INTEGRAL MODELS OF SHIMURA VARIETIES OF PEL TYPE

SANDRA ROZENSZTAJN

UMPA, ÉNS de Lyon, UMR 5669 du CNRS, 46, allée d'Italie, 69364 Lyon Cedex 07, France
E-mail address: sandra.rozensztajn@ens-lyon.fr

1 PEL data for integral models

1.1 PEL Shimura data

We start with a set $\mathcal{D}$ of Shimura data of PEL type, as explained in [Del71, § 4.9] or [GN, § 4.3]. That is, $\mathcal{D}$ consists of:

(1) a finite semisimple $\mathbf{Q}$-algebra B, endowed with a positive involution $*$.

(2) a finite B-module V, endowed with a non-degenerate bilinear alternating pairing $\langle \cdot, \cdot \rangle$, such that $\langle bx, y \rangle = \langle x, b^* y \rangle$ for all $b \in B$ and $(x, y) \in V$.

(3) an $\mathbf{R}$-morphism $h : \mathbf{C} \to \mathrm{End}_B(V)_\mathbf{R}$ such that complex conjugation on $\mathbf{C}$ corresponds by h to the adjunction in $\mathrm{End}_B(V)_\mathbf{R}$ with respect to the pairing $\langle \cdot, \cdot \rangle$, and such that $(u, v) \mapsto \langle u, h(i)v \rangle$ is a symmetric definite positive pairing over $V_\mathbf{R}$.

Let G be the reductive group over $\mathbf{Q}$ defined by:

$$G(R) = \{g \in \mathrm{GL}(V \otimes_\mathbf{Q} R), \exists \mu \in R^\times, \forall x, y \in V \otimes R,$$
$$\langle gx, gy \rangle = \mu \langle x, y \rangle \text{ and the action of } g \text{ is } B\text{-linear}\}$$

We can attach to h a morphism $\mu_h : \mathbf{C}^\times \to G_\mathbf{C}$ that induces a decomposition $V_\mathbf{C} = V_0 \oplus V_1$, where $\mu_h(z)$ acts by z on V_1 and by 1 on V_0. The reflex field E of the Shimura data is then the subfield of $\bar{\mathbf{Q}}$ generated by the traces of the elements of B acting on V_0.

Let $\mathcal{X}$ be the $G(\mathbf{R})$-conjugacy class of $h^{-1} : \mathbf{C}^\times \to G_\mathbf{R}$. Then for each compact open subgroup K of $G(\mathbf{A}_f)$ that is small enough, consider the analytic space $G(\mathbf{Q})\backslash\mathcal{X} \times (G(\mathbf{A}_f)/K)$. We recall now some of the results of [Del71] (see also a detailled explanation in [GN]). There is an algebraic variety $\mathrm{Sh}(G, \mathcal{X})_K$ such that $\mathrm{Sh}(G, \mathcal{X})_K(\mathbf{C}) = G(\mathbf{Q})\backslash\mathcal{X} \times (G(\mathbf{A}_f)/K)$. The variety $\mathrm{Sh}(G, \mathcal{X})_K$ actually has a model over the reflex field E, and under additional conditions, this model is actually unique. We call it the canonical model of the Shimura variety $\mathrm{Sh}(G, \mathcal{X})_K$ ([Del71, Définition 3.13]). In the situation we are studying, where the Shimura data is of PEL type, the canonical model can be constructed as a union of connected components of a moduli space parameterizing abelian varieties with given polarization, endomorphisms and level structure, that can be expressed in terms of the Shimura data $\mathcal{D}$.

https://doi.org/10.1017/9781108649711.004 Published online by Cambridge University Press

1.2 New data

We now want to construct an integral model at p of the Shimura variety $\mathrm{Sh}(G,\mathcal{X})_K$, that is a smooth model over the ring $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$. We need some extra data and assumptions in order to ensure that this is possible, and in order to define this integral model as a moduli space of abelian varieties.

We add to the Shimura data $\mathcal{D}$ the following element: let $\mathcal{O}_B$ be a $\mathbf{Z}_{(p)}$-order in B that is stable under the involution $*$ of B and becomes maximal after tensorization with $\mathbf{Z}_p$.

Moreover, we require additional conditions:

(1) B is unramified at p, that is $B_{\mathbf{Q}_p} = B \otimes_{\mathbf{Q}} \mathbf{Q}_p$ is isomorphic to a product of matrix algebras over unramified extensions of $\mathbf{Q}_p$.

(2) there exists a $\mathbf{Z}_p$-lattice Λ in $V_{\mathbf{Q}_p}$ that is stable under $\mathcal{O}_B$, and such that the pairing $\langle\cdot,\cdot\rangle$ induces a perfect duality of Λ with itself.

Note that condition (1) implies that $\mathcal{O}_B \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is isomorphic to a product of matrix algebras over rings of integers of unramified extensions of $\mathbf{Q}_p$.

Example 1.1. Let B be an imaginary quadratic extension of $\mathbf{Q}$. Then condition (1) simply means that p does not ramify in B. We can choose for $\mathcal{O}_B$ the $\mathbf{Z}_{(p)}$-order generated by the ring of integers of B.

1.3 The reductive group

Consider the reductive group G attached to the Shimura data $\mathcal{D}$. Because of the additional conditions, $G_{\mathbf{Q}_p}$ is unramified, as we can define a smooth reductive model $\mathcal{G}$ of $G_{\mathbf{Q}_p}$ over $\mathbf{Z}_p$. Indeed, fix a lattice Λ as in §1.2, (2). Let C_0 be the subgroup of $G(\mathbf{Q}_p)$ that stabilizes the lattice Λ. Then C_0 is the hyperspecial subgroup of the $\mathbf{Z}_p$-points of $\mathcal{G}$.

Remark 1.2. A consequence of [Kot92, Lemma 7.2] is that, when the Shimura data is of type A or C (see §1.4), the lattice Λ as in §1.2 (2) is essentially unique, as two such lattices differ by the action of an element of $G(\mathbf{Q}_p)$. As a consequence, the subgroup C_0 of $G(\mathbf{Q}_p)$ is uniquely defined up to conjugation in these cases.

1.4 Classification

For later use, we recall the classification of Shimura data. Let $\mathcal{C} = \mathrm{End}_B(V)$. It is endowed with an involution $*$ coming from the involution of B, and by construction $G(\mathbf{Q}) = \{x \in \mathcal{C}, xx^* \in \mathbf{Q}^\times\}$. Let $\mathcal{C}_{\mathbf{R}} = \mathcal{C} \otimes_{\mathbf{Q}} \mathbf{R}$. Then we have three families of Shimura data:

Type A: $\mathcal{C}_{\mathbf{R}}$ is a product of copies of $M_n(\mathbf{C})$ for some n.
Type C: $\mathcal{C}_{\mathbf{R}}$ is a product of copies of $M_{2n}(\mathbf{R})$ for some n.
Type D: $\mathcal{C}_{\mathbf{R}}$ is a product of copies of $M_n(\mathbf{H})$ for some n.

Unitary groups correspond to type A. More precisely, if the data is of type A, then the subgroup G_1 of G such that $G_1(\mathbf{Q}) = \{x \in \mathcal{C}, x^*x = 1\}$ is the restriction of scalars to $\mathbf{Q}$ of some inner form of a unitary group. The group of symplectic similitudes GSp corrsponds

https://doi.org/10.1017/9781108649711.004 Published online by Cambridge University Press

to type C, and orthogonal groups in even dimension to type D. As type D can be more complicated and we are mostly interested in the case of unitary groups, we will sometimes state results only for types A and C.

2 Preliminaries

In this Section we fix $\mathcal{D}$ a set of Shimura data as in §1.

2.1 Polarized abelian schemes with an action of $\mathcal{O}_B$

Let S be an $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-scheme.

Definition 2.1. Let R be a subring of $\mathbf{Q}$. An R-isogeny between two abelian schemes A and A' over S is an isomorphism in the category where the objects are abelian schemes and the set of morphisms from A to A' is $\mathrm{Hom}(A, A') \otimes_{\mathbf{Z}} R$. An R-polarization of A is a polarization of A that is also an R-isogeny from A to the dual abelian scheme A^t.

Definition 2.2. We say that (A, λ, ι) is a $\mathbf{Z}_{(p)}$-polarized abelian scheme with an action of $\mathcal{O}_B$ if:

(1) A is an abelian scheme over S.

(2) λ is a $\mathbf{Z}_{(p)}$-polarization.

(3) ι is an injective ring homorphism $\mathcal{O}_B \to \mathrm{End}(A) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$ which respects involutions on both sides: the involution $*$ on the left side, and the Rosati involution $\dagger$ coming from λ on the right side.

2.2 The determinant condition of Kottwitz

We now have to find a way to explain how $\mathcal{O}_B$ acts on the abelian scheme. More precisely we want to be able to express the fact that $\mathcal{O}_B$ acts on $\mathrm{Lie}\, A$ the same way it acts on V_0.

2.2.1 The determinant condition for projective modules

We fix once and for all a generating family $\alpha_1, \dots \alpha_t$ of $\mathcal{O}_B$ as a $\mathbf{Z}_{(p)}$-module.

Let R be an algebra over $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$, and M be a finitely generated projective R-module. Suppose that $\mathcal{O}_B$ acts on M by R-linear endomorphisms. We then say that M is a R-module with an action of $\mathcal{O}_B$. A special case of such a module with an action of $\mathcal{O}_B$ is given by V_0 (see definition in §1.1).

We consider the action of $\mathcal{O}_B[X_1, \dots X_t]$ on $M \otimes_R R[X_1, \dots X_t]$. We denote by $\mathrm{det}_M \in R[X_1, \dots X_t]$ the determinant of the element $X_1\alpha_1 + \dots X_t\alpha_t$ for this action. Here the ring R is understood.

It is clear that det_M is functorial in R : that is, if $f : R \to R'$ is a homomorphism of $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-algebras, and $M' = M \otimes_R R'$, then $\mathcal{O}_B$ acts on M' by R'-linear endomorphisms and $\mathrm{det}_{M'} = f(\mathrm{det}_M)$.

We have the following result:

Lemma 2.3. $\det_{V_0} \in \left(\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}\right)[X_1, \ldots X_t]$.

Proof. By definition of the reflex field E, all the elements $\det(b; V_0)$ lie in E, so the coefficients of $\det_{V_0}$ are in E. Let F be a number field such that the action of B on V_0 is defined on F. Then the image of $\mathcal{O}_B$ in the algebra of $g \times g$ matrices over F is an $\mathcal{O}_F \otimes \mathbf{Z}_{(p)}$-order. Hence $\det(b, V_0)$ is in $\mathcal{O}_F \otimes \mathbf{Z}_{(p)}$ for all $b \in \mathcal{O}_B$. So the coefficients of $\det_{V_0}$ are integral over $\mathbf{Z}_{(p)}$. This proves the Lemma. □

Lemma 2.4. *Let k be a field, and V and W be two finite-dimensional k-vector spaces with an action of $\mathcal{O}_B$. Then V and W are isomorphic if and only if* $\det_V = \det_W$.

Proof. Let us denote $\mathcal{O}_B \otimes_{\mathbf{Z}} k$ by A. Then A is a finite dimensional semisimple algebra over k. Indeed, if $\mathrm{char}(k) = 0$ then $A = B \otimes_{\mathbf{Q}} k$, and if $\mathrm{char}(k) = p$ then A is a product of matrix algebras over extensions of $\mathbb{F}_p$, as $\mathcal{O}_B \otimes \mathbf{Z}_p$ is a maximal order of $B \otimes \mathbf{Q}_p$, which is itself a product of matrix algebras over unramified extensions of $\mathbf{Q}_p$. Moreover, V and W are isomorphic as k-vector spaces with an action of $\mathcal{O}_B$ if and only if they are isomorphic as A-modules.

We write $A = A_1 \times \cdots \times A_n$ where the A_i are simple k-algebras. We consider V and W as A-modules. Then we have decompositions $V = V_1 \times \cdots \times V_n$ and $W = W_1 \times \cdots \times W_n$ where V_i and W_i are A_i-modules. As A_i is simple it has only one isomorphism class of irreducible representation. Hence V and W are isomorphic if and only if $\dim V_i = \dim W_i$ for all i, and it is clear that this information can be recovered from $\det_V$ and $\det_W$. □

Definition 2.5. If R is an $(\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)})$-algebra, and M an R-module with an action of $\mathcal{O}_B$, then we say that M satisfies the determinant condition if $\det_M$ equals the image of $\det_{V_0}$ in $R[X_1, \ldots X_t]$.

We then show how the isomorphism class varies under specialization:

Lemma 2.6. *Suppose R is an $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-algebra that is a local ring with residue field k, and let M be a finitely generated free R-module with an action of $\mathcal{O}_B$. Then M satisfies the determinant condition if and only if $M \otimes_R k$ does.*

Proof. We need only prove that M satisfies the determinant condition when $M \otimes_R k$ does. Note that if $R \subset R'$, then the R'-module $R' \otimes_R M$ satisfies the determinant condition if and only if M satisfies the condition. Note also that as $\mathcal{O}_B$ is finitely generated over $\mathbf{Z}_p$, there exists a noetherian subring $R_0 \subset R$, and a free R_0-module M_0 with an action of $\mathcal{O}_B$ such that $M = M_0 \otimes_{R_0} R$. So we can assume that R is noetherian if needed.

Fix K a finite extension of $\mathbf{Q}_p$ containing E such that there exists a K-vector space W with an action of B satisfying the determinant condition. As $\mathcal{O}_B \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is compact, W contains an $\mathcal{O}_K$-lattice $\mathcal{L}$ with an action of $\mathcal{O}_B$, and $\mathcal{L}$ satisfies the determinant condition. Suppose first that R is in fact an $\mathcal{O}_K$-algebra. Then by Lemma 2.4 $M \otimes_R k$ is isomorphic to $\mathcal{L} \otimes_{\mathcal{O}_K} k$ as a k-module with action of $\mathcal{O}_B$, as $\det_{M \otimes_R k} = \det_{\mathcal{L} \otimes_{\mathcal{O}_K} k}$. Let us show now that $\mathcal{L} \otimes_{\mathcal{O}_K} R$ is a projective $\mathcal{O}_B \otimes_{\mathbf{Z}} R$-module. It is enough to show it when $R = \mathcal{O}_K$. But $\mathcal{O}_B \otimes_{\mathbf{Z}} \mathcal{O}_K$ is a product of matrix algebras over extensions of $\mathbf{Z}_p$. So the result holds by Morita equivalence, as $\mathcal{L}$ is torsion-free. So the $\mathcal{O}_B \otimes_{\mathbf{Z}} R$-linear isomorphism

https://doi.org/10.1017/9781108649711.004 Published online by Cambridge University Press

$\mathcal{L} \otimes_{\mathcal{O}_K} k \to M \otimes_R k$ lifts to an $\mathcal{O}_B \otimes_{\mathbf{Z}} R$-linear morphism $\mathcal{L} \otimes_{\mathcal{O}_K} R \to M$. Now forget the action of $\mathcal{O}_B$. Nakayama's Lemma implies that this morphism is an isomorphism. As $\mathcal{L} \otimes_{\mathcal{O}_K} R$ satisfies the determinant condition, so does M.

Fix now F a finite extension of E such that the action of B on V_0 can be realized on F, that is, there exists an F-vector space $\mathcal{L}$ with an action of $\mathcal{O}_B$ satisfying the determinant condition. Suppose now that R is an F-algebra. Then by the same reasoning as before, we show that M satisfies the determinant condition if $M \otimes_R k$ does, using this time the fact that $\mathcal{L}$ is a projective $\mathcal{O}_B \otimes_{\mathbf{Z}} F$-module.

Assume now that we have an R-algebra R' which is either an $\mathcal{O}_K$-algebra or an F-algebra, which is such that $R \subset R'$, and such that for any maximal ideal $\mathfrak{m}'$ of R', we have $\mathfrak{m} \subset (\mathfrak{m}' \cap R)$. Then if the determinant condition holds for $M \otimes_R k$, it holds for $M \otimes_R (R'/\mathfrak{m}')$ for all $\mathfrak{m}'$ as $k \subset R'/\mathfrak{m}'$, hence it holds for $M \otimes_R R'_{\mathfrak{m}'}$ by the study of the two previous cases, and so it holds also for M as $R \subset R'$, and $R' \subset \prod_{\mathfrak{m}'} R'_{\mathfrak{m}'}$ (by [Mat89, Theorem 4.6]).

So we need only find such an R'. If p is invertible in R, we set $R' = R \otimes_E F$. If p is not invertible in R, we set $R' = \hat{R} \otimes_{\mathbf{Z}_p} \mathcal{O}_K$, where $\hat{R}$ is the p-adic completion of R (in this situation we need to assume that R is noetherian). □

We conclude that $\det_M$ depends only on specializations:

Corollary 2.7. *Let R be an $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-algebra such that* $\operatorname{spec} R$ *is connected, and let M be a finitely generated projective R-module with an action of $\mathcal{O}_B$. Then M satisfies the determinant condition if and only if there exists a maximal ideal $\mathfrak{m}$ of R, with residue field k, such that $M \otimes_R k$ satisfies the determinant condition.*

2.2.2 The determinant condition for abelian schemes with an action of $\mathcal{O}_B$

Let (A, λ, ι) be a polarized abelian scheme with an action of $\mathcal{O}_B$ over the base scheme S. Then $\mathcal{O}_B$ acts on $\operatorname{Lie} A$, which is a locally free $\mathcal{O}_S$-module. For each open affine subset U of S, we can define $\det_{\operatorname{Lie} A}(U) \in \Gamma(U, \mathcal{O}_S)[X_1, \dots X_t]$ as in §2.2.1. By functoriality of the definition of $\det$, these sections are compatible, hence glue to define a global section $\det_{\operatorname{Lie} A} \in \Gamma(S, \mathcal{O}_S)[X_1, \dots X_t]$. As $\Gamma(S, \mathcal{O}_S)$ is naturally an $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-algebra it makes sense to compare $\det_{\operatorname{Lie} A}$ to the image of $\det_{V_0}$ in $\Gamma(S, \mathcal{O}_S)[X_1, \dots X_t]$. Following [Kot92], we set the following definition:

Definition 2.8. The triple (A, λ, ι) satisfies the determinant condition of Kottwitz if $\det_{\operatorname{Lie} A}$ is the image of $\det_{V_0}$.

One consequence of the definition is the following: the dimension of $\operatorname{Lie} A$ and hence that of A is equal to that of V_0.

2.2.3 Some geometric properties of the determinant condition

We state what geometric consequences we can deduce from Proposition 2.6 and Corollary 2.7:

Proposition 2.9. *Let S be an $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-scheme, and S_0 a closed subscheme of S with nilpotent definition ideal. Let (A, λ, ι) be an abelian scheme over S with an action of $\mathcal{O}_B$. Suppose that the base change of (A, λ, ι) to S_0 satisfies the determinant condition. Then so does (A, λ, ι).*

https://doi.org/10.1017/9781108649711.004 Published online by Cambridge University Press

Proposition 2.10. *Let S be an $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-scheme, and (A, λ, ι) an abelian scheme over S with an action of $\mathcal{O}_B$. Then there is a closed subscheme T of S that is a union of connected components, such that for any closed point x of S, $(A_x, \lambda_x, \iota_x)$ satisfies the determinant condition if and only if x is a point of T.*

2.2.4 *The example of unitary groups*

In the case of unitary groups over $\mathbf{Q}$, we give a condition on $\operatorname{Lie} A$ that is equivalent to the determinant condition of Kottwitz and that is simpler to state.

Let B be a quadratic imaginary extension of $\mathbf{Q}$, and let τ be in B such that $\mathcal{O}_B = \mathbf{Z}[\tau]$. Fix a prime p that is unramified in B.

Let R be an $\mathcal{O}_{B,(p)}$-algebra, and M a locally free R-module with an action of $\mathcal{O}_B$. Then we have a decomposition $M = M^+ \oplus M^-$ where M^+ and M^- are also locally free. Here M^+ is defined as the submodule of M where the action of τ from the action of $\mathcal{O}_B$ and the action of the image of τ in R coincide, and M^- is the submodule where these action differ by conjugation in $\mathcal{O}_B$. When R is connected we can then define the type of M as the pair of integers $(\operatorname{rk} M^+, \operatorname{rk} M^-)$.

Then, if S is an $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-scheme, an abelian scheme A with an action of $\mathcal{O}_B$ over the base S satisfies the determinant condition if and only if $\operatorname{Lie} A$ has the same type as the B-module V_0.

2.3 Level structures

2.3.1 *Tate modules*

Denote by $\mathbf{A}_f^p$ the ring of finite adeles of $\mathbf{Q}$ away from p, and by $\hat{\mathbf{Z}}^{(p)}$ the ring $\prod_{\ell \neq p} \mathbf{Z}_\ell$. We denote $V \otimes_{\mathbf{Q}} \mathbf{A}_f^p$ by $V^{(p)}$.

Let (A, λ, ι) be a polarized abelian scheme with an action of $\mathcal{O}_B$, defined on an $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-schemes. Let s be any geometric point of S, and consider the Tate modules: $T(A_s) = \varprojlim_N A_s[N]$, $T^{(p)}(A_s) = \varprojlim_{N \text{ prime to } p} A_s[N] = T(A_s) \otimes_{\hat{\mathbf{Z}}} \hat{\mathbf{Z}}^{(p)}$ and $V^{(p)}(A_s) = H_1(A_s, \mathbf{A}_f^p) = T^{(p)}(A_s) \otimes_{\hat{\mathbf{Z}}^{(p)}} \mathbf{A}_f^p$.

They are endowed with a non-degenerate bilinear form, coming from the polarization λ, and an action of $\mathcal{O}_B$, coming from the action of $\mathcal{O}_B$ on A itself.

Let f be a separable isogeny from A to A' with kernel C. Then f induces a morphism $T(f) : T(A_s) \to T(A'_s)$ which is injective with cokernel isomorphic to C_s. If f is of prime-to-p degree then f induces $T^{(p)}(f) : T^{(p)}(A_s) \to T^{(p)}(A'_s)$ which is injective with cokernel C_s. We also have an isomorphism $V^{(p)}(f) : V^{(p)}(A_s) \to V^{(p)}(A'_s)$.

Suppose f is a separable R-isogeny for some subring R of $\mathbf{Q}$. Then $V(f)$ is still well-defined but f does not necessarily map $T(A_s)$ into $T(A'_s)$. In fact f maps $T(A_s)$ into $T(A'_s)$ if and only if f is an isogeny in the usual sense.

2.3.2 *Level subgroups*

Definition 2.11. A level subgroup of G is a compact open subgroup of $G(\mathbf{A}_f^p)$.

In particular, for us a level structure is always "away from p".

https://doi.org/10.1017/9781108649711.004 Published online by Cambridge University Press

Example 2.12. Fix a lattice $L \subset V$ that is self-dual for $\langle \cdot, \cdot \rangle$. Fix an integer N prime to p. We define the principal level subgroup of level N of $G(\mathbf{A}_f^p)$ as the compact open subgroup $\{g \in G(\mathbf{A}_f^p), (g-1)(L \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}^{(p)}) \subset N(L \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}^{(p)})\}$.

2.3.3 *Definition of the level structures*

Let S be an $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-scheme, and s a geometric point of S. Let (A, λ, ι) be a polarized abelian scheme with an action of $\mathcal{O}_B$. We say that a map η^p from $V^{(p)}$ to $V^{(p)}(A_s)$ respects the structures on both sides if it respects the bilinear forms up to a scalar in $(\mathbf{A}_f^p)^\times$, and if it is compatible with the $\mathcal{O}_B$-action on both sides.

Let g be in $G(\mathbf{A}_f^p)$. If η^p respects the structures on both sides then so does $\eta^p \circ g$. Hence $G(\mathbf{A}_f^p)$ acts on the set of such maps.

Definition 2.13. Let K^p be a level subgroup. A level structure of level K^p on (A, λ, ι) is a choice of a geometric point s for each connected component of S, and for each s a choice of a K^p-orbit $\bar{\eta}^p$ of morphisms $\eta^p : V^{(p)} \to V^{(p)}(A_s)$ respecting the structures on both sides and such that the orbit is fixed under the action of $\pi_1(s, S)$.

Remark 2.14. The last condition ensures that a level structure is in fact independant of the choice of s. Moreover, a level structure exists at some point s if and only if for any geometric point s' in the same connected component as s there exists a level structure at s'.

3 The integral model as a moduli scheme

3.1 Definition of the moduli problem

Let us fix a set of Shimura data $\mathcal{D}$ as in §1. We also fix a compact open subgroup K^p of $G(\mathbf{A}_f^p)$. We will define a moduli problem classifying abelian schemes with an action of $\mathcal{O}_B$ and K^p-level structure.

Definition 3.1. Let $\mathcal{F}_{K^p}$ be the following category fibered in groupoids over the category $(Sch/\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)})$ of $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-schemes:

- The objects over a scheme S are quadruples $\underline{A} = (A, \lambda, \iota; \bar{\eta}^p)$, where (A, λ, ι) is a $\mathbf{Z}_{(p)}$-polarized projective abelian scheme over S with an action of $\mathcal{O}_B$ which respects the determinant condition of Kottwitz (Definition 2.8 of §2.2), and $\bar{\eta}^p$ is a level structure of level K^p over each connected component of S.
- The morphisms from $\underline{A} = (A, \lambda, \iota; \bar{\eta}^p)$ to $\underline{A'} = (A', \lambda', \iota'; \bar{\eta}'^p)$ over S are given by a $\mathbf{Z}_{(p)}$-isogeny $f : A \to A'$ compatible with the action of $\mathcal{O}_B$ and the level structures, that is:
    (1) there exists a locally constant function r on S with values in $\mathbf{Z}_{(p)}^\times$ such that $\lambda = r(f^t \circ \lambda' \circ f)$.
    (2) f induces a morphism from $\mathrm{End}\,(A) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$ to $\mathrm{End}\,(A') \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$, that we still denote by f; then for all $b \in \mathcal{O}_B$, $f \circ \iota(b) = \iota'(b)$.

https://doi.org/10.1017/9781108649711.004 Published online by Cambridge University Press

(3) $\bar{\eta}'^p = V^{(p)}(f) \circ \bar{\eta}^p$, where we denote by $V^{(p)}(f)$ the morphism induced by f from $V^{(p)}(A_s)$ to $V^{(p)}(A'_s)$ for any s.

3.2 Known results about the moduli problem

We summarize here what is known about the moduli problem. We will explain these results in more detail in the rest of the Chapter.

3.2.1 Representability

Theorem 3.2. *$\mathcal{F}_{K^p}$ is a smooth Deligne-Mumford stack and it is representable by a quasi-projective scheme when K^p is small enough.*

The part of the Theorem concerning representability will be proved in § 4, where we will also explain what "small enough" means. The part about smoothness will be proved in § 5.

When $\mathcal{F}_{K^p}$ is representable by a scheme, we denote this scheme by S_{K^p}. In some cases we know a little more about S_{K^p} ([Kot92, end of § 5]:

Proposition 3.3. *Suppose that* $\mathrm{End}_B(V)$ *is a division algebra and let K^p be small enough that $\mathcal{F}_{K^p}$ is representable by a scheme S_{K^p}. Then S_{K^p} is projective over* $\operatorname{spec} \mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$.

3.2.2 Hecke operators

Theorem 3.4. *The family of schemes S_{K^p}, for K^p the small enough compact open subgroups of $G(\mathbf{A}_f^p)$, form a tower of schemes with finite smooth transition morphisms. The group $G(\mathbf{A}_f^p)$ acts on the tower via Hecke operators.*

This is the object of §6.

3.2.3 Generic fiber

The generic fiber of the schemes we have constructed are isomorphic (except for some special cases) to the Shimura varieties S_K in characteristic zero that were defined in the first Chapter, when $K = K^p C_0$, where C_0 is the hyperspecial subgroup of $G(\mathbf{Q}_p)$ defined in §1.3. Moreover this is compatible with the action of Hecke operators. More precisely we will see in §7:

Theorem 3.5. *When the Shimura data is of type A or C we have the following isomorphism for each compact open subgroup K^p of $G(\mathbf{A}_f^p)$:*

$$S_{K^p} \otimes_{\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}} E \xrightarrow{\sim} S_{K^p C_0}$$

Moreover, the induced isomorphism between the towers $(S_{K^p} \otimes E)_{K^p}$ and $(S_{K^p C_0})_{K^p}$ is compatible with the action of $G(\mathbf{A}_f^p)$ on both sides.

4 Representability of the moduli problem

4.1 Statement of the Theorem

The goal of this Section is the proof of the following Theorem (which is a more precise version of Theorem 3.2):

https://doi.org/10.1017/9781108649711.004 Published online by Cambridge University Press

Theorem 4.1. *For all level subgroups K^p, $\mathcal{F}_{K^p}$ is a Deligne-Mumford stack. Moreover if K^p is small enough so that it is contained in a principal level subgroup of level $N \geq 3$, the functor $\mathcal{F}_{K^p}$ is representable by a quasi-projective scheme over $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$.*

We will prove this Theorem by comparing our moduli problem to the case of the Siegel modular varieties, that is the case where the endomorphism ring is trivial, which is already known by the results of [MFK94]. This is the proof outlined in [Kot92]. Another strategy to study the representability of the moduli problem would be via Artin's criterion. This has the advantage of being more direct and not to rely on the difficult results of [MFK94], but it has the drawback that it only shows that the moduli problem is representable by an algebraic space when the level is small enough. To prove that it is in fact representable by a quasi-projective scheme one then has to use additional arguments. A detailed proof using this strategy can be found in [Lan13].

4.2 The Siegel case

We first study the so-called Siegel case. In this case the scheme we obtain is the Siegel moduli space of abelian varieties.

4.2.1 Siegel Shimura data

We define the Siegel Shimura data $\mathcal{D}$ as follows: we take $B = \mathbf{Q}$, $\mathcal{O}_B = \mathbf{Z}_{(p)}$, and $V = \mathbf{Q}^{2g}$ endowed with the standard symplectic form. In this case the group G is the similitude symplectic group GSp_{2g}, and there exists only one conjugacy class of maps h satisfying the conditions of §1.1 (3). The reflex field is $\mathbf{Q}$. As $B = \mathbf{Q}$, this means that we can forget about the action of $\mathcal{O}_B$ in the definition of the moduli problem.

4.2.2 The result of [MFK94]

We describe the result of [MFK94] concerning the case of the Siegel Shimura data. Fix an integer $g \geq 1$ and an integer $N \geq 3$.

Definition 4.2. Let $\mathcal{A}(N)$ be the category fibered in groupoids on $\mathbf{Z}[1/N]$-schemes such that: For any $\mathbf{Z}[1/N]$-scheme S, the set of objects of $\mathcal{A}_S$ is the set of triples $\underline{A} = (A, \lambda; \alpha)$ where A is a projective abelian scheme of dimension g, λ is a principal polarization, and α is a symplectic similitude (with multiplicator in $(\mathbf{Z}/N\mathbf{Z})^\times$) between $(\mathbf{Z}/N\mathbf{Z})^{2g}_S$ and $A[N]_S$. Here we consider $(\mathbf{Z}/N\mathbf{Z})^{2g}$ to be endowed with the standard symplectic form. If $\underline{A}$, $\underline{A'}$ are objects of $\mathcal{A}_S$, then the morphisms from $\underline{A}$ to $\underline{A'}$ are the isomorphisms $f : A \to A'$ such that $\lambda = f^t \circ \lambda' \circ f$ and $\alpha' = f \circ \alpha$.

The main result is the following ([MFK94], Theorem 7.9, see also [MB85], Theorem 3.2 of Chapter VII):

Proposition 4.3. *The category fibered in groupoids $\mathcal{A}(N)$ is representable by a (smooth) quasi-projective scheme.*

4.2.3 Reformulation of the moduli problem

In order to compare more easily our situation to that studied in [MFK94] we give another formulation of the moduli problem. As $\mathbf{Q}^{2g} = V$ is endowed with the standard symplectic

https://doi.org/10.1017/9781108649711.004 Published online by Cambridge University Press

form, $\mathbf{Z}^{2g}$ is a self-dual lattice. We write L for $\mathbf{Z}^{2g} \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}^{(p)}$. Let $K(N)$ be the principal level subgroup of level N for some N prime to p, as defined in Example 2.12, with respect to the lattice L. Let $\mathcal{F}_{K(N)}$ be the category fibered in groupoids as in Definition 3.1, with the Siegel Shimura data.

Proposition 4.4. *In the case of the Siegel Shimura data, when N is prime to p, $\mathcal{F}_{K(N)}$ and the restriction of $\mathcal{A}(N)$ to the subcategory of $\mathbf{Z}_{(p)}$-schemes are isomorphic.*

Proof. In the proof, we abbreviate $\mathcal{A}(N)$ by $\mathcal{A}$ and $\mathcal{F}_{K(N)}$ by $\mathcal{F}$. Let S be a $\mathbf{Z}_{(p)}$-scheme, and let $\mathcal{A}_S$ and $\mathcal{F}_S$ be the categories of objects of $\mathcal{A}$ and $\mathcal{F}$ over S. We must construct an equivalence of categories between $\mathcal{A}_S$ and $\mathcal{F}_S$. We can assume that S is connected, and fix a geometric point s of S.

Let $(A, \lambda; \alpha)$ be an object of $\mathcal{A}_S$. We want to construct an object of $\mathcal{F}_S$. The morphism $\alpha : (\mathbf{Z}/N\mathbf{Z})_S^{2g} \to A[N]_S$ gives a symplectic similitude $\alpha_s : (\mathbf{Z}/N\mathbf{Z})^{2g} \to A[N]_s$ that is invariant under the action of $\pi_1(s, S)$. This morphism α_s then extends to a symplectic similitude $\eta_0^p : L \to T^{(p)}(A_s)$, which also induces a symplectic similitude $\eta^p : V^{(p)} \to V^{(p)}(A_s)$. The map η^p is not uniquely defined, but its $K(N)$-conjugation class is uniquely determined by α_s, and so is invariant under the action of $\pi_1(s, S)$. Hence the orbit $\bar{\eta}^p$ of η^p defines a level structure $\bar{\eta}^p$ extending α. Hence to each $(A, \lambda; \alpha)$ we can attach $(A, \lambda; \bar{\eta}^p)$ which is an object of $\mathcal{F}_S$. Morphisms in $\mathcal{A}_S$ also define morphisms in $\mathcal{F}_S$, hence our construction is functorial.

Let us show now that this functor is faithful. Let $\underline{A} = (A, \lambda; \alpha)$ and $\underline{A'} = (A', \lambda'; \alpha')$ be objects in $\mathcal{A}_S$. Let $f : A \to A'$ be a $\mathbf{Z}_{(p)}$-isogeny which is a morphism in the category $\mathcal{F}_S$. Then f is an isomorphism, as by construction of the level structures f induces an isomorphism from $T^{(p)}(A)$ to $T^{(p)}(A')$. Moreover we necessarily have $\lambda' = f^t \circ \lambda \circ f$, as λ and λ' are principal polarizations. Hence f is a morphism in the category $\mathcal{A}_S$.

Finally let us show that this functor is essentially surjective. We need to see that any object in $\mathcal{F}_S$ is isomorphic to an object coming from $\mathcal{A}_S$. Let $\underline{A} = (A, \lambda; \bar{\eta}^p)$ be an object of $\mathcal{F}_S$. We need to find $\underline{A'} = (A', \lambda', \bar{\eta}^p\prime)$ and a morphism $\underline{A} \to \underline{A'}$ in $\mathcal{F}_S$ such that λ' is a principal polarization and $\bar{\eta}'^p : V^{(p)} \to V^{(p)}(A'_s)$ induces a symplectic similitude between $(\mathbf{Z}/N\mathbf{Z})_S^{2g}$ and $A[N]_S$.

Observe first that we need only find an object $\underline{A'}$ in $\mathcal{F}_S$ with a map $\underline{A'} \to \underline{A}$ (or, equivalently, with a map $\underline{A} \to \underline{A'}$) such that η'^p induces an isomorphism between L and $T^{(p)}(A'_s)$. Indeed such a level structure then induces a $\pi_1(s, S)$-invariant symplectic similitude $\alpha_s : (\mathbf{Z}/N\mathbf{Z})^{2g} \to A[N]_s$ that gives us the isomorphism $\alpha : (\mathbf{Z}/N\mathbf{Z})_S^{2g} \to A[N]_S$. Moreover, if we have such an $\underline{A'}$ we can change the polarization λ' to make it principal. We know that the bilinear forms on L and $T^{(p)}(A'_s)$ differ by a scalar a in $(\mathbf{A}_f^p)^\times$. By multiplying λ' by some prime-to-p integer n, we multiply the bilinear form on $T^{(p)}(A'_s)$ by n, and so we change a into na. So we can assume that a is in $(\mathbf{A}_f^p)^\times \cap \hat{\mathbf{Z}}^{(p)}$. Let ℓ be a prime not dividing p. We know that the isogeny λ' is divisible by ℓ if and only if the pairing on $T_\ell A_s$ coming from λ' is divisible by ℓ. Hence we can divide λ' by some integer n prime to p so that the pairing on L and the pairing on $T^{(p)}(A'_s)$ coming from the new λ' differ by an element of $\hat{\mathbf{Z}}^{(p)\times}$. But then the new polarization λ' induces an isomorphism $T^{(p)}(A'_s) \to T^{(p)}(A'^t_s)$ and so is principal, as we already know that its degree is prime to p.

Now we try to find $\underline{A'} = (A', \lambda', \bar{\eta}^{p\prime})$ and a morphism $\underline{A} \to \underline{A'}$ in $\mathcal{F}_S$ such that $\bar{\eta}'^p : V^{(p)} \to V^{(p)}(A'_s)$ induces a symplectic similitude between L and $T^{(p)}(A'_s)$. We note first that $T^{(p)}(A_s)$ and $\eta^p(L)$ are commensurable lattices in $V^{(p)}(A_s)$.

Fix a submodule $M \subset T^{(p)}(A_s)$ with finite index which is invariant under the action of $\pi_1(s, S)$. Then, from the interpretation of the Tate module as a π_1, we see that there exists A' and a prime-to-p isogeny $f : A' \to A$ such that $f(T^{(p)}(A'_s)) = M$. We can define a polarization on A' by $\lambda' = f^t \circ \lambda \circ f$. If moreover $M \subset \eta^p(L)$, then we can define a level structure on A' by the condition that $\eta^p = f \circ \eta'^p$. Then $\underline{A'} = (A', \lambda'; \bar{\eta}'p)$ is an object of $\mathcal{F}_S$ and f induces a map $\underline{A'} \to \underline{A}$ in $\mathcal{F}_S$.

We apply this to $M = T^{(p)}(A_s) \cap \eta^p(L)$. The construction gives us an $\underline{A'}$ with a map to $\underline{A}$ which satisfies moreover that $T^{(p)}(A'_s) \subset \eta'^p(L)$. So now we need only treat the case where $T^{(p)}(A_s) \subset \eta^p(L)$.

Fix a submodule $M \subset V^{(p)}(A_s)$ that is invariant under the action of $\pi_1(s, S)$, and suppose that $T^{(p)}(A_s) \subset M$ with finite index. Then there exists an étale subgroup scheme $C \subset A$ with C_s isomorphic to $M/T^{(p)}(A_s)$, such that the isogeny $f : A \to A' = A/C$ induces an isomorphism from M to $T^{(p)}(A'_s)$. There is some prime-to-p integer n such that (nf^{-1}) is an isogeny from A' to A, then we can endow the abelian scheme A' with a polarization $\lambda' = (nf^{-1})^t \circ \lambda \circ (nf^{-1})$. We can define a level structure on (A', λ') by $\bar{\eta}'p = f \circ \bar{\eta}^p$. Assume now that $T^{(p)}(A_s) \subset \eta^p(L) = M$. Then f defines a morphism in $\mathcal{F}_S$ from $(A, \lambda; \bar{\eta}^p)$ to $(A', \lambda'; \bar{\eta}'p)$. Moreover $\bar{\eta}'p$ has the property that $\bar{\eta}'p(L) = T^{(p)}(A'_s)$. So this treats the case where $T^{(p)}(A_s) \subset \eta^p(L)$. □

4.3 From the Siegel case to the PEL case

Fix a Shimura data $\mathcal{D}$ as in §1. We denote by $\mathrm{GSp}(V)$ the symplectic similitude group attached to the vector space V and the given alternating pairing on it, forgetting the action of B. Then the reductive group G from the data $\mathcal{D}$ is naturally a subgroup of $\mathrm{GSp}(V)$.

We fix a level subgroup $K^p \subset \mathrm{GSp}(\mathbf{A}_f^p)$, and set $K_G^p = K^p \cap G(\mathbf{A}_f^p)$ which is a level subgroup of $G(\mathbf{A}_f^p)$. We can define two moduli problems as in Definition 3.1. One is attached Shimura data $\mathcal{D}$, and the level subgroup K_G^p, we denote it by $\mathcal{F}$. The other is attached to the Siegel Shimura data, and the level subgroup K^p, we denote it by $\mathcal{S}$.

We have a natural transformation from $\mathcal{F}$ to $\mathcal{S}$, which is defined by sending the quadruple $(A, \lambda, \iota; \bar{\eta})$ over an S-scheme to $(A, \lambda; \tilde{\bar{\eta}})$, where $\tilde{\bar{\eta}}$ is the K^p-orbit generated by $\bar{\eta}$.

We have to prove that the functor $\mathcal{F}$ is relatively representable over $\mathcal{S}$, and that $\mathcal{F}$ is projective over $\mathcal{S}$. More precisely we will show:

Proposition 4.5. *Let K^p be such that $\mathcal{S}$ is representable by a scheme. Then $\mathcal{F}$ is relatively representable over $\mathcal{S}$ by a scheme that is projective over $\mathcal{S}$.*

By §4.2, $\mathcal{S}$ is representable by a scheme for example when K^p is a principal level subgroup of level $N \geq 3$ prime to p.

4.3.1 Construction of a scheme

In this Paragraph we fix an $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-scheme S, an abelian scheme A over S, a $\mathbf{Z}_{(p)}$-polarization λ of A, and a K^p-level structure $\bar{\eta}^p$ on (A, λ).

We will need the following result:

Lemma 4.6. *Let S be a locally noetherian scheme, and A a projective abelian scheme over S. Then the functor from S-schemes to the category of sets that attaches $End(A_T)$ to T is representable by a union of projective schemes over S. We will denote by $\mathcal{E}$ the scheme representing the functor $T \mapsto End(A_T) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$. It is also a union of projective schemes over S.*

Proof. This follows from the theory of Hilbert schemes, as an endomorphism of A_T is a special case of a subscheme of $A \times_T A$. A detailed proof of this Lemma can be found in [Hid04, Section 6.1]. □

In our special case, the abelian scheme A is endowed with a prime-to-p polarization λ. So $\mathcal{E}$ naturally comes with an involution r, which is the Rosati involution. Let $m = 2n$ and $a_1, \dots a_m$ be a set of generators of $\mathcal{O}_B$ as a $\mathbf{Z}_{(p)}$-algebra with $a_{n+i} = a_i^*$. We define a closed subscheme Z of $\mathcal{E}^m$ as follows: let T be an S-scheme, and $(x_1, \dots x_m) \in \mathcal{E}^m(T)$. Then $(x_1, \dots x_m)$ is in Z if and only if any $\mathbf{Z}_{(p)}$-polynomial relation verified by $(a_1, \dots a_m)$ is also verified by $(x_1, \dots x_m)$ and by $(r(x_1), \dots r(x_m))$, where $r(x_i) = x_{n+i}$. The abelian scheme A_Z is then endowed with an algebra homomorphism $\mathcal{O}_B \to \mathrm{End}(A_Z) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$, which is compatible with the Rosati involution. That is, A_Z is a polarized abelian scheme with an action of $\mathcal{O}_B$ as in §2.1.

We know by Proposition 2.10 that the locus where the $\mathcal{O}_B$-action on A_Z satisfies the determinant condition is a union of connected components of Z.

Moreover, A_Z is also endowed with a K^p-level structure, by base change from the level structure on A. We want to understand the locus where this level structure comes from a K_G^p-level structure. Let Z' be a connected component of Z, and fix a geometric point z of Z'. Then the K^p-level structure corresponds to a $\pi_1(z, Z')$-invariant K^p-orbit $\bar{\eta}^p$ of symplectic similitudes $V^{(p)} \to V^{(p)}(A_z)$. Consider the condition: there exists an element f in $\bar{\eta}^p$ which is $\mathcal{O}_B$-equivariant, and such that its K_G^p-orbit is $\pi_1(z, Z')$- invariant. This condition does not depend on the choice of z on Z' by Remark 2.14. This condition is satisfied if and only if $\bar{\eta}^p$ comes from a K_G^p-level structure on $A_{Z'}$, and then is comes from a unique such structure, as two elements f as in the condition differ by an element of K_G^p. So the locus where the K^p-level structure comes from a K_G^p-level structure is a union of connected components of Z.

We denote by $X_{\mathcal{D}}$ the union of the connected components of Z where the determinant condition holds and the K^p-level structure comes from a K_G^p-level structure. Let $A_{X_{\mathcal{D}}}$ be the abelian scheme over $X_{\mathcal{D}}$ coming from A. As follows from the construction of $X_{\mathcal{D}}$, we have:

Lemma 4.7. *The abelian scheme $A_{X_{\mathcal{D}}}$ is naturally endowed with a structure of a polarized abelian scheme with K_G^p-level structure.*

4.3.2 Comparing $\mathcal{F}$ to $\mathcal{S}$

We now show the relative representability of $\mathcal{F}$ over $\mathcal{S}$ when $\mathcal{S}$ is representable by a scheme. We fix a scheme S, and a morphism $S \to \mathcal{S}$, and consider the functor $\mathcal{F}' = \mathcal{F} \times_{\mathcal{S}} S$. We have to show that $\mathcal{F}'$ is representable by a scheme.

https://doi.org/10.1017/9781108649711.004 Published online by Cambridge University Press

The given morphism $S \to \mathcal{S}$ amounts to an equivalence class of triples $(\mathcal{A}, \lambda; \bar{\eta})$ where $\mathcal{A}$ is an abelian variety over S, endowed with a prime-to-p polarization λ, and a level structure $\bar{\eta}$ of level K^p. We choose a representative of this equivalence class. We can then construct a scheme $X_{\mathcal{D}}$ over S as in §4.3.1.

We then define a natural transformation $\mathcal{F}' \to X_{\mathcal{D}}$. Let T be an S-scheme. An element of $\mathcal{F}'(T)$ is an equivalence class of quadruples $(A, \lambda, \iota; \bar{\eta})$, such that its image by the forgetful functor $\mathcal{F} \to \mathcal{S}$ is in the same equivalence class as $(\mathcal{A}, \lambda; \bar{\eta})_T$. That is, there is a prime-to-p isogeny $f : A \to \mathcal{A}_T$, compatible with the polarizations and the level structures. Then f induces an isomorphism between $\mathrm{End}(A) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$ and $\mathrm{End}(\mathcal{A}_T) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$. We use this isomorphism to define a morphism $\iota : \mathcal{O}_B \to \mathrm{End}(\mathcal{A}_T) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$. Hence we get a point in $X_{\mathcal{D}}(T)$.

We have to show that this construction is well-defined, that is, it does not depend on the choice of $(A, \lambda, \iota; \bar{\eta})$ in the equivalence class. But this comes from the fact that any element of $(\mathcal{A}, \lambda; \bar{\eta})_T$ has no non-trivial automorphism, as we have chosen the level such that $\mathcal{S}$ is representable by a scheme.

Lemma 4.8. *This natural transformation is an isomorphism.*

Proof. We only have to find a natural transformation $X_{\mathcal{D}} \to \mathcal{F}'$ that is a quasi-inverse to the transformation we have just defined. But this is Lemma 4.7. □

Hence $\mathcal{F}'$ is representable by the scheme $X_{\mathcal{D}}$. The connected components of $X_{\mathcal{D}}$ are projective over the scheme representing $\mathcal{S}$, which is itself quasi-projective over $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$. To finish the proof of Proposition 4.5, we only have to show that $X_{\mathcal{D}}$ has only a finite number of connected components. But this comes from the fact that $\mathcal{F}$ is locally of finite presentation over $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$, as can be seen using the criterion of Proposition 4.15 of [LMB00].

4.4 Reduction to the case of principal level structures

We now finish the proof of Theorem 4.1 for the set of Shimura data $\mathcal{D}$. Fix a lattice $L \subset V$ as in Example 2.12, and denote by $K_G(N) \subset G(\mathbf{A}_f^p)$ the principal level subgroup of level N relative to this choice of L. If $N \geq 3$ and N is prime to p, then Proposition 4.5 implies that $\mathcal{F}_{K_G(N)}$ is representable by a scheme.

Let K^p be a level subgroup of $G(\mathbf{A}_f^p)$, and let K'^p level subgroup that is contained in K^p. Then we have a functor $\mathcal{F}_{K'^p} \to \mathcal{F}_{K^p}$ that sends the object $(A, \lambda, \iota; \bar{\eta}^p)/S$ to the object $(A, \lambda, \iota; \tilde{\bar{\eta}}^p)/S$ where $\tilde{\bar{\eta}}^p$ is the K^p-orbit generated by $\bar{\eta}^p$ (see also §6).

Assume now that K'^p is $K_G(N)$ for some $N \geq 3$ and prime to p, so that $\mathcal{F}_{K'^p}$ is a scheme, and in particular a Deligne-Mumford stack. This functor makes the scheme $\mathcal{F}_{K'^p}$ an étale presentation of the stack $\mathcal{F}_{K^p}$. Hence $\mathcal{F}_{K^p}$ is a Deligne-Mumford stack by [LMB00, Proposition 4.3.1].

If $K^p \subset K_G(N)$ for an $N \geq 3$, then $\mathcal{F}_{K^p}$ is representable by an algebraic space. This follows from Lemma 4.9 below and [LMB00, Corollary 8.1.1]: a Deligne-Mumford stack where the objects have only the trivial automorphism is representable by an algebraic space. But then we have a finite morphism $\mathcal{F}_{K^p} \to \mathcal{F}_{K_G(N)}$, which is hence schematic, so $\mathcal{F}_{K^p}$ is representable by a scheme as $\mathcal{F}_{K_G(N)}$ is.

https://doi.org/10.1017/9781108649711.004 Published online by Cambridge University Press

We used the following rigidity lemma:

Lemma 4.9. *Let K^p be a level subgroup contained in a principal level subgroup $K_G(N)$ with $N \geq 3$. Then for any scheme S over $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$ and any object $\underline{A}$ of $\mathcal{F}_{K^p,S}$, $\underline{A}$ has only the trivial automorphism.*

Proof. It follows from the fact that an automorphism of a polarized abelian variety over an algebraically closed field that acts as the identity on the N-torsion subgroup for some $N \geq 3$ is the identity automorphism (see [Ser], or [Mil86] in the book [CS86] for a proof). □

5 Smoothness

Theorem 5.1. *The stack $\mathcal{F}_{K^p}$ is a smooth Deligne-Mumford stack. When K^p is small enough so that S_{K^p} is a scheme, then it is a smooth scheme.*

We need only prove this when K^p is small enough so that S_{K^p} is a scheme, as the transition morphisms between the S_{K^p} with varying level subgroups are étale. As S_{K^p} is locally of finite presentation, we only have to prove that S_{K^p} is formally smooth. We make use of the infinitesimal lifting criterion for smoothness, noting that as $\mathcal{O}_E \otimes \mathbf{Z}_{(p)}$ is noetherian, we need only test artinian algebras. That is, it suffices to show:

Proposition 5.2. *Let R be an artinian $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-algebra. Let $S_0 = \operatorname{spec} R_0$ and $S = \operatorname{spec} R$ such that $R_0 = R/I$ with $I^2 = 0$. If $(A_0, \lambda_0, \iota_0; \bar{\eta}_0)$ on S_0 satisfies the determinant condition of Kottwitz, then it lifts to a $(A, \lambda, \iota; \bar{\eta})$ on S that also satisfies the determinant condition.*

5.1 First reductions

Let us first take care of the level structure:

Lemma 5.3. *If $(A_0, \lambda_0, \iota_0)$ lifts to (A, λ, ι) then any level structure $\bar{\eta}_0$ on $(A_0, \lambda_0, \iota_0)$ lifts to a level structure $\bar{\eta}$ on (A, λ, ι).*

Lifting $\bar{\eta}$ amounts to lifting some sections of $A_0[N]$ to sections of $A[N]$, for a family of integers N prime to p. As $A[N]$ is étale over S, this is automatic.

We now take the determinant condition out of the picture: if $(A_0, \lambda_0, \iota_0)$ on S_0 satisfying the determinant condition of Kottwitz lifts to (A, λ, ι) on S, then the lift automatically satisfies the determinant condition by Proposition 2.9.

Moreover we also know the following result, which is a consequence of the "rigidity lemma" ([MFK94, Theorem 6.1]):

Lemma 5.4. *Let A and B be two abelian schemes over S, then the restriction* $\operatorname{Hom}(A, B) \to \operatorname{Hom}(A_0, B_0)$ *is injective.*

From this we can deduce that if λ_0 and ι_0 both extend to a lifting A of A_0, then the compatibility condition between involutions is automatically satisfied.

https://doi.org/10.1017/9781108649711.004 Published online by Cambridge University Press

5.2 The theory of Grothendieck-Messing

Consider the situation as in the hypothesis of Proposition 5.2. As R is artinian, on each component of S, p is either invertible or nilpotent. The generic fiber of the moduli space is smooth, as it is the canonical model of a Shimura variety (see §7), so the existence of a lift in the case where p is invertible is already known. So we can assume that p is nilpotent on S. Hence we can use the theory of Grothendieck-Messing to study the problem of lifting A_0. Let us recall the part of the theory relevant to the situation. The complete constructions and proofs can be found in [Mes72].

There is a functor from the category of abelian schemes over S_0 to the category of locally free sheaves on S associating to an abelian scheme A_0/S_0 the evaluation of the Dieudonné crystal $\mathbb{D}(A_0)$ on the inclusion $S_0 \to S$, that we will denote by $\mathbb{D}(A_0)_S$. For any abelian variety A/S lifting A_0, $\mathbb{D}(A_0)_S$ is canonically isomorphic to $\mathcal{H}^1_{DR}(A/S)$.

In the case where A_0 is a $\mathbf{Z}_{(p)}$-polarized abelian scheme with an action of $\mathcal{O}_B$, $\mathbb{D}(A_0)_S$ also has an action of $\mathcal{O}_B$. Moreover the polarization induces a morphism $\mathbb{D}(A_0)_S \to \mathbb{D}(A_0^t)_S = \mathbb{D}(A_0)_S^*$, which is an isomorphism because the polarization is separable, and which is compatible with the action of $\mathcal{O}_B$ on both sides. Hence the polarization induces a non-degenerate alternating pairing on $\mathbb{D}(A_0)_S$ that is skew-hermitian with respect to $\mathcal{O}_B$.

A submodule of $\mathbb{D}(A_0)_S$ is said to be admissible if it is locally a direct factor, and reduces to $(\mathrm{Lie}\, A_0)^*$ on S_0.

Theorem 5.5 (Grothendieck-Messing). *There is an equivalence of categories between the category of abelian schemes over S and the category of pairs (A_0, F), where A_0 is an abelian scheme over S_0 and F an admissible submodule of $\mathbb{D}(A_0)_S$, given by $A \mapsto (A_{|S_0}, (\mathrm{Lie}\, A)^*)$.*

In order for the lifting A of A_0 to be polarized with an action of $\mathcal{O}_B$, it is enough that $(\mathrm{Lie}\, A)^*$ is an $\mathcal{O}_B$-stable totally isotropic submodule of $\mathbb{D}(A_0)_S$.

We are then reduced to the following linear algebra problem: Let M be a projective module of rank $2g$ over R with an action of $\mathcal{O}_B$ and a non-degenerate alternating pairing that is skew-hermitian with respect to $\mathcal{O}_B$. Let $M_0 = M \otimes_R R_0$, and let $N_0 \subset M_0$ be a locally direct factor submodule of M_0 of rank g stable under the action of $\mathcal{O}_B$ and totally isotropic for the alternating pairing. Then we need to find a lifting of N_0 to a submodule N of M that has the same properties.

The way to find such a submodule depends of the type of the group G. Details can be found in [LR87] and [Zin82]. We will only treat a simple example: the case of unitary groups over $\mathbf{Q}$.

5.3 An example: unitary groups over Q

Let B be an imaginary quadratic extension of $\mathbf{Q}$, with involution the complex conjugation, and suppose that the prime p is split in B. Then $A = \mathcal{O}_B \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is $\mathbf{Z}_p \times \mathbf{Z}_p$ and the involution exchanges the factors. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Then $e_i M$ is totally isotropic for $i = 1, 2$ as $e_1^* = e_2$. Moreover a submodule Q of an A-module is A-stable if and only if $Q = e_1 Q \oplus e_2 Q$.

We can further simplify the problem: Let A_0 be the universal abelian scheme over S_{K^p}. We know that $\mathcal{H}^1_{DR}(A_0/S)$ and $(\mathrm{Lie}\, A_0)^*$ are locally free modules on S_{K^p}. As smoothness is a local question on S_{K^p}, we can assume that the exact sequence $0 \to (\mathrm{Lie}\, A_0)^* \to \mathcal{H}^1_{DR}(A_0/S) \to \mathrm{Lie}(A_0^t) \to 0$ is split and that these modules are in fact free, This amounts to assuming that we have a decomposition $M_0 = N_0 \oplus P_0$, with N_0 and P_0 free. This implies that M is a free R-module with basis any lifting of a basis of M_0. Let us denote $e_i N_0$ by $N_{0,i}$ for $i = 1, 2$. Then the $N_{0,i}$ are projective. We can also assume that they are free, by the same reasoning as before.

Let us choose a basis of M_0 consisting of the union of a basis of $N_{0,i}$, $i = 1, 2$, and a basis of P_0. We can lift the basis of $N_{0,i}$ to a family in $e_i M$, which gives us free liftings $N_i \subset e_i M$ of $N_{0,i}$. They are totally isotropic, but not necessarily orthogonal. As the bilinear form is non-degenerate, we can modify the lifting of the basis of $N_{0,2}$ such that N_2 is orthogonal to N_1, and still $N_1 \subset e_1 M$. Then $N = N_1 \oplus N_2$ is the lifting of N_0 we were looking for. Indeed, N is A-stable, totally isotropic, projective (even free), and M/N is projective, as it is isomorphic to the submodule P of M generated by any lifting of the chosen basis of P_0.

6 Hecke operators

We explain here the relation between the Shimura varieties when the level varies and the action of the Hecke operators.

6.1 The tower of Shimura varieties

Let K^p and K'^p be compact open subgroups of $G(\mathbf{A}_f^p)$, such that $K^p \subset K'^p$. Then we have a natural morphism from $\mathcal{F}_{K^p}$ to $\mathcal{F}_{K'^p}$ which sends a quadruple $(A, \lambda, \iota; \bar{\eta})$ over the base S to the quadruple $(A, \lambda, \iota; \bar{\eta}')$, where $\bar{\eta}'$ is the K'^p-orbit generated by η. Hence we have a morphism of moduli schemes $S_{K^p} \to S_{K'^p}$. As in the characteristic zero case, we then have a whole tower of integral models $(S_{K^p})_{K^p}$.

If K^p is a normal subgroup of K'^p, then $S_{K^p} \to S_{K'^p}$ is an étale Galois covering of Galois group K'^p/K^p. More generally, for all $K^p \subset K'^p$ compact open subgroups of $G(\mathbf{A}_f^p)$, the morphism $S_{K^p} \to S_{K'^p}$ is finite étale and surjective.

The tower is smooth in the following sense: each of the schemes is smooth for K^p small enough (as explained in §5), and the maps in the tower are also smooth.

6.2 Action of the Hecke operators

We also have Hecke operators: the group $G(\mathbf{A}_f^p)$ acts on the tower via its action on the level structure. That is: for each $g \in G(\mathbf{A}_f^p)$, g maps $\mathcal{F}_{K^p}$ to $\mathcal{F}_{g^{-1}K^p g}$ by sending $(A, \lambda, \iota; \bar{\eta})$ to $(A, \lambda, \iota; \bar{\eta} \circ g)$.

7 Relation to the generic fiber

We will now see how the scheme S_{K^p} relates to the Shimura variety $\mathrm{Sh}(G, X)_K(\mathbf{C}) = G(\mathbf{Q})\backslash \mathcal{X} \times G(\mathbf{A}_f)/K$ and to its canonical model.

7.1 Modular definition of the canonical model

We first recall the construction of the canonical model. Let K be a compact open subgroup of $G(\mathbf{A}_f)$, and let $\mathrm{Sh}(G,X)_K(\mathbf{C}) = G(\mathbf{Q})\backslash\mathcal{X} \times G(\mathbf{A}_f)/K$.

We can obtain a canonical model of this Shimura variety via a moduli space, as follows:

Definition 7.1. Let $\mathcal{Q}$ be the following category fibered in groupoids over the category (Sch/E) of E-schemes:

- The objects over a scheme S are quadruples $\underline{A} = (A, \lambda, \iota; \bar{\eta})$, where (A, λ, ι) is a polarized projective abelian scheme over S with an action of $\mathcal{O}_B$ which respects the determinant condition of Kottwitz (Definition 2.8 of §2.2), and $\bar{\eta}$ is a level structure of level K over each connected component of S, that is, a K-orbit of isomorphisms between $V \otimes \mathbf{A}_f$ and $H_1(A_s, \mathbf{A}_f)$, for s a geometric point of S, that is invariant under the action of $\pi_1(s, S)$ (so that the definition does not depend on the choice of s).
- The morphisms from $\underline{A}$ to $\underline{A'}$ over S are given by a $\mathbf{Q}$-isogeny $f : A \to A'$ compatible with the action of $\mathcal{O}_B$ and the level structures, that is:
    (1) there exists a locally constant function r on S with values in $\mathbf{Q}^\times$ such that $\lambda = r(f^t \circ \lambda' \circ f)$.
    (2) f induces a morphism from $\mathrm{End}\,(A) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$ to $\mathrm{End}\,(A') \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$, that we still denote by f; then for all $b \in \mathcal{O}_B$, $f \circ \iota(b) = \iota'(b)$.
    (3) $\bar{\eta}' = V(f) \circ \bar{\eta}$, where we denote by $V(f)$ the morphism induced from $V(A_s)$ to $V({A'}_s)$.

The functor $\mathcal{Q}$ is representable by a scheme S_K when K is small enough. Then S_K is a disjoint union of canonical models over E of the Shimura variety $\mathrm{Sh}(G,X)_K$. More precisely, as is explained in [Kot92, § 8]:

Proposition 7.2.

$$S_K = \sqcup_{\ker^1(\mathbf{Q},G)} Sh(G', \mathcal{X})_K$$

where $\ker^1(\mathbf{Q}, G)$ *is the set of locally trivial elements of* $H^1(\mathbf{Q}, G)$ *and parametrizes the inner forms* G' *of* G *that are locally isomorphic to* G *at every place.*

The failure of the Hasse principle is essentially harmless, as follows from the study of $\ker^1(\mathbf{Q}, G)$ in [Kot92, § 7]. Recall that if the data is of type A, then G is the restriction of scalars to $\mathbf{Q}$ of a unitary group. We denote by n the dimension of the hermitian space giving rise to this group.

Proposition 7.3. *When* G *is of type* C*, or of type* A *with even* n*,* $\ker^1(\mathbf{Q}, G)$ *is trivial. When* G *is of type* A *with odd* n*, all the groups* G' *are isomorphic to* G*.*

In particular, under the hypotheses of the proposition, S_K is in fact isomorphic to a finite union of copies of the canonical model of the Shimura variety $\mathrm{Sh}(G, \mathcal{X})_K$.

https://doi.org/10.1017/9781108649711.004 Published online by Cambridge University Press

7.2 Relationship to the integral model

Let C_0 be the hyperspecial maximal compact open subgroup of $G(\mathbf{A}_f)$ at p defined in §1.3, that stabilizes a lattice $\Lambda \subset V_{\mathbf{Q}_p}$. Note that when the Shimura data is of type A or C, then by Remark 1.2 there is no ambiguity about C_0 up to conjugation by an element of $G(\mathbf{Q}_p)$. If K^p is a compact open subgroup of $G(\mathbf{A}_f^p)$, then $K = K^p C_0$ is a compact open subgroup of $G(\mathbf{A}_f)$.

Theorem 7.4. *We have then the following isomorphism when the Shimura data is of type A or C:*

$$S_{K^p} \otimes E \xrightarrow{\sim} S_{K^p C_0}$$

Moreover, the induced isomorphism between the towers $(S_{K^p} \otimes E)_{K^p}$ and $(S_{K^p C_0})_{K^p}$ is compatible with the action of $G(\mathbf{A}_f^p)$ on both sides.

It follows from this result that the generic fiber of the integral model S_{K^p} is a union of copies of the canonical model of the Shimura variety $\mathrm{Sh}(G, \mathcal{X})_K$.

Let $\mathcal{F}_{K^p}$ the category we introduced in Definition 3.1 in order to define the moduli problem for the integral model of the Shimura variety, relative to the level subgroup K^p. In what follows we will abbreviate $\mathcal{F}_{K^p}$ by $\mathcal{F}$. We write $\mathcal{F}_{|E}$ for the restriction $\mathcal{F}$ to the set of E-schemes. Hence if $\mathcal{F}$ is representable by the $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-scheme S_{K^p} then $\mathcal{F}_{|E}$ is representable by $S_{K^p} \otimes E$.

Theorem 7.4 is a consequence of the following Proposition (which is similar to Proposition 4.4):

Proposition 7.5. *When the Shimura data is of type A or C, the categories fibered in groupoids $\mathcal{Q}$ and $\mathcal{F}_{|E}$ are isomorphic. The isomorphism is compatible with the action of the Hecke operators on the towers on both sides when K^p varies.*

Proof. Let S be an E-scheme. Let us explain how to define an equivalence of categories from $\mathcal{F}_{|E,S}$ to $\mathcal{Q}_S$. Let $\underline{A} = (A, \lambda, \iota; \bar{\eta}^p)$ be an object of $\mathcal{F}_{|E,S}$. The problem is in the definition of $\bar{\eta}'$: we already have a K^p-orbit of isomorphisms $\bar{\eta}$ between $V \otimes \mathbf{A}_f^p$ and $H_1(A_s, \mathbf{A}_f^p)$ and we have to extend it to the whole of $\mathbf{A}_f$. That is, we have to find a C_0-orbit of isomorphisms between Λ and $H_1(A_s, \mathbf{Z}_p)$.

Observe that V and $H_1(A_s, \mathbf{Q})$ are isomorphic B-modules, as they become so after tensorization by $\mathbf{Q}_\ell$ for any $\ell \neq p$ (this follows from the existence of the level structure outside p). Then $V \otimes \mathbf{Q}_p$ and $H_1(A_s, \mathbf{Q}_p)$ are isomorphic as B-modules. Moreover both have self-dual $\mathcal{O}_B$-lattices. Now we use the condition on the Shimura data: as it is of type A or C, we know by [Kot92, Lemma 7.2] that the lattices Λ and $H_1(A_s, \mathbf{Z}_p)$ are isomorphic as hermitian modules with an action of $\mathcal{O}_B$. Moreover the C_0-orbit of the isomorphism is then well-defined independently of choices. Hence we can uniquely extend the level structure $\bar{\eta}^p$ to $\bar{\eta}$. □

References

[CS86] Gary Cornell and Joseph H. Silverman, editors. *Arithmetic geometry*. Springer-Verlag, New York, 1986. Papers from the conference held at the University of Connecticut, Storrs, Connecticut, July 30–August 10, 1984.

[Del71] Pierre Deligne. Travaux de Shimura. In *Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389*, pages 123–165. Lecture Notes in Math., Vol. 244. Springer, Berlin, 1971.

[GN] Alain Genestier and Bao Chau Ngo. Lectures on Shimura varieties. First chapter of this book.

[Hid04] Haruzo Hida. *p-adic automorphic forms on Shimura varieties*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2004.

[Kot92] Robert E. Kottwitz. Points on some Shimura varieties over finite fields. *J. Amer. Math. Soc.*, 5(2):373–444, 1992.

[Lan13] Kai-Wen Lan. *Arithmetic compactifications of PEL-type Shimura varieties*, volume 36 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2013.

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[LR87] R. P. Langlands and M. Rapoport. Shimuravarietäten und Gerben. *J. Reine Angew. Math.*, 378:113–220, 1987.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[MB85] Laurent Moret-Bailly. Pinceaux de variétés abéliennes. *Astérisque*, (129):266, 1985.

[Mes72] William Messing. *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*. Lecture Notes in Mathematics, Vol. 264. Springer-Verlag, Berlin, 1972.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[Mil86] J. S. Milne. Abelian varieties. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 103–150. Springer, New York, 1986.

[Ser] Jean-Pierre Serre. Rigidité du foncteur de Jacobi d'échelon $n \geq 3$. Appendice à l'exposé 17 du séminaire Cartan 1960–1961.

[Zin82] Thomas Zink. Über die schlechte Reduktion einiger Shimuramannigfaltigkeiten. *Compositio Math.*, 45(1):15–107, 1982.