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NUCLEARITY AND BANACH SPACES

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Summary

Let E be a nuclear space provided with a topology different from the weak topology. Let $\{A_i: i \in I\}$ be a fundamental system of equicontinuous subsets of the topological dual E' of E. If $\{F_i: i \in I\}$ is a family of infinite dimensional Banach spaces with separable predual, there is a fundamental system $\{B_i: i \in I\}$ of weakly closed absolutely convex equicontinuous subsets of E' such that E'_{B_i} is norm-isomorphic to F_i, for each $i \in I$. Other results related with the one above are also given.

The linear spaces we use are defined over the field K of the real or complex numbers. If $\langle E, F\rangle$ is a dual pair we use $\sigma(E, F)$ to denote the topology on E of the uniform convergence over the finite sets of F. If A is contained in E, A° denotes the polar set of A in F. When we write "space" we mean "separated locally convex topological vector space". $G[\mathcal{T}]$ means the space G provided with the topology $\mathcal{T}$. The neighbourhoods of the origin in G which appear are always supposed to be closed and absolutely convex. Let U be a neighbourhood of the origin in G and let H be the set $\{x \in G: \lambda x \in U, \forall \lambda \in K\}$. If φ is the canonical mapping: $G \to G/H$, G_U denotes the normed space over G/H with closed unit ball $\varphi(U)$ and $G_{(U)}$ denotes the completion of G_U. If A is a bounded closed absolutely convex subset of G we write G_A to represent the normed space over the linear hull of A with the gauge of A as norm. If $x \in G$ and $u \in G'$, G' being the topological dual of G, we write $\langle x, u\rangle$ for the value of u on x. If H is a Hilbert space, and $y, z \in H$, then $(y|z)$ is the inner product of y by z.

Given l^p, $1 \leq p < \infty$, in (1) A. Grothendieck proved that if U is a neighbourhood of the origin in a nuclear space E there is a neighbourhood of the origin $V \subset U$ in E, such that $E_{(V)}$ is a subspace of l^p. This result was improved by H. Jarchow, (2), by showing that E is the reduced projective limit of a product of complemented subspaces of l^p. In (5), S. Saxon proved that if U is a neighbourhood of the origin in the Fréchet space s, the space of rapidly decreasing sequences, and F is an infinite dimensional Banach space with Schauder basis, then there is a neighbourhood of the origin V in s such that $s_{(V)}$ is norm-isomorphic to F.

All the results stated above can be obtained using our Theorem 1. To give its proof we shall need the following result of Ovsepian and Pelczyanski, (3): (a) *In every separable Banach space X of infinite dimension there is a* *bi-orthogonal sequence* $(x_n, u_n)_{n=1}^{\infty}$ *such that the linear combinations of the* $\{x_n\}_{n=1}^{\infty}$ *are dense in E, for every x in E if* $\langle x, u_n\rangle = 0$ *for all n then* $x = 0$ *and* $\sup_n \|x_n\| \cdot \|u_n\| < \infty$.

Theorem 1. *Let F be an infinite dimensional separable Banach space and let U be a neighbourhood of the origin in a nuclear space E. If* E_U *is infinite dimensional there is in E a neighbourhood of the origin V,* $V \subset U$, *such that* $E_{(V)}$ *is norm isomorphic to F.*

Proof. Since E is nuclear, let A_1 and A_2 be closed absolutely convex equicontinuous sets of $E'[\sigma(E', E)]$ such that $U^\circ \subset A_1 \subset A_2$, E'_{A_1} and E'_{A_2} are Hilbert spaces and the canonical mapping J from E'_{A_1} into E'_{A_2} is of type l^1 (see Pietsch, (4, p. 133)). Since E_{A_1} is infinite dimensional we can put

$$J(x) = \sum_{n=1}^{\infty} \lambda_n (x|e_n) f_n, \quad \text{for every} \quad x \in E'_{A_1},$$

where $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ are orthonormal families in E'_{A_1} and E'_{A_2}, respectively, $\lambda_n > 0$, $n = 1, 2, \ldots$, and $\lambda = \Sigma_{n=1}^{\infty} \lambda_n < \infty$. Since U^0 is bounded in E'_{A_1}

$$\sup \left\{ \sum_{n=1}^{\infty} |(y|e_n)|^2 : y \in U^\circ \right\} = h < \infty.$$

According to result (a) it is possible to find a bi-orthogonal system $(x_n, u_n)_{n=1}^{\infty}$ in F such that $\{x_n\}_{n=1}^{\infty}$ is total in F, $\{u_n\}_{n=1}^{\infty}$ is total in $F'[\sigma(F', F)]$, $\|x_n\| = 1$, $\|u_n\| \leq M$, $n = 1, 2, \ldots$.

Let T be the linear mapping from E into F defined in the following way

$$T(x) = \sum_{n=1}^{\infty} M(h\lambda\lambda_n)^{1/2} \langle f_n, x\rangle x_n.$$

We shall go along the following four steps in order to prove the theorem: T is well defined, T is continuous, $T(E)$ is dense in F and $T^{-1}(B)$ is contained in U, where B is the closed unit ball of F.

Clearly, A_2° is a neighbourhood of the origin in E and A_2 is a neighbourhood of the origin in E'_{A_2}. Since $\{f_n\}_{n=1}^{\infty}$ is an orthonormal family in E'_{A_2}

$$\sum_{n=1}^{\infty} |\langle f_n, x\rangle|^2 \leq k < \infty$$

for every $x \in A_2^\circ$.

$$\|T(x)\| = \left\| \sum_{n=1}^{\infty} M(h\lambda\lambda_n)^{1/2} \langle f_n, x\rangle x_n \right\|$$

$$\leq M(h\lambda)^{1/2} \left(\sum_{n=1}^{\infty} \lambda_n \right)^{1/2} \left(\sum_{n=1}^{\infty} |\langle f_n, x\rangle|^2 \right)^{1/2} \leq M\lambda(hk)^{1/2}, \quad \forall\, x \in A_2^\circ,$$

therefore, T is well defined and continuous.

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Let T' be the dual mapping of T from $F'[\sigma(F',F)]$ into $E'[\sigma(E',E)]$. If $z\in E$, $u\in F'$, then

$$\langle u, T(z)\rangle=\sum_{n=1}^{\infty} M(h\lambda\lambda_n)^{1/2}\langle f_n, z\rangle\langle u, x_n\rangle$$
$$=\left\langle\sum_{n=1}^{\infty} M(h\lambda\lambda_n)^{1/2}\langle u, x_n\rangle f_n, z\right\rangle=\langle T'(u), z\rangle$$

and, therefore,

$$T'(u)=\sum_{n=1}^{\infty} M(h\lambda\lambda_n)^{1/2}\langle u, x_n\rangle f_n.$$

To see that $T(E)$ is dense in F it is sufficient to show that T' is injective. Since $\{x_1, x_2, \ldots, x_n, \ldots\}$ is a bounded set of F, there is a positive number α such that

$$|\langle u, x_n\rangle|\leq\alpha \quad \text{for every positive integer } n.$$

On the other hand, $\{f_n\}_{n=1}^{\infty}$ is an orthonormal family in the Hilbert space E'_{A_2} and thus $T'(u)\in E'_{A_2}$. Then $T'(u)=0$ if and only if $\lambda_n^{1/2}\langle u, x_n\rangle=0$, $n=1,2,\ldots$ Since $\{x_n\}_{n=1}^{\infty}$ is total in F, $T'(u)=0$ if and only if $u=0$.

Finally, we show that $T'(B^\circ)\supset U^\circ$. If $y\in U^\circ$, then

$$\sum_{n=1}^{\infty}\lambda_n^{1/2}|(y|e_n)|\leq\left(\sum_{n=1}^{\infty}\lambda_n\right)^{1/2}\left(\sum_{n=1}^{\infty}|(y|e_n)|^2\right)^{1/2}\leq\lambda^{1/2}h^{1/2}.$$

On the other hand,

$$y=J(y)=\sum_{n=1}^{\infty}\lambda_n(y|e_n)f_n=\sum_{n=1}^{\infty}\{(h\lambda)^{-1/2}\lambda_n^{1/2}(y|e_n)\}\cdot\{(h\lambda\lambda_n)^{1/2}f_n\}$$

and, therefore y lies in the closed absolutely convex hull of $\{(h\lambda\lambda_n)^{1/2}f_n\}_{n=1}^{\infty}$ in $E'[\sigma(E',E)]$ since

$$\sum_{n=1}^{\infty}(h\lambda)^{-1/2}\lambda_n^{1/2}|(y|e_n)|\leq(h\lambda)^{-1/2}\left(\sum_{n=1}^{\infty}\lambda_n\right)^{1/2}\left(\sum_{n=1}^{\infty}|(y|e_n)|^2\right)^{1/2}$$
$$\leq(h\lambda)^{-1/2}\lambda^{1/2}h^{1/2}=1.$$

Since $\|u_n\|\leq M$, $n=1,2,\ldots$, then $M^{-1}u_n\in B^\circ$ and

$$T'(M^{-1}u_n)=\sum_{p=1}^{\infty}M(h\lambda\lambda_p)^{1/2}\langle M^{-1}u_n, x_p\rangle f_p=(h\lambda\lambda_n)^{1/2}f_n$$

hence $T'(B^\circ)$ contains the closed absolutely convex hull of $\{(h\lambda\lambda_n)^{1/2}f_n\}_{n=1}^{\infty}$, in $E'[\sigma(E',E)]$, since B° is $\sigma(F',F)$-compact and, therefore $T'(B^\circ)$ is a closed absolutely convex set of $E'[\sigma(E',E)]$. Then $T'(B^\circ)\supset U^\circ$ and therefore $T^{-1}(B)\subset U$. We take now $V=T^{-1}(B)$ and the theorem is proved.

Theorem 2. *Let F be an infinite dimensional Banach space with separable predual G. Let E be a nuclear space and let A_1 be an equicontinuous weakly compact absolutely convex subset of E'. If E'_{A_1} is of infinite dimension, there is an equicontinuous weakly compact absolutely convex subset A_2 in E', containing A_1, such that E'_{A_2} is norm isomorphic to F and the canonical mapping from E'_{A_1} into E'_{A_2} is nuclear.*

Proof. Let M be a weakly compact equicontinuous absolutely convex subset of E', such that $M \supset A_1$ and the canonical mapping from E'_{A_1} into E'_M is nuclear. By the proof of Theorem 1, we can find a continuous linear mapping T from E into G, such that $T^{-1}(B)$ is contained in M°, where B is the closed unit ball of G. If T' is the dual mapping from F into E' of T it is enough to take $A_2 = T'(B^\circ)$.

Theorem 3. *Let $\{F_i : i \in I\}$ be a family of infinite dimensional Banach spaces with separable predual. Let $E[\mathcal{T}]$ be a nuclear space such that $\mathcal{T} \neq \sigma(E, E')$. If there is in E a basis $\{U_i : i \in I\}$ of neighbourhoods of the origin, then there is in $E'[\sigma(E', E)]$ a fundamental system of equicontinuous closed absolutely convex subsets $\{B_i : i \in I\}$ such that E'_{B_i} is norm-isomorphic to F_i for every $i \in I$.*

Proof. Since $\mathcal{T} \neq \sigma(E, E')$ the space $E'_{U_i^\circ}$ is of infinite dimension, $i \in I$. Applying Theorem 2, we can find, for every $i \in I$, a subset B_i, equicontinuous weakly compact absolutely convex in E' such that $B_i \supset U_i^\circ$ and E'_{B_i} is norm-isomorphic to F_i.

Theorem 4. *Let $\{E_i : i \in I\}$ be a family of infinite dimensional separable Banach spaces. Let $E[\mathcal{T}]$ be a complete nuclear space such that $\mathcal{T} \neq \sigma(E, E')$. If there is in $E[\mathcal{T}]$ a neighbourhood basis $\{U_i : i \in I\}$ of the origin, then E is a reduced projective limit $E = \lim_{\leftarrow} g_{ji} F_i$, where F_i is a space topologically isomorphic to E_i.*

Proof. For every $i \in I$, we can find, by Theorem 1, a neighbourhood $V_i \subset U_i$ such that $E_{(V_i)}$ is norm-isomorphic to E_i. If φ_i is the canonical mapping from E onto E_{V_i} and V_j, $j \in I$, is such that $V_j \subset V_i$, let g_{ji} be the continuous linear mapping from $E_{(V_i)}$ into $E_{(V_j)}$ such that $\varphi_j = g_{ji} \circ \varphi_i$. Obviously, $E = \lim_{\leftarrow} g_{ji} E_{(V_i)}$.

Corollary. *Let $E[\mathcal{T}]$ be a Fréchet space such that $\mathcal{T} \neq \sigma(E, E')$. Let $\{E_n\}_{n=1}^\infty$ be a sequence of separable infinite dimensional Banach spaces. Then $E[\mathcal{T}]$ is nuclear if and only if it is topologically isomorphic to the reduced projective limit of the sequence $\{E_n\}_{n=1}^\infty$, $E = \lim_{\leftarrow} g_{mn} E_n$ such that g_{mn} is a nuclear mapping when $m < n$.*

Proof. It is sufficient to take a decreasing fundamental sequence $\{U_n\}_{n=1}^\infty$ of neighbourhoods of the origin in E, $V_n \subset U_n$, $E_{(V_n)}$ is norm-

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isomorphic to E_n, $V_n \supset V_{n+1}$, and using the notations of the proof of Theorem 4, g_{mn} is nuclear. The converse is immediate.

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