{\frac{1}{k+t}}|T(t)|^{\frac{2k}{k+t}}|T(t)^*|^{\frac{1}{k+t}})^{\frac{1}{k+1}} \geq |T(t)^*|^{\frac{2}{k+t}} \\
&\Leftrightarrow T(t) \text{ belongs to class } A\left(\frac{k}{k+t}, \frac{1}{k+t}\right) \\
&\Rightarrow T(t) \text{ belongs to class } A(k).
\end{aligned}$$

By Theorem 6 and Theorem R_6 we have,

$$\begin{aligned}
\sigma_a(T(t) \setminus \{0\}) &= \sigma_{na}(T(t) \setminus \{0\}) \\
&= \tau_t(\sigma_{na}(T) \setminus \{0\}) \\
&= \tau_t(\sigma_a(T) \setminus \{0\}) \\
&= \tau_t(\sigma_a(T)) \setminus \{0\}
\end{aligned}$$

Moreover, if $0 \in \sigma_a(T(t))$ then there exists a sequence $\{y_n\}$ of unit vectors such that $U|T|^{k+t}y_n \to 0$ as $n \to \infty$. Hence, $\||T|^k y_n\|^2 = (U|T|^{k+t}y_n, U|T|^{k-t}y_n) \to 0$, so that, $\lim_{n\to\infty} |T|^k y_n = 0$. It follows that $\lim_{n\to\infty} Ty_n = 0$ and hence $0 \in \sigma_a(T)$.

On the other hand, if $0 \in \sigma_a(T)$ then we have $0 \in \sigma_a(T(t))$ since,

$$\|U|T|^{k+t}y_n\| = \|U|T||T|^{k+t-1}y_n\| \leq \||T|^{k+t-1}\|\|Ty_n\| \to 0 \text{ as } n \to \infty.$$

Hence we obtain, $\sigma_a(T(t)) = \tau_t(\sigma_a(T))$ for all $t \in [0, 1]$ and by Theorem R_5, we have $\sigma(T(t)) = \tau_t(\sigma(T))$ for all $t \in [0, 1]$. Putting $t = 1$, we get

$$\sigma(U|T|^{k+1}) = \{\rho^{k+1}e^{i\theta} : \rho e^{i\theta} \in \sigma(T)\}. \tag{20}$$

By (1) of Theorem R_4 and (20) we have,

$$\begin{aligned}\sigma(WU||T|^kT|) &= \sigma(|T|^kU|T|) = \sigma(U|T|^{k+1})\\ &= \{\rho^{k+1}e^{i\theta} : \rho e^{i\theta} \in \sigma(T)\}.\end{aligned}$$

By Theorem 1, $\hat{T} = WU||T|^kT|^{\frac{1}{k+1}}$ is hyponomal. Hence by Theorem R_4 we get,

$$\begin{aligned}\sigma(\hat{T}) &= \sigma\big(WU||T|^kT|^{\frac{1}{k+1}}\big) = \big\{(\rho^{k+1})^{\frac{1}{k+1}}e^{i\theta} : \rho^{k+1}e^{i\theta} \in \sigma(U|T|^{k+1})\big\}\\ &= \big\{\rho e^{i\theta} : e^{i\theta}\rho^{k+1} \in \sigma(U|T|^{k+1})\big\}\\ &= \sigma(T).\end{aligned}$$

□

In the following corollaries we assume that T satisfies the following Limit Condition.

Limit Condition. For each $\lambda \in \sigma_a(T)$ and a corresponding sequence $\{y_n\}$ of unit vectors, $\hat{T}$ satisfies the condition that $\lim_{n\to\infty} \||\hat{T}|^2 y_n\| = |\lambda|^2$, where T is a class $A(k)$ operator and $\hat{T}$ is its hyponormal operator transform.

Corollary 8. *Let T be a class $A(k)$ operator such that the Limit Condition is satisfied. Then $\|T\| = \|\hat{T}\| = r(T)$ where $r(T)$ denotes the spectral radius of T.*

Proof.

$$\begin{aligned}\|\hat{T}\| &= \sup\{\|\hat{T}y\| : \|y\| = 1\}\\ &= \sup\big\{(|\hat{T}|^2y, y)^{\frac{1}{2}} : \|y\| = 1\big\}\\ &= \sup\big\{(||T|^kT|^{\frac{2}{k+1}}y, y)^{\frac{1}{2}} : \|y\| = 1\big\}\\ &\geq \sup\{\|Ty\| : \|y\| = 1\}\\ &= \|T\|.\end{aligned}$$

Since $\hat{T}$ is hyponormal, $\|\hat{T}\| = r(\hat{T})$. Hence we have,

$$\begin{aligned}\|T\| &\leq \|\hat{T}\|\\ &= r(\hat{T})\\ &= \sup\{|\lambda| : \lambda \in \sigma(\hat{T}\}\\ &= \sup\{|\lambda| : \lambda \in \sigma(T)\}\\ &= r(T).\end{aligned}$$

Since every class $A(k)$ operator is normaloid, $\|T\| = r(T)$. So $\|T\| = r(T) = r(\hat{T}) = \|\hat{T}\|$ □

Corollary 9. *Let T be a class $A(k)$ operator with a single limit point in its spectrum such that the Limit Condition is satisfied, then the residual spectrum of T is empty.*

Proof. By Theorem 7 $\sigma(T) = \sigma(\hat{T})$. Hence $\sigma(\hat{T})$ has a single limit point. Since $\hat{T}$ is hyponormal with a single limit point in its spectrum it is normal [**16**]. For a hyponormal operator the residual spectrum is empty. Since $\sigma_p(T) = \sigma_p(\hat{T})$ the residual spectrum of T is also empty. □

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COROLLARY 10. *A generalised nilpotent class $A(k)$ operator satisfying the Limit Condition is necessarily zero.*

Proof. Since $\hat{T}$ is hyponormal, $\sigma(\hat{T})$ contains a scalar μ such that $|\mu| = \|\hat{T}\|$ **[4]**. For every positive integer n, it follows that [**10**, Theorem 33.1],

$$\|T\|^n = \|\hat{T}\|^n = \|\mu\|^n = \|\mu^n\| \leq \|T^n\| \leq \|T\|^n.$$

Hence $\|T\|^n = \|T^n\|$. By hypothesis, $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = 0$. It follows that $\|T\| = 0$. Hence $T = 0$. □

An operator $T \in \mathbf{L}(\mathbf{H})$ is said to satisfy *Single-Valued Extension Property* (SVEP) if for any open subset V in $\mathbf{C}$, the function $T - \lambda : \Theta(V, H) \to \Theta(V, H)$ defined by pointwise multiplication, is one-to-one. Here $\Theta(V, H)$ denotes the Fréchet space of $\mathbf{H}$-valued analytical functions on V with respect to the uniform topology. An operator $T \in L(H)$ is said to satisfy the property (β) if for every open subset $\mathbf{G}$ of $\mathbf{C}$ and every sequence $f_n : \mathbf{G} \to \mathbf{H}$ of $\mathbf{H}$-valued analytic functions such that $(T - \lambda)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of $\mathbf{G}$, $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of $\mathbf{G}$. This was first introduced by Bishop [**5**].

To prove that a class $A(k)$ operator T has property (β) we need the following Theorem which is a modified form of [**14**, Lemma 2.5].

Theorem R_7 [14]. *Let $\boldsymbol{D}$ be an open subset of $\boldsymbol{C}$ and $f_n : \boldsymbol{D} \to \boldsymbol{H}(n = 1, 2, \ldots)$ vector valued analytic functions such that $|\mu| f_n(\mu) \to 0$ uniformly on every compact subset of $\boldsymbol{D}$. Then $f_n(\mu) \to 0$ again uniformly on every compact subset of $\boldsymbol{D}$.*

Proof of Theorem R_7.

Let us fix an arbitrary $\lambda \in \mathbf{D}$. It suffices to show that there exists a constant $r > 0$ such that $\{|\mu - \lambda| \leq r\} \subset \mathbf{D}$ and $f_n(\mu) \to 0$ uniformly on $\{|\mu - \lambda| \leq r\}$. If $\lambda \neq 0$, then we need merely to take r such that $0 \notin \{|\mu - \lambda| \leq r\} \subset \mathbf{D}$. We consider the case in which $\lambda = 0$. Take any constant $r > 0$ such that $\{|\mu| \leq r\} \subset \mathbf{D}$. Then for each $n = 1, 2, \ldots,$ we can find an ω_n with $|\omega_n| = r$ such that $\|f_n(\mu)\| \leq \|f_n(\omega_n)\|$ on $\{|\mu| \leq r\}$ by the maximum principle. Thus

$$\|f_n(\mu)\| = \frac{1}{|\omega_n|} |\omega_n| \|f_n(\mu)\| \leq \frac{1}{r} \|\omega_n f_n(\omega_n)\| \to 0$$

uniformly on $\{|\mu| \leq r\}$. □

THEOREM 11. *A class $A(k)$ operator T has property (β) if $\lim_{n\to\infty} |T|^k f_n(\mu)$ and $\lim_{n\to\infty} |\hat{T}|^k f_n(\mu)$ both exist and $\lim_{n\to\infty}[\||\hat{T}|^2 f_n(\mu)\| - \||\mu|^2 f_n(\mu)\|] = 0$.*

Proof. Let $\mathbf{D}$ be an open neighborhood of $\lambda \in \mathbf{C}$ and $f_n(n = 1, 2, \ldots)$ vector-valued analytic functions on $\mathbf{D}$ such that $(T - \mu)f_n(\mu) \to 0$ uniformly on every compact subset of $\mathbf{D}$.

We may assume that $\sup_n \|f_n(\mu)\| < +\infty$ on every compact subset of $\mathbf{D}$. In fact, let M_n be a positive number such that $\|f_n(\mu)\| \leq M_n$. Then by replacing $f_n(\mu)$ with $\frac{f_n(\mu)}{M_n+1}$, we have $\sup_n \|f_n(\mu)\| \leq 1$ and $(T - \mu)f_n(\mu) \to 0$ uniformly on every compact subset of $\mathbf{D}$. By hypothesis, $(T - \mu)f_n(\mu) \to 0$ uniformly on every compact subset of $\mathbf{D}$. Since $|\|Tf_n(\mu)\| - \|\mu f_n(\mu)\|| \leq \|(T - \mu)f_n(\mu)\|$ we obtain

$$\lim_{n\to\infty} (\|Tf_n(\mu)\| - \|\mu f_n(\mu)\|) = 0. \tag{21}$$

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Since T belongs to class $A(k)$,

$$\begin{aligned}\|Tf_n(\mu)\|^2 - \|\mu f_n(\mu)\|^2 &= (|T|^2 f_n(\mu), f_n(\mu)) - (|\mu|^2 f_n(\mu), f_n(\mu))\\ &\leq \big((T^*|T|^{2k}T)^{\frac{1}{k+1}} f_n(\mu), f_n(\mu)\big) - (|\mu|^2 f_n(\mu), f_n(\mu))\\ &= \big(||T|^k T|^{\frac{2}{k+1}} f_n(\mu), f_n(\mu)\big) - (|\mu|^2 f_n(\mu), f_n(\mu))\\ &= (|\hat{T}|^2 f_n(\mu), f_n(\mu)) - (|\mu|^2 f_n(\mu), f_n(\mu))\\ &\leq [\||\hat{T}|^2 f_n(\mu)\| - \||\mu|^2 f_n(\mu)\|]\|f_n(\mu)\| \to 0,\end{aligned}$$

by assumption. Hence

$$\lim_{n\to\infty} [(|\hat{T}|^2 f_n(\mu), f_n(\mu)) - (|\mu|^2 f_n(\mu), f_n(\mu))] = 0. \tag{22}$$

Also

$$\begin{aligned}\|(|\hat{T}|^2 - |\mu|^2) f_n(\mu)\|^2 &= \||\hat{T}|^2 f_n(\mu)\| - 2|\mu|^2(|\hat{T}|^2 f_n(\mu), f_n(\mu)) + |\mu|^4\|f_n(\mu)\|^2\\ &= \||\hat{T}|^2 f_n(\mu)\| - \||\mu|^2 f_n(\mu)\|^2\\ &\quad - 2|\mu|^2((|\hat{T}|^2 - |\mu|^2) f_n(\mu), f_n(\mu)) \to 0,\end{aligned}$$

uniformly as $n \to \infty$. That is,

$$\lim_{n\to\infty} \big(|\hat{T}|^2 - |\mu|^2\big) f_n(\mu) = 0, \tag{23}$$

$$\lim_{n\to\infty} \big(|\hat{T}| - |\mu|\big) f_n(\mu) = 0. \tag{24}$$

By hypothesis $\lim_{n\to\infty} |\hat{T}|^k f_n(\mu)$ exists and hence

$$\lim_{n\to\infty} (|\hat{T}|^k - |\mu|^k) f_n(\mu) = 0.$$

By (21) and (22)

$$\left\|(|\hat{T}|^2 - |T|^2)^{\frac{1}{2}} f_n(\mu)\right\|^2 = (|\hat{T}|^2 f_n(\mu), f_n(\mu)) - (|T|^2 f_n(\mu), f_n(\mu)) \to 0.$$

Hence $(|\hat{T}|^2 - |T|^2) f_n(\mu) \to 0$ uniformly. By (23) $\lim_{n\to\infty}(|T|^2 - |\mu|^2) f_n(\mu) = 0$; $\lim_{n\to\infty}(|T| - |\mu|) f_n(\mu) = 0$. Hence $\lim_{n\to\infty}(|T|^k - |\mu|^k) f_n(\mu) = 0$.

Since $\hat{T}|\hat{T}|^k = |T|^k T$, we have

$$\begin{aligned}(\hat{T} - \mu)|\hat{T}|^k f_n(\mu) &= (|T|^k T f_n(\mu) - \mu|\hat{T}|^k f_n(\mu))\\ &= |T|^k (T - \mu) f_n(\mu) + \mu(|T|^k - |\mu|^k) f_n(\mu)\\ &\quad + \mu(|\mu|^k - |\hat{T}|^k) f_n(\mu) \to 0,\end{aligned}$$

uniformly. According to Putinar **[15]**, every hyponormal operator has property (β) and hence $\hat{T}$ has property (β). Hence, $|\hat{T}|^k f_n(\mu) \to 0$ uniformly and $|\hat{T}| f_n(\mu) \to 0$ uniformly as $n \to \infty$. By (24) we have $|\mu| f_n(\mu) \to 0$ uniformly and by Theorem R_7 we obtain $f_n(\mu) \to 0$ uniformly. Thus T has property (β) and hence the Single Valued Extension Property. □

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