16

The spaces $\mathcal{M}(A)$ and $\mathcal{H}(A)$

In this chapter, we introduce the notion of *complementary space*, which generalizes the classic geometric notion of orthogonal complement. This notion of complementary space is central in the theory of $\mathcal{H}(b)$ spaces. In Section 16.1, we study the bounded (contractively or isometrically) embeddings. This leads to the definition of $\mathcal{M}(A)$ spaces. Then, in Section 16.2, we characterize the relations between two $\mathcal{M}(A)$ spaces. In Section 16.3, we describe the linear functional on $\mathcal{M}(A)$. In Section 16.4, we give our first definition of complementary space based on an operatorial point of view. As we will see in the next chapter, this operatorial point of view seems particularly interesting in the context of $\mathcal{H}(b)$ spaces and Toeplitz operators. In Section 16.5, we describe the relation between $\mathcal{H}(A)$ and $\mathcal{H}(A^*)$. This relation, though very simple, is probably one of the most useful results in the theory of $\mathcal{H}(b)$ spaces. The overlapping space is introduced and described in Section 16.6. In Sections 16.7 and 16.8, we give useful results concerning some decomposition of $\mathcal{M}(A)$ and $\mathcal{H}(A)$ spaces. In Section 16.9, we introduce our second definition of complementary space and show that it coincides with the first one. Finally, in the last section, we show how the Julia operator can be used to connect this notion of complementary spaces to the more familiar geometric structure of orthogonal complements.

16.1 The space $\mathcal{M}(A)$

Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $\mathcal{H}_1 \subset \mathcal{H}_2$. We do not necessarily assume that $\mathcal{H}_1$ inherits the Hilbert structure of $\mathcal{H}_2$. They can have different Hilbert space structures. The assumption $\mathcal{H}_1 \subset \mathcal{H}_2$ ensures that the inclusion mapping

$$\begin{array}{rccc} i: & \mathcal{H}_1 & \longrightarrow & \mathcal{H}_2 \\ & x & \longmapsto & x \end{array}$$

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is well defined. If this mapping is bounded, i.e. if there is a constant $c > 0$ such that

$$\|x\|_{\mathcal{H}_2} \le c\,\|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1), \tag{16.1}$$

we say that $\mathcal{H}_1$ is *boundedly* contained in $\mathcal{H}_2$ and write $\mathcal{H}_1 \Subset \mathcal{H}_2$. If the mapping i is a contraction, i.e. $c \le 1$, we say that $\mathcal{H}_1$ is *contractively* included in $\mathcal{H}_2$ and write $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$. Finally, if

$$\|x\|_{\mathcal{H}_2} = \|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1),$$

we say that $\mathcal{H}_1$ is *isometrically* contained in $\mathcal{H}_2$. If it happens that the set identity $\mathcal{H}_1 = \mathcal{H}_2$ holds and, moreover, $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same Hilbert space structure, i.e. $\|x\|_{\mathcal{H}_2} = \|x\|_{\mathcal{H}_1}$ for all possible x, then we write $\mathcal{H}_1 \doteq \mathcal{H}_2$. It is important to distinguish between the set identity $\mathcal{H}_1 = \mathcal{H}_2$ and the Hilbert space identity $\mathcal{H}_1 \doteq \mathcal{H}_2$.

A very special case of the above phenomenon is when $\mathcal{H}_1$ is a closed subspace of $\mathcal{H}_2$ and inherits its Hilbert space structure. In this case, $\mathcal{H}_1$ is isometrically embedded inside $\mathcal{H}_2$. In the next section, we will look at this phenomenon from a slightly different angle.

The inequality (16.1) reveals some facts about the topologies of $\mathcal{H}_1$ and $\mathcal{H}_2$. If $\mathcal{E}$ is a closed (or open) subset of $\mathcal{H}_2$, then $\mathcal{E} \cap \mathcal{H}_1$ is closed (or open) in $\mathcal{H}_1$ with respect to the topology of $\mathcal{H}_1$. However, the topology of $\mathcal{H}_1$ is usually richer. In other words, the topology of $\mathcal{H}_1$ is finer than the topology it inherits from $\mathcal{H}_2$. That is why, if Λ is a continuous function on $\mathcal{H}_2$, then its restriction to $\mathcal{H}_1$ remains continuous. We will treat this fact in more detail in Section 16.3. As a special case, if $\mathcal{E} \subset \mathcal{H}_1 \subset \mathcal{H}_2$ is closed in $\mathcal{H}_2$, then $\mathcal{E}$ is also closed in $\mathcal{H}_1$. However, if $\mathcal{E}$ is closed in $\mathcal{H}_1$, we cannot conclude that it is also closed in $\mathcal{H}_2$. The following result reveals the relation between different closures of a set in $\mathcal{H}_1$.

Lemma 16.1 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, assume that $\mathcal{H}_1$ is boundedly embedded into $\mathcal{H}_2$, and let $\mathcal{E} \subset \mathcal{H}_1$. Then*

$$\mathrm{Clos}_{\mathcal{H}_2}(\mathrm{Clos}_{\mathcal{H}_1}\mathcal{E}) = \mathrm{Clos}_{\mathcal{H}_2}\mathcal{E}.$$

Proof For simplicity, put $\mathcal{F} = \mathrm{Clos}_{\mathcal{H}_1}\mathcal{E}$. Since $\mathcal{E} \subset \mathcal{F}$, we have

$$\mathrm{Clos}_{\mathcal{H}_2}\mathcal{E} \subset \mathrm{Clos}_{\mathcal{H}_2}\mathcal{F}.$$

To prove the converse, let $x \in \mathrm{Clos}_{\mathcal{H}_2}\mathcal{F}$ and fix any $\varepsilon > 0$. Then there exists $y \in \mathcal{F}$ such that $\|x - y\|_{\mathcal{H}_2} \le \varepsilon/2$. But, since $y \in \mathcal{F}$ and $\mathcal{F} = \mathrm{Clos}_{\mathcal{H}_1}\mathcal{E}$, there exists $z \in \mathcal{E}$ such that $\|y - z\|_{\mathcal{H}_1} \le \varepsilon/2C$, where C is the constant of embedding of $\mathcal{H}_1$ into $\mathcal{H}_2$, i.e.

$$\|x\|_{\mathcal{H}_2} \le C\,\|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

Therefore, we have $\|y - z\|_{\mathcal{H}_2} \le \varepsilon/2$ and then

$$\|x - z\|_{\mathcal{H}_2} \le \|x - y\|_{\mathcal{H}_2} + \|y - z\|_{\mathcal{H}_2} \le \varepsilon.$$

Therefore, $x \in \mathrm{Clos}_{\mathcal{H}_2}\mathcal{E}$. □

Suppose that $\mathcal{H}_1$ is a Hilbert space, $\mathcal{H}_2$ is a set and $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a set bijection between $\mathcal{H}_1$ and $\mathcal{H}_2$. Then the map A can be served to transfer the Hilbert space structure of $\mathcal{H}_1$ to $\mathcal{H}_2$. It is enough to define

$$\langle Ax, Ay\rangle_{\mathcal{H}_2} = \langle x, y\rangle_{\mathcal{H}_1} \tag{16.2}$$

for all $x, y \in \mathcal{H}_1$. The algebraic operations on $\mathcal{H}_2$ are defined similarly. If $\mathcal{H}_2$ is a linear space and A is an algebraic isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$, the latter requirement is already fulfilled. In this case, (16.2) puts an inner product, maybe a new one, on $\mathcal{H}_2$.

The above construction sounds very elementary. Nevertheless, it has profound consequences. In fact, it is the main ingredient in the definition of $\mathcal{H}(b)$ spaces. To move in this direction, suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and that $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. By the first homomorphism theorem, the operator A induces an isomorphism between the quotient space $\mathcal{H}_1/\ker A$ and $\mathcal{R}(A)$. Hence, by (16.2), the identity

$$\langle Ax, Ay\rangle_{\mathcal{R}(A)} = \langle x + \ker A,\ y + \ker A\rangle_{\mathcal{H}_1/\ker A} \qquad (x, y \in \mathcal{H}_1) \tag{16.3}$$

gives a Hilbert space structure on $\mathcal{R}(A)$. We denote this Hilbert space by $\mathcal{M}(A)$. The norm of $x + \ker A$ in $\mathcal{H}_1/\ker A$ is originally defined by

$$\|x + \ker A\|_{\mathcal{H}_1/\ker A} = \inf_{z \in \ker A} \|x + z\|_{\mathcal{H}_1}.$$

But, for each $z \in \ker A$,

$$\begin{aligned}\|x + z\|^2_{\mathcal{H}_1} &= \|P_{(\ker A)^\perp} x + (z + P_{\ker A} x)\|^2_{\mathcal{H}_1}\\ &= \|P_{(\ker A)^\perp} x\|^2_{\mathcal{H}_1} + \|z + P_{\ker A} x\|^2_{\mathcal{H}_1},\end{aligned}$$

and thus we easily see that

$$\|x + \ker A\|_{\mathcal{H}_1/\ker A} = \|P_{(\ker A)^\perp} x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

Hence, by the polarization identity (1.16), we have

$$\langle x + \ker A,\ y + \ker A\rangle_{\mathcal{H}_1/\ker A} = \langle P_{(\ker A)^\perp} x,\ P_{(\ker A)^\perp} y\rangle_{\mathcal{H}_1} \qquad (x, y \in \mathcal{H}_1).$$

Moreover, by (1.27),

$$\langle P_{(\ker A)^\perp} x,\ P_{(\ker A)^\perp} y\rangle_{\mathcal{H}_1} = \langle x,\ P_{(\ker A)^\perp} y\rangle_{\mathcal{H}_1} = \langle P_{(\ker A)^\perp} x,\ y\rangle_{\mathcal{H}_1}.$$

Therefore, the definition (16.3) reduces to

$$\begin{aligned}\langle Ax, Ay\rangle_{\mathcal{M}(A)} &= \langle P_{(\ker A)^\perp} x,\ P_{(\ker A)^\perp} y\rangle_{\mathcal{H}_1}\\ &= \langle x,\ P_{(\ker A)^\perp} y\rangle_{\mathcal{H}_1}\\ &= \langle P_{(\ker A)^\perp} x,\ y\rangle_{\mathcal{H}_1}\end{aligned} \tag{16.4}$$

for each $x, y \in \mathcal{H}_1$. In particular, for each $x \in \mathcal{H}_1$,

$$\|Ax\|_{\mathcal{M}(A)} = \|P_{(\ker A)^\perp} x\|_{\mathcal{H}_1}. \tag{16.5}$$

Moreover, if at least one of x or y is orthogonal to $\ker A$, then, by (16.4),

$$\langle Ax, Ay\rangle_{\mathcal{M}(A)} = \langle x, y\rangle_{\mathcal{H}_1}. \tag{16.6}$$

The rather trivial inequality

$$\|Ax\|_{\mathcal{M}(A)} \leq \|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1), \tag{16.7}$$

which is a direct consequence of (16.5), will also be frequently used. The preceding formulas should be kept in mind throughout the text.

On $\mathcal{R}(A)$ we now have two inner products. One is inherited from $\mathcal{H}_2$ and the new one imposed by A. In the following, when we write $\mathcal{M}(A)$ we mean that $\mathcal{R}(A)$ is endowed with the latter structure. If this is not the case, we will explicitly mention which structure is considered on $\mathcal{R}(A)$. Let us explore the relation between these two structures. Since A is a bounded operator, we have

$$\|Ax\|_{\mathcal{H}_2} = \|AP_{(\ker A)^\perp} x\|_{\mathcal{H}_2} \leq \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}\, \|P_{(\ker A)^\perp} x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

Therefore, by (16.5),

$$\|Ax\|_{\mathcal{H}_2} \leq \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}\, \|Ax\|_{\mathcal{M}(A)} \qquad (x \in \mathcal{H}_1). \tag{16.8}$$

This inequality means that the inclusion map

$$\begin{array}{rccc} i: & \mathcal{M}(A) & \longrightarrow & \mathcal{H}_2\\ & w & \longmapsto & w \end{array}$$

is continuous and its norm is at most $\|A\|$. In fact, by (16.7),

$$\|Ax\|_{\mathcal{H}_2} \leq \|i\|\, \|Ax\|_{\mathcal{M}(A)} \leq \|i\|\, \|x\|_{\mathcal{H}_1} \qquad (x \in A).$$

Thus, considering (16.8), we deduce that

$$\|i\|_{\mathcal{L}(\mathcal{M}(A),\mathcal{H}_2)} = \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}. \tag{16.9}$$

Moreover,

$$i^* = AA^*. \tag{16.10}$$

Indeed, let $y \in \mathcal{H}_2$ and $Ax \in \mathcal{M}(A)$, with $x \in \mathcal{H}_1$ and $x \perp \ker A$. Then we have

$$\langle Ax, i^*y\rangle_{\mathcal{M}(A)} = \langle Ax, y\rangle_{\mathcal{H}_2} = \langle x, A^*y\rangle_{\mathcal{H}_1} = \langle Ax, AA^*y\rangle_{\mathcal{M}(A)},$$

which proves that $i^*y = AA^*y$. We will see in Section 16.8 that, in a sense, the operator i^* plays the role of an orthogonal projection of $\mathcal{H}_2$ onto $\mathcal{M}(A)$.

If A is invertible, then the relations (16.7), (16.8) and

$$\|x\|_{\mathcal{H}_1} = \|A^{-1}Ax\|_{\mathcal{H}_1} \leq \|A^{-1}\| \, \|Ax\|_{\mathcal{H}_2}$$

imply that the norms in $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{M}(A)$ (which as a set is equal to $\mathcal{H}_2$) are equivalent, i.e.

$$\|x\|_{\mathcal{H}_1} \asymp \|Ax\|_{\mathcal{H}_2} \asymp \|Ax\|_{\mathcal{M}(A)}. \tag{16.11}$$

If A is a bounded operator, the above construction puts $\mathcal{M}(A)$ boundedly inside $\mathcal{H}_2$. If A is a contraction, i.e. $\|A\| \leq 1$, then $\mathcal{M}(A)$ is contractively contained in $\mathcal{H}_2$; and if $\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{H}_2}$, $w \in \mathcal{M}(A)$, then $\mathcal{M}(A)$ is isometrically contained in $\mathcal{H}_2$. Based on the conventions made in Section 16.1, we emphasize that, for $A, B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, the notation $\mathcal{M}(A) \doteq \mathcal{M}(B)$ means not only that the algebraic equality $\mathcal{M}(A) = \mathcal{M}(B)$ holds, but also that the Hilbert space structures coincide, i.e.

$$\langle w_1, w_2 \rangle_{\mathcal{M}(A)} = \langle w_1, w_2 \rangle_{\mathcal{M}(B)}$$

for all possible elements w_1 and w_2. Clearly, in the light of the polarization identity, the latter is equivalent to

$$\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{M}(B)}$$

for all possible elements w.

The relation (16.5) contains all the information regarding the definition of the structure of $\mathcal{M}(A)$. In short, the structure of $\mathcal{M}(A)$ is the same as that of $\mathcal{H}_1/\ker A$. This fact is explained in another language in the following result.

Theorem 16.2 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and define*

$$\begin{array}{rccc} \mathbb{A}: & \mathcal{H}_1 & \longrightarrow & \mathcal{M}(A) \\ & x & \longmapsto & Ax. \end{array}$$

Then $\mathbb{A}$ is a bounded operator, i.e. $\mathbb{A} \in \mathcal{L}(\mathcal{H}_1, \mathcal{M}(A))$, and, moreover, $\mathbb{A}^*$ is an isometry on $\mathcal{M}(A)$.

Proof The inequality (16.7) can be rewritten as

$$\|\mathbb{A}x\|_{\mathcal{M}(A)} = \|Ax\|_{\mathcal{M}(A)} \leq \|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

This means that $\mathbb{A}$ is a bounded operator. In order to show that $\mathbb{A}^*$ is an isometry on $\mathcal{M}(A)$, by Corollary 7.23, it is enough to show that $\mathbb{A}$ is a surjective partial isometry. That $\mathbb{A}$ is surjective is a trivial consequence of the definition of $\mathcal{M}(A)$. Moreover, $\ker \mathbb{A} = \ker A$. Hence, by (16.5),

$$\|\mathbb{A}x\|_{\mathcal{M}(A)} = \|Ax\|_{\mathcal{M}(A)} = \|P_{(\ker A)^\perp} x\|_{\mathcal{H}_1} = \|P_{(\ker \mathbb{A})^\perp} x\|_{\mathcal{H}_1}$$

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for each $x \in \mathcal{H}_1$. Thus, $\mathbb{A}$ is a partial isometry (see the original definition (7.14)). □

The definition of spaces $\mathcal{M}(A)$ is closely related to the notion of bounded embeddings introduced at the beginning of this section. Indeed, if $\mathcal{M}$ is a Hilbert space that is boundedly contained in another Hilbert space $\mathcal{H}$, then the inclusion map

$$\begin{array}{rccc} i: & \mathcal{M} & \longrightarrow & \mathcal{H} \\ & x & \longmapsto & x \end{array}$$

is bounded from $\mathcal{M}$ into $\mathcal{H}$. Now, since for any $x \in \mathcal{M} = \mathcal{M}(i)$, we have

$$\|x\|_{\mathcal{M}(i)} = \|i(x)\|_{\mathcal{M}(i)} = \|x\|_{\mathcal{M}},$$

the space $\mathcal{M}$ coincides with $\mathcal{M}(i)$, that is

$$\mathcal{M} \doteq \mathcal{M}(i).$$

Conversely, if $\mathcal{M} \doteq \mathcal{M}(A)$, where $A : \mathcal{H}_1 \longrightarrow \mathcal{H}$ is bounded, then $\mathcal{M}$ is boundedly contained in $\mathcal{H}$. Thus, we get the following result.

Theorem 16.3 *Let $\mathcal{M}$ and $\mathcal{H}$ be two Hilbert spaces. Then the following assertions are equivalent.*

(i) *The space $\mathcal{M}$ is boundedly contained in $\mathcal{H}$ (respectively contractively; respectively isometrically).*

(ii) *There exists a bounded operator $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ (respectively a contraction; respectively an isometry) such that*

$$\mathcal{M} \doteq \mathcal{M}(A). \tag{16.12}$$

In the next section, we examine the problem of uniqueness in the representation of $\mathcal{M}$ given by (16.12). See also Exercise 16.2.2.

The following result shows that, if $A \in \mathcal{L}(\mathcal{H})$ is an orthogonal projection, then in fact we do not obtain a new structure on $\mathcal{M}(A)$. The Hilbert space structure of $\mathcal{M}(A)$ is precisely the one it has in the first place as a closed subspace of $\mathcal{H}$.

Lemma 16.4 *Let M be a closed subspace of $\mathcal{H}$, and let $P_M \in \mathcal{L}(\mathcal{H})$ denote the orthogonal projection on M. Then*

$$\mathcal{M}(P_M) \doteq M,$$

i.e. $\mathcal{M}(P_M) = M$ and $\|w\|_{\mathcal{M}(P_M)} = \|w\|_{\mathcal{H}}$ for all $w \in M$.

Proof The identity $\mathcal{M}(P_M) = M$ is an immediate consequence of the definition of an orthogonal projection. Remember that $\ker P_M = M^\perp$, and since M is closed, $(M^\perp)^\perp = M$. Hence, by (16.5),

$$\|P_M x\|_{\mathcal{M}(P_M)} = \|P_{(\ker P_M)^\perp} x\|_{\mathcal{H}} = \|P_M x\|_{\mathcal{H}} \qquad (x \in \mathcal{H}_1).$$ □

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Lemma 16.5 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H})$. Then*

$$\|Bw\|_{\mathcal{M}(BA)} \le \|w\|_{\mathcal{M}(A)} \qquad (w \in \mathcal{M}(A)).$$

Proof It is clear that $B\,\mathcal{M}(A) \subset \mathcal{M}(BA)$. Put $w = Ax$, $x \in \mathcal{H}_1$. Hence, by (16.5),

$$\|Bw\|_{\mathcal{M}(BA)} = \|P_{(\ker BA)^\perp} x\|_{\mathcal{H}_1} \quad \text{and} \quad \|w\|_{\mathcal{M}(A)} = \|P_{(\ker A)^\perp} x\|_{\mathcal{H}_1}.$$

But, since $\ker BA \supset \ker A$, we have

$$\|P_{(\ker BA)^\perp} x\|_{\mathcal{H}_1} \le \|P_{(\ker A)^\perp} x\|_{\mathcal{H}_1}.$$

Therefore, we deduce that $\|Bw\|_{\mathcal{M}(BA)} \le \|w\|_{\mathcal{M}(A)}$. □

Exercises

Exercise 16.1.1 Let $\mathcal{H}$ be a set endowed with two inner products whose corresponding norms are complete and equivalent, i.e.

$$c\,\|x\|_1 \le \|x\|_2 \le C\,\|x\|_1 \qquad (x \in \mathcal{H}),$$

where c and C are positive constants. Show that $(\mathcal{H}, \langle\,\cdot,\cdot\,\rangle_1)$ is boundedly contained in $(\mathcal{H}, \langle\,\cdot,\cdot\,\rangle_2)$, and vice versa.

Exercise 16.1.2 Let $(X, \mathcal{A})$ be a measurable space, and let μ and ν be two positive measures on the σ-algebra $\mathcal{A}$. Suppose that

$$\mu(E) \le \nu(E) \tag{16.13}$$

for all $E \in \mathcal{A}$. Show that $L^2(\nu)$ is contractively contained in $L^2(\mu)$.

Hint: The assumption (16.13) can be rewritten as

$$\int_X \chi_E \, d\mu \le \int_X \chi_E \, d\nu,$$

where χ_E is the characteristic function of E. Take linear combinations with positive coefficients, and then apply the monotone convergence theorem to obtain

$$\int_X \varphi \, d\mu \le \int_X \varphi \, d\nu$$

for all positive measurable functions φ. Hence, deduce $\|f\|_{L^2(\mu)} \le \|f\|_{L^2(\nu)}$.

Exercise 16.1.3 Let $\varphi \in L^\infty(\mathbb{T})$, and consider the multiplication operator

$$\begin{array}{rccc} M_\varphi : & L^2(\mathbb{T}) & \longrightarrow & L^2(\mathbb{T}) \\ & f & \longmapsto & \varphi f, \end{array}$$

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which was studied in Section 7.2. Show that

$$\|\varphi f\|_{\mathcal{M}(M_\varphi)} = \left(\frac{1}{2\pi} \int_{\mathbb{T}\setminus E} |f(e^{it})|^2 \, dt \right)^{1/2} \qquad (f \in L^2(\mathbb{T}))$$

and that

$$\langle \varphi f, \varphi g \rangle_{\mathcal{M}(M_\varphi)} = \frac{1}{2\pi} \int_{\mathbb{T}\setminus E} f(e^{it}) \, \overline{g(e^{it})} \, dt \qquad (f, g \in L^2(\mathbb{T})),$$

where $E = \{\zeta \in \mathbb{T} : \varphi(\zeta) = 0\}$. The first identity reveals that $\mathcal{M}(M_\varphi) = \varphi L^2(\mathbb{T})$ is contractively contained in $L^2(\mathbb{T})$. Under what condition is $\mathcal{M}(M_\varphi)$ isometrically contained in $L^2(\mathbb{T})$?

Exercise 16.1.4 Let Θ be an inner function for the open unit disk, and let

$$\begin{array}{rccc} M_\Theta : & H^2(\mathbb{D}) & \longrightarrow & H^2(\mathbb{D}) \\ & f & \longmapsto & \Theta f. \end{array}$$

Show that

$$\|\Theta f\|_{\mathcal{M}(M_\Theta)} = \|f\|_{H^2(\mathbb{D})} = \|\Theta f\|_{H^2(\mathbb{D})} \qquad (f \in H^2(\mathbb{D})).$$

Thus $\mathcal{M}(M_\Theta) = \Theta H^2$ is isometrically contained in $H^2(\mathbb{D})$.
Hint: M_Θ is injective and $|\Theta| = 1$ almost everywhere on $\mathbb{T}$.

Exercise 16.1.5 Let $A \in \mathcal{L}(H_1, H_2)$ and $\alpha \in \mathbb{C}$, $\alpha \neq 0$. Show that

$$\|w\|_{\mathcal{M}(\alpha A)} = \frac{\|w\|_{\mathcal{M}(A)}}{|\alpha|} \qquad (w \in \mathcal{M}(A)).$$

16.2 A characterization of $\mathcal{M}(A) \subset \mathcal{M}(B)$

If the operators $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$ are such that $\mathcal{M}(A) \Subset \mathcal{M}(B)$, then we surely have $\mathcal{M}(A) \subset \mathcal{M}(B)$. Conversely, if the set inclusion $\mathcal{M}(A) \subset \mathcal{M}(B)$ holds, then the inclusion mapping

$$\begin{array}{rccc} i : & \mathcal{M}(A) & \longrightarrow & \mathcal{M}(B) \\ & w & \longmapsto & w \end{array}$$

is well defined. But, in fact, more is true. The way that the structures of $\mathcal{M}(A)$ and $\mathcal{M}(B)$ are defined forces i to be a bounded operator and thus $\mathcal{M}(A)$ is boundedly contained in $\mathcal{M}(B)$.

Lemma 16.6 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$ be such that $\mathcal{M}(A) \subset \mathcal{M}(B)$. Then $\mathcal{M}(A) \Subset \mathcal{M}(B)$.*

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Proof We need to show that the inclusion $i : \mathcal{M}(A) \longrightarrow \mathcal{M}(B)$ is a bounded operator. The justification is based on the closed graph theorem. Let $(w_n)_{n\geq 1}$ be a sequence in $\mathcal{R}(A)$ that converges to w in $\mathcal{M}(A)$ and to w' in $\mathcal{M}(B)$. Note that $iw_n = w_n$. Since $\mathcal{M}(A)$ and $\mathcal{M}(B)$ are both boundedly embedded into H, the sequence $(w_n)_{n\geq 1}$ also tends to w and to w' in the norm of H. Then, by uniqueness of the limit, we must have $w = w'$. Hence, the closed graph theorem implies that i is continuous. □

Lemma 16.6 shows that the new notation $\Subset$ is not needed in the study of $\mathcal{M}(A)$ spaces. However, we emphasize that $\mathcal{M}(A) = \mathcal{M}(B)$ is not equivalent to $\mathcal{M}(A) \doteq \mathcal{M}(B)$. The identity $\mathcal{M}(A) = \mathcal{M}(B)$ implies that

$$c\,\|w\|_{\mathcal{M}(B)} \leq \|w\|_{\mathcal{M}(A)} \leq C\,\|w\|_{\mathcal{M}(B)},$$

while in the definition of $\mathcal{M}(A) \doteq \mathcal{M}(B)$ we assumed that

$$\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{M}(B)}.$$

To use Lemma 16.6, we naturally ask under what conditions the set inclusion $\mathcal{M}(A) \subset \mathcal{M}(B)$ holds. Let us treat a sufficient condition. Suppose that there is a bounded operator $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, with $\|C\| \leq c$, such that $A = BC$. Since, for each $x \in \mathcal{H}_1$, $Ax = B(Cx)$, we have the set inclusion $\mathcal{M}(A) \subset \mathcal{M}(B)$. Thus, by Lemma 16.6, $\mathcal{M}(A) \Subset \mathcal{M}(B)$. Moreover, by (16.7) and the fact that $\|C\| \leq c$, we have

$$\|Ax\|_{\mathcal{M}(B)} = \|BCx\|_{\mathcal{M}(B)} \leq \|Cx\|_{\mathcal{H}_2} \leq c\,\|x\|_{\mathcal{H}_1}.$$

By (16.5), replacing x by $P_{(\ker A)^\perp} x$ gives us

$$\|Ax\|_{\mathcal{M}(B)} \leq c\,\|Ax\|_{\mathcal{M}(A)} \qquad (x \in \mathcal{H}_1).$$

Hence, the norm of i is less than or equal to c. This means that $\mathcal{M}(A)$ is boundedly contained in $\mathcal{M}(B)$ and, in particular, if $c = 1$, $\mathcal{M}(A)$ is contractively contained in $\mathcal{M}(B)$. What is surprising is that the existence of C is also necessary for the bounded inclusion of $\mathcal{M}(A)$ in $\mathcal{M}(B)$.

Theorem 16.7 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$, and let $c > 0$. Then the following are equivalent.*

(i) $AA^* \leq c^2 BB^*$.
(ii) *There is an operator $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, with $\|C\| \leq c$, such that $A = BC$.*
(iii) *We have $\mathcal{M}(A) \subset \mathcal{M}(B)$ with*

$$\|w\|_{\mathcal{M}(B)} \leq c\,\|w\|_{\mathcal{M}(A)} \qquad (w \in \mathcal{M}(A)),$$

i.e. the inclusion $i : \mathcal{M}(A) \longrightarrow \mathcal{M}(B)$ is a bounded operator of norm less than or equal to c.

In particular, $\mathcal{M}(A) \hookrightarrow \mathcal{M}(B)$ if and only if $AA^* \leq BB^*$.

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Proof (i) $\Longleftrightarrow$ (ii) This is the content of Theorem 7.11.

(ii) $\Longrightarrow$ (iii) This was discussed above.

(iii) $\Longrightarrow$ (ii) Take an element $w = Ax \in \mathcal{M}(A)$, with some $x \in \mathcal{H}_1$. Hence, for each $x \in \mathcal{H}_1$, there is a $y \in \mathcal{H}_2$ such that

$$Ax = By. \tag{16.14}$$

The element y is not necessarily unique. However, if $By = By'$, with $y, y' \in \mathcal{H}_2$, then $B(y - y') = 0$ and thus $y - y' \in \ker B$. In other words, we have $P_{(\ker B)^\perp} y = P_{(\ker B)^\perp} y'$. Therefore, the mapping

$$\begin{array}{rccc} C: & \mathcal{H}_1 & \longrightarrow & \mathcal{H}_2 \\ & x & \longmapsto & P_{(\ker B)^\perp} y, \end{array}$$

with $y \in \mathcal{H}_2$ given by (16.14), is well defined and

$$BCx = BP_{(\ker B)^\perp} y = By = Ax \qquad (x \in \mathcal{H}_1).$$

This means that the definition of C is adjusted such that the identity $A = BC$ holds. Moreover, by (16.5) and (16.7) and our assumption,

$$\begin{aligned} \|Cx\|_{\mathcal{H}_2} &= \|P_{(\ker B)^\perp} y\|_{\mathcal{H}_2} \\ &= \|By\|_{\mathcal{M}(B)} \\ &= \|Ax\|_{\mathcal{M}(B)} \\ &\le c\,\|Ax\|_{\mathcal{M}(A)} \\ &\le c\,\|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1). \end{aligned}$$

Hence, C is a bounded operator of norm less than or equal to c. □

We gather some important corollaries below. The first one follows immediately from Theorem 16.7.

Corollary 16.8 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$. Then the following statements hold.*

(i) $\mathcal{M}(A) \doteq \mathcal{M}(B)$ *if and only if* $AA^* = BB^*$.

(ii) $\mathcal{M}(A) \doteq \mathcal{M}(|A|)$, *where* $|A| = (AA^*)^{1/2}$.

If the linear manifold $\mathcal{R}(A)$ is closed in H, then it inherits the Hilbert space structure of H. One may wonder if this Hilbert space structure coincides with the one we put on $\mathcal{R}(A)$ and called it $\mathcal{M}(A)$. The following corollary answers this question.

Corollary 16.9 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$. Then $\mathcal{R}(A)$ is a closed subspace of $\mathcal{H}$ and $\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{H}}$, for each $w \in \mathcal{M}(A)$, if and only if A is a partial isometry. In this case, we have*

$$\mathcal{M}(A) \doteq \mathcal{M}(AA^*).$$

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Proof If A is a partial isometry, then, by Theorem 7.22, $P = AA^*$ is an orthogonal projection and thus $|A| = P$. Hence, by Corollary 16.8(ii), $\mathcal{M}(A) \doteq \mathcal{M}(P)$. This means that $\mathcal{R}(A) = \mathcal{R}(P)$ and $\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{M}(P)}$, for each $w \in \mathcal{M}(A)$. But, by Lemma 16.4, $\mathcal{R}(P)$ is a closed subspace of H and $\|w\|_{\mathcal{M}(P)} = \|w\|_{\mathcal{H}}$ for each $w \in \mathcal{M}(P)$.

Now, suppose that $M = \mathcal{R}(A)$ is a closed subspace of H. Then the identity $\mathcal{M}(A) = \mathcal{M}(P_M)$ is trivial. Then, by Lemma 16.4 and our assumptions, we have $\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{H}} = \|w\|_{\mathcal{M}(P_M)}$, for each $w \in \mathcal{M}(A)$. In other words, we have the stronger relation $\mathcal{M}(A) \doteq \mathcal{M}(P_M)$. Hence, by Corollary 16.8(i),

$$AA^* = P_M P_M^* = P_M.$$

Therefore, again by Theorem 7.22, A is a partial isometry. In this case, the relations

$$\mathcal{M}(A) \doteq \mathcal{M}(P_M) \doteq \mathcal{M}(AA^*)$$

were implicitly established above. □

In Theorem 16.7, the condition $AA^* \leq c^2 BB^*$ was studied. The following result is a slightly more generalized version of one part of this theorem. It answers the following natural question. If $\mathcal{M}(A)$ and $\mathcal{M}(B)$ are boundedly contained in H, and if $C : H \longrightarrow H$ is a bounded operator, under what conditions does C map continuously $\mathcal{M}(B)$ into $\mathcal{M}(A)$?

Corollary 16.10 *Let $A, B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$, and let $C \in \mathcal{L}(\mathcal{H})$. Then:*

(i) $\mathcal{M}(B) \subset C\mathcal{M}(A)$ *if and only if there exists $c > 0$ such that*

$$BB^* \leq c^2 CAA^*C^*;$$

(ii) $C\mathcal{M}(A) \subset \mathcal{M}(B)$ *if and only if there exists $c > 0$ such that*

$$CAA^*C^* \leq c^2 BB^*. \tag{16.15}$$

Moreover, if the inequality (16.15) *is satisfied, then the mapping*

$$\begin{array}{rccc} \mathbf{C}: & \mathcal{M}(A) & \longrightarrow & \mathcal{M}(B) \\ & w & \longmapsto & Cw \end{array}$$

is a well-defined operator in $\mathcal{L}(\mathcal{M}(A), \mathcal{M}(B))$ and

$$\|\mathbf{C}\|_{\mathcal{L}(\mathcal{M}(A),\mathcal{M}(B))} \leq c.$$

Proof (i) By Theorem 16.7 and Lemma 16.6, the operator inequality $BB^* \leq c^2 CAA^*C^*$ is equivalent to the fact that $\mathcal{M}(B) \subset \mathcal{M}(CA)$. But $\mathcal{M}(CA) = C\mathcal{M}(A)$, which gives the first assertion.

(ii) The proof has the same flavor. Using once more Theorem 16.7 and Lemma 16.6, we see that the operator inequality (16.15) is equivalent to the set inclusion $\mathcal{M}(CA) \subset \mathcal{M}(B)$ and, since $\mathcal{M}(CA) = C\mathcal{M}(A)$, that gives

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the desired equivalence. It remains to check that $\mathbf{C}$ is a bounded operator of norm less than or equal to c. Using Theorem 16.7 once more, we see that the condition (16.15) implies that

$$\|w\|_{\mathcal{M}(B)} \le c\|w\|_{\mathcal{M}(CA)} \qquad (w \in \mathcal{M}(CA)).$$

Now put $w = Cx$, $x \in \mathcal{M}(A)$, and then apply Lemma 16.5 to get

$$\|Cx\|_{\mathcal{M}(B)} \le c\,\|Cx\|_{\mathcal{M}(CA)} \le c\,\|x\|_{\mathcal{M}(A)} \qquad (x \in \mathcal{M}(A)). \qquad \square$$

Without any serious difficulty, we will denote the operator $\mathbf{C}$ also by C. In particular, the relation

$$CAA^*C^* \le BB^*$$

ensures that C is a contraction from $\mathcal{M}(A)$ into $\mathcal{M}(B)$.

Corollary 16.11 *Let $A, B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$, and let $C \in \mathcal{L}(\mathcal{H})$ be such that*

$$CAA^*C^* = BB^*. \tag{16.16}$$

Assume further that C is injective. Then, the mapping C is a unitary operator from $\mathcal{M}(A)$ onto $\mathcal{M}(B)$.

Proof According to Corollary 16.8, equation (16.16) implies that $\mathcal{M}(CA) = \mathcal{M}(B)$, that is $C\mathcal{M}(A) = \mathcal{M}(B)$ and

$$\|w\|_{\mathcal{M}(CA)} = \|w\|_{\mathcal{M}(B)}$$

for any $w \in \mathcal{M}(CA)$. Hence, C maps $\mathcal{M}(A)$ onto $\mathcal{M}(B)$ and for any $w \in \mathcal{M}(A)$ we have

$$\|Cw\|_{\mathcal{M}(CA)} = \|Cw\|_{\mathcal{M}(B)}. \tag{16.17}$$

If we write $w = Ax$, with $x \in \mathcal{H}_1 \ominus \ker A$, then

$$\|Cw\|_{\mathcal{M}(CA)} = \|CAx\|_{\mathcal{M}(CA)} = \|P_{(\ker CA)^\perp}x\|_{\mathcal{H}_1}.$$

But we always have $\ker A \subset \ker CA$ and, since C is assumed to be injective, the reverse inclusion is also true. Hence $\ker A = \ker CA$ and thus $P_{(\ker CA)^\perp}x = x$. Then we get

$$\|Cw\|_{\mathcal{M}(B)} = \|x\|_{\mathcal{H}_1} = \|Ax\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{M}(A)}.$$

Hence C is unitary operator from $\mathcal{M}(A)$ onto $\mathcal{M}(B)$. $\square$

If $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$ is injective, then B puts an isometric isomorphic copy of $\mathcal{H}_2$ in $\mathcal{H}$, which we denote by $\mathcal{M}(B)$. This fact is an immediate consequence of the definition of $\mathcal{M}(B)$. This result is mentioned below in further detail.

Corollary 16.12 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$. Then the following are equivalent.*

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(i) $AA^* = BB^*$ *and* B *is injective.*
(ii) $\mathcal{M}(A) \doteq \mathcal{M}(B)$ *and* B *is injective.*
(iii) B *is an isometry from* $\mathcal{H}_2$ *onto* $\mathcal{M}(A)$.

Proof The equivalence (i) $\Longleftrightarrow$ (ii) follows from Corollary 16.8.
(ii) $\Longleftrightarrow$ (iii) By hypothesis, $\mathcal{M}(A) = \mathcal{M}(B)$ and

$$\|Bx\|_{\mathcal{M}(A)} = \|Bx\|_{\mathcal{M}(B)} \qquad (x \in \mathcal{H}_2).$$

But, since B is injective, we have

$$\|Bx\|_{\mathcal{M}(B)} = \|x\|_{\mathcal{H}_2} \qquad (x \in \mathcal{H}_2).$$

Thus,

$$\|Bx\|_{\mathcal{M}(A)} = \|x\|_{\mathcal{H}_2} \qquad (x \in \mathcal{H}_2).$$

This identity shows that B is an isometry from H_2 onto $\mathcal{M}(A)$.
(iii) $\Longrightarrow$ (ii) By assumption, we have $\mathcal{R}(A) = \mathcal{R}(B)$ and

$$\|Bx\|_{\mathcal{M}(A)} = \|x\|_{\mathcal{H}_2} \qquad (x \in \mathcal{H}_2).$$

That B is an isometry implies $\ker B = \{0\}$. Hence,

$$\|Bx\|_{\mathcal{M}(B)} = \|x\|_{\mathcal{H}_2}$$

for every $x \in (\ker B)^\perp = \mathcal{H}_2$. Thus,

$$\|Bx\|_{\mathcal{M}(A)} = \|Bx\|_{\mathcal{M}(B)} \qquad (x \in \mathcal{H}_2).$$

This means that $\mathcal{M}(A) \doteq \mathcal{M}(B)$. □

Exercises

Exercise 16.2.1 Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$. Show that $\mathcal{M}(|A|) \doteq \mathcal{M}(|B|)$ if and only if $|A| = |B|$.
Hint: Use Corollary 16.8(i).

Exercise 16.2.2 Let $\mathcal{M}$ and $\mathcal{H}$ be two Hilbert spaces and assume that $\mathcal{M}$ is boundedly contained in $\mathcal{H}$. Show that there is a unique positive operator $T \in \mathcal{L}(\mathcal{H})$ such that $\mathcal{M} \doteq \mathcal{M}(T)$.
Hint: For the existence, consider $i = i_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$, the inclusion mapping from $\mathcal{M}$ into $\mathcal{H}$. Then, use Corollary 16.8(ii) to show that $\mathcal{M} \doteq \mathcal{M}(|i|)$, where $|i| = (ii^*)^{1/2}$. For the uniqueness, assume that there exist two positive operators $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ such that $\mathcal{M} \doteq \mathcal{M}(T_1) = \mathcal{M}(T_2)$. Use Corollary 16.8(i) to deduce that

$$T_1 T_1^* = T_2 T_2^*$$

and conclude using the positivity of T_1 and T_2 and the uniqueness of the positive square root.

Exercise 16.2.3 Let $\mathcal{E}$, $\mathcal{H}$ and $\mathcal{H}_*$ be Hilbert spaces such that $\mathcal{E} \subset \mathcal{H}_*$, and let $T : \mathcal{H} \longrightarrow \mathcal{H}_*$ be a bounded operator. Show that the following assertions are equivalent:

(i) $\mathcal{E} \doteq \mathcal{M}(T)$;
(ii) $TT^* = ii^*$, where $i : \mathcal{E} \longrightarrow \mathcal{H}_*$ is the embedding operator.

Hint: Use the fact that $\mathcal{E} \doteq \mathcal{M}(i)$ and apply Corollary 16.8.

16.3 Linear functionals on $\mathcal{M}(A)$

Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Suppose that

$$\Lambda : \mathcal{H}_2 \longrightarrow \mathbb{C}$$

is a bounded linear functional on $\mathcal{H}_2$. Then, by Riesz's theorem (Theorem 1.24), there is a unique $w \in \mathcal{H}_2$ such that

$$\Lambda z = \langle z, w \rangle_{\mathcal{H}_2} \qquad (z \in \mathcal{H}_2).$$

According to (16.8), the inclusion map

$$i : \mathcal{M}(A) \longrightarrow \mathcal{H}_2$$

is continuous. Hence,

$$\Lambda \circ i : \mathcal{M}(A) \longrightarrow \mathbb{C}$$

is a bounded linear functional on $\mathcal{M}(A)$. Thus, again by Riesz's theorem, there is a unique $w' \in \mathcal{M}(A)$ such that

$$(\Lambda \circ i)(z) = \langle z, w' \rangle_{\mathcal{M}(A)} \qquad (z \in \mathcal{M}(A)).$$

We naturally proceed to find the relation between w and w'. Note that $\Lambda \circ i$ is precisely the restriction of Λ to $\mathcal{M}(A)$, which, according to our general convention, we also denote by $\boldsymbol{\Lambda}$.

Theorem 16.13 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Let $w \in \mathcal{H}_2$, and let*

$$\Lambda z = \langle z, w \rangle_{\mathcal{H}_2} \qquad (z \in \mathcal{H}_2)$$

be the corresponding bounded linear functional on H_2. Then its restriction

$$\boldsymbol{\Lambda} : \mathcal{M}(A) \longrightarrow \mathbb{C}$$

is a bounded linear functional on $\mathcal{M}(A)$ and

$$\boldsymbol{\Lambda}(Ax) = \langle Ax, AA^* w \rangle_{\mathcal{M}(A)} \qquad (x \in \mathcal{H}_1).$$

Moreover,

$$\|\mathbf{\Lambda}\|_{\mathcal{M}(A)^*} = \|A^*w\|_{\mathcal{H}_1}.$$

Remark: We recall that, by Riesz's theorem,

$$\|\Lambda\|_{H_2^*} = \|w\|_{\mathcal{H}_2}.$$

Proof By the definition of the adjoint operator, we have

$$\Lambda(Ax) = \langle Ax, w\rangle_{\mathcal{H}_2} = \langle x, A^*w\rangle_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

But, by Theorem 1.30,

$$A^*w \in \mathcal{R}(A^*) \subset (\ker A)^{\perp}.$$

Hence, by (16.6),

$$\langle x, A^*w\rangle_{\mathcal{H}_1} = \langle Ax, AA^*w\rangle_{\mathcal{M}(A)} \qquad (x \in \mathcal{H}_1).$$

Therefore, we can write

$$\mathbf{\Lambda}(Ax) = \langle Ax, AA^*w\rangle_{\mathcal{M}(A)} \qquad (x \in \mathcal{H}_1).$$

This representation shows that

$$\|\mathbf{\Lambda}\|_{\mathcal{M}(A)^*} = \|AA^*w\|_{\mathcal{M}(A)}.$$

However, by (16.5) and the fact that $A^*w \in (\ker A)^{\perp}$, we have

$$\|AA^*w\|_{\mathcal{M}(A)} = \|A^*w\|_{\mathcal{H}_1}. \qquad \square$$

16.4 The complementary space $\mathcal{H}(A)$

If A is a Hilbert space contraction, then $AA^* \leq I$ and thus $(I-AA^*)^{1/2}$ is well defined (see Exercise 2.4.5). Therefore, we can consider the linear manifold $\mathcal{R}((I-AA^*)^{1/2})$ and put a Hilbert space structure on it, as explained in the previous section and denoted by $\mathcal{M}((I-AA^*)^{1/2})$. We call

$$\mathcal{H}(A) = \mathcal{M}((I-AA^*)^{1/2})$$

the *complementary space* of $\mathcal{M}(A)$, and the intersection $\mathcal{M}(A) \cap \mathcal{H}(A)$ is called the *overlapping space*. In the rest of this chapter we study $\mathcal{H}(A)$ and its relation to $\mathcal{M}(A)$.

Lemma 16.14 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ be a contraction. Then $\mathcal{H}(A)$ is a closed subspace of $\mathcal{H}$ and $\|w\|_{\mathcal{H}(A)} = \|w\|_{\mathcal{H}}$, for each $w \in \mathcal{H}(A)$, if and only if A is a partial isometry. In this case, the set identity*

$$\mathcal{H}(A) = \mathcal{R}(I-AA^*)$$

holds.

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Proof By Corollary 16.9, $\mathcal{H}(A)$ is a closed subspace of $\mathcal{H}$ and $\|w\|_{\mathcal{H}(A)} = \|w\|_{\mathcal{H}}$, for each $w \in \mathcal{H}(A)$, if and only if $(I - AA^*)^{1/2}$ is a partial isometry. But, by Theorem 7.22, this happens if and only if $I - AA^*$ is an orthogonal projection. Clearly, $I - AA^*$ is an orthogonal projection if and only if AA^* is an orthogonal projection. Finally, again by Theorem 7.22, AA^* is an orthogonal projection if and only if A is a partial isometry.

In this case, since $I - AA^*$ is an orthogonal projection, we have $(I - AA^*)^{1/2} = I - AA^*$, and thus the set identity $\mathcal{H}(A) = \mathcal{R}(I - AA^*)$ holds. □

For an operator $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, we clearly have the set inclusions

$$\mathcal{R}(AA^*) \subset \mathcal{R}(A) \subset \mathcal{H}_2.$$

Therefore, $\mathcal{R}(AA^*)$ is a linear submanifold of $\mathcal{M}(A)$. We show that, with respect to the topology of $\mathcal{M}(A)$, in a sense $\mathcal{R}(AA^*)$ is a large set.

Lemma 16.15 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Then the linear manifold $\mathcal{R}(AA^*)$ is dense in $\mathcal{M}(A)$.*

Proof To show that $\mathcal{R}(AA^*)$ is dense in $\mathcal{M}(A)$, we use a standard Hilbert space technique. If 0 is the only vector in $\mathcal{M}(A)$ that is orthogonal to $\mathcal{R}(AA^*)$, then this linear manifold is dense in $\mathcal{M}(A)$. Thus let $w \in \mathcal{M}(A)$ be such that

$$\langle w, z \rangle_{\mathcal{M}(A)} = 0$$

for all $z \in \mathcal{R}(AA^*)$. We proceed to show that $w = 0$. By definition, $w = Ax$, for some $x \in \mathcal{H}_1$, and $z = AA^*y$, where y runs through $\mathcal{H}_2$. Remember that $A^*y \perp \ker A$. Hence, by (16.6),

$$\begin{aligned} 0 &= \langle w, z \rangle_{\mathcal{M}(A)} \\ &= \langle Ax, AA^*y \rangle_{\mathcal{M}(A)} \\ &= \langle x, A^*y \rangle_{\mathcal{H}_1} \\ &= \langle Ax, y \rangle_{\mathcal{H}_2} \\ &= \langle w, y \rangle_{\mathcal{H}_2} \end{aligned}$$

for all $y \in \mathcal{H}_2$. Therefore, $w = 0$. □

We now write Lemma 16.15 for $\mathcal{H}(A)$ spaces. This is the version that we mostly need.

Corollary 16.16 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a Hilbert space contraction. Then the linear manifold $\mathcal{R}(I - AA^*)$ is dense in $\mathcal{H}(A)$. Moreover, for each $z \in \mathcal{H}_2$ and $w \in \mathcal{H}(A)$,*

$$\|(I - AA^*)z\|^2_{\mathcal{H}(A)} = \|(I - AA^*)^{1/2}z\|^2_{\mathcal{H}_2} = \|z\|^2_{\mathcal{H}_2} - \|A^*z\|^2_{\mathcal{H}_1}$$

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and

$$\langle w,\ (I - AA^*)z\rangle_{\mathcal{H}(A)} = \langle w, z\rangle_{\mathcal{H}_2}.$$

Proof For the first part, it is enough to apply Lemma 16.15 to the self-adjoint operator $(I - AA^*)^{1/2} \in \mathcal{L}(\mathcal{H}_2)$ and see that $\mathcal{R}(I - AA^*)$ is dense in $\mathcal{H}(A)$.

To prove the first identity, note that $(I - AA^*)^{1/2}z \perp \ker(I - AA^*)^{1/2}$. Thus, by (16.5),

$$\begin{aligned}\|(I - AA^*)z\|^2_{\mathcal{H}(A)} &= \|(I - AA^*)^{1/2}z\|^2_{\mathcal{H}_2}\\ &= \langle (I - AA^*)^{1/2}z,\ (I - AA^*)^{1/2}z\rangle_{\mathcal{H}_2}\\ &= \langle (I - AA^*)z,\ z\rangle_{\mathcal{H}_2}\\ &= \|z\|^2_{H_2} - \|A^*z\|^2_{\mathcal{H}_1}.\end{aligned}$$

For the second relation, we write $w = (I - AA^*)^{1/2}w'$, where $w' \perp \ker (I - AA^*)^{1/2}$. Hence, by (16.6),

$$\begin{aligned}\langle w,\ (I - AA^*)z\rangle_{\mathcal{H}(A)} &= \langle (I - AA^*)^{1/2}w',\ (I - AA^*)z\rangle_{\mathcal{H}(A)}\\ &= \langle w',\ (I - AA^*)^{1/2}z\rangle_{\mathcal{H}_2}\\ &= \langle (I - AA^*)^{1/2}w',\ z\rangle_{\mathcal{H}_2}\\ &= \langle w, z\rangle_{\mathcal{H}_2}.\end{aligned}$$

This completes the proof. □

Given an element $y \in \mathcal{H}$, we sometimes need to know if it belongs to a given complementary space $\mathcal{H}(A)$ or not. The following result is a characterization of this type.

Theorem 16.17 *Let A be a contraction on a Hilbert space H and let $y \in \mathcal{H}$. Then $y \in \mathcal{H}(A)$ if and only if*

$$\sup_{0 \le r < 1} \|(I - r^2AA^*)^{-1/2}y\|_{\mathcal{H}} < +\infty.$$

Moreover, if $y = (I - AA^*)^{1/2}x$ with $x \perp \ker(I - AA^*)^{1/2}$, then

$$\lim_{r\to 1} \|(I - r^2AA^*)^{-1/2}y - x\|_{\mathcal{H}} = 0,$$

and if $y_1, y_2 \in \mathcal{H}(A)$, then

$$\langle y_1, y_2\rangle_{\mathcal{H}(A)} = \lim_{r\to 1}\langle (I - r^2AA^*)^{-1/2}y_1,\ (I - r^2AA^*)^{-1/2}y_2\rangle_{\mathcal{H}}. \quad (16.18)$$

In particular, for each $y \in \mathcal{H}(A)$,

$$\|y\|_{\mathcal{H}(A)} = \lim_{r\to 1} \|(I - r^2AA^*)^{-1/2}y\|_{\mathcal{H}}.$$

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Proof Put $B_r = I - r^2 AA^*$, $0 \le r < 1$, and $B = I - AA^*$. Then the conditions of Theorem 7.10 are clearly satisfied and thus we deduce that y belongs to the range of $B^{1/2}$, which is $\mathcal{H}(A)$, if and only if

$$\sup_{0 \le r < 1} \|B_r^{-1/2} y\|_{\mathcal{H}} = \sup_{0 \le r < 1} \|(I - r^2 AA^*)^{-1/2} y\|_{\mathcal{H}} < +\infty.$$

The first equality was also established in Theorem 7.10. Now, if $y_i \in \mathcal{H}(A)$, $i = 1, 2$, then $y_i = (I - AA^*)^{1/2} x_i$, with $x_i \perp \ker(I - AA^*)^{1/2}$, and, by the first equality, $(I - r^2 AA^*)^{-1/2} y_i$ converges to x_i in H, as $r \longrightarrow 1$. Hence,

$$\langle y_1, y_2 \rangle_{\mathcal{H}(A)} = \langle x_1, x_2 \rangle_{\mathcal{H}} = \lim_{r \to 1} \langle (I - r^2 AA^*)^{-1/2} y_1,\ (I - r^2 AA^*)^{-1/2} y_2 \rangle_{\mathcal{H}}.$$

□

Exercises

Exercise 16.4.1 Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Show that

$$\|w\|_{\mathcal{M}(A)} \le \|A\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)} \|w\|_{\mathcal{M}(AA^*)} \qquad (w \in \mathcal{M}(AA^*)).$$

Hint: Write $w = AA^* x$, where $x \perp \ker AA^*$.
Remark: This means that $\mathcal{M}(AA^*)$ is boundedly contained in $\mathcal{M}(A)$. This fact also follows from Lemma 16.6.

Exercise 16.4.2 Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a Hilbert space contraction. Show that

$$\|w\|_{\mathcal{H}(A)} \le \|w\|_{\mathcal{M}(I - AA^*)} \qquad (w \in \mathcal{M}(I - AA^*)).$$

Hint: Apply Exercise 16.4.1 to the operator $(I - AA^*)^{1/2}$.
Remark: This means that $\mathcal{M}(I - AA^*)$ is contractively contained in $\mathcal{H}(A)$.

16.5 The relation between $\mathcal{H}(A)$ and $\mathcal{H}(A^*)$

In this section we explore the relation between $\mathcal{H}(A)$ and $\mathcal{H}(A^*)$. In particular, we obtain a frequently used identity that exhibits the bridge between the inner products in $\mathcal{H}(A)$ and $\mathcal{H}(A^*)$.

Theorem 16.18 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a contraction, and let $w \in \mathcal{H}_2$. Then $w \in \mathcal{H}(A)$ if and only if $A^* w \in \mathcal{H}(A^*)$. Moreover, if $w_1, w_2 \in \mathcal{H}(A)$, then*

$$\langle w_1, w_2 \rangle_{\mathcal{H}(A)} = \langle A^* w_1, A^* w_2 \rangle_{\mathcal{H}(A^*)} + \langle w_1, w_2 \rangle_{\mathcal{H}_2}.$$

In particular, for each $w \in \mathcal{H}(A)$,

$$\|w\|^2_{\mathcal{H}(A)} = \|A^* w\|^2_{\mathcal{H}(A^*)} + \|w\|^2_{\mathcal{H}_2}.$$

Proof We recall the intertwining relation (7.12):

$$A^*(I - AA^*)^{1/2} = (I - A^*A)^{1/2}A^*.$$

Hence, the set inclusion $A^*\mathcal{H}(A) \subset \mathcal{H}(A^*)$ follows immediately. This is equivalent to saying that

$$w \in \mathcal{H}(A) \quad \Longrightarrow \quad A^*w \in \mathcal{H}(A^*).$$

To prove the inverse, let $w \in \mathcal{H}_2$ be such that $A^*w \in \mathcal{H}(A^*)$. Thus, by definition, there is $x \in \mathcal{H}_1$ such that

$$A^*w = (I - A^*A)^{1/2}x.$$

By the intertwining relation, the trivial identity

$$w = (I - AA^*)w + AA^*w = (I - AA^*)w + A(I - A^*A)^{1/2}x$$

can be rewritten as

$$w = (I - AA^*)^{1/2}[(I - AA^*)^{1/2}w + Ax]. \tag{16.19}$$

Hence, $w \in \mathcal{H}(A)$. In other words, we also have

$$A^*w \in \mathcal{H}(A^*) \quad \Longrightarrow \quad w \in \mathcal{H}(A).$$

To prove the identity for the inner products, let $w_1, w_2 \in \mathcal{H}(A)$. Hence, there are $y_1, y_2 \in \mathcal{H}_2$ such that

$$w_k = (I - AA^*)^{1/2}y_k \qquad (k = 1, 2).$$

Without loss of generality, we assume that $y_k \perp \ker(I - AA^*)$. This assumption has two consequences: first,

$$\langle w_1, w_2\rangle_{\mathcal{H}(A)} = \langle y_1, y_2\rangle_{\mathcal{H}_2},$$

and second, $A^*y_k \perp \ker(I - A^*A)$ (see Exercise 1.8.3). But, by the intertwining relation, we have

$$A^*w_k = (I - A^*A)^{1/2}A^*y_k \qquad (k = 1, 2).$$

Therefore, we also have

$$\langle A^*w_1, A^*w_2\rangle_{\mathcal{H}(A^*)} = \langle A^*y_1, A^*y_2\rangle_{\mathcal{H}_1}.$$

Now, a direct calculation shows that

$$\begin{aligned}
\langle w_1, w_2\rangle_{H_2} &= \langle (I - AA^*)^{1/2}y_1,\ (I - AA^*)^{1/2}y_2\rangle_{\mathcal{H}_2}\\
&= \langle (I - AA^*)y_1,\ y_2\rangle_{\mathcal{H}_2}\\
&= \langle y_1, y_2\rangle_{\mathcal{H}_2} - \langle A^*y_1, A^*y_2\rangle_{\mathcal{H}_1}\\
&= \langle w_1, w_2\rangle_{\mathcal{H}(A)} - \langle A^*w_1, A^*w_2\rangle_{\mathcal{H}(A^*)}.
\end{aligned}$$

This completes the proof. □

Applying Theorem 16.18 to the operator A^* gives the following result.

Corollary 16.19 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a contraction. Then a vector $x \in \mathcal{H}_1$ belongs to $\mathcal{H}(A^*)$ if and only if $Ax \in \mathcal{H}(A)$. Moreover, if $x_1, x_2 \in \mathcal{H}(A^*)$, then*

$$\langle x_1, x_2 \rangle_{\mathcal{H}(A^*)} = \langle Ax_1, Ax_2 \rangle_{\mathcal{H}(A)} + \langle x_1, x_2 \rangle_{\mathcal{H}_1}.$$

In particular, for each $x \in \mathcal{H}(A^*)$,

$$\|x\|^2_{\mathcal{H}(A^*)} = \|Ax\|^2_{\mathcal{H}(A)} + \|x\|^2_{\mathcal{H}_1}.$$

16.6 The overlapping space $\mathcal{M}(A) \cap \mathcal{H}(A)$

As we mentioned in Section 16.4, the intersection $\mathcal{M}(A) \cap \mathcal{H}(A)$ is called the overlapping space. We first show that the overlapping space is precisely the image of $\mathcal{H}(A^*)$ under the operator A. Then we exploit this observation to characterize the trivial overlapping space.

Lemma 16.20 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a Hilbert space contraction. Then we have the set identity*

$$\mathcal{M}(A) \cap \mathcal{H}(A) = A\mathcal{H}(A^*).$$

Moreover, the operator A acts as a contraction from $\mathcal{H}(A^*)$ into $\mathcal{H}(A)$.

Proof By Corollary 16.19, $A\mathcal{H}(A^*) \subset \mathcal{H}(A)$. Moreover, by definition, we have $A\mathcal{H}(A^*) \subset \mathcal{M}(A)$. Hence, $A\mathcal{H}(A^*) \subset \mathcal{M}(A) \cap \mathcal{H}(A)$. To prove the other inclusion, let $w \in \mathcal{M}(A) \cap \mathcal{H}(A)$. Therefore, $w = Ax$, for some $x \in \mathcal{H}_1$, and $Ax \in \mathcal{H}(A)$. Thus, again by Corollary 16.19, we necessarily have $x \in \mathcal{H}(A^*)$, and this means $w = Ax \in A\mathcal{H}(A^*)$. If we apply Theorem 16.18 to A^*, then, for each $w \in \mathcal{H}(A^*)$, we have

$$\|w\|^2_{\mathcal{H}(A^*)} = \|Aw\|^2_{\mathcal{H}(A)} + \|w\|^2_{\mathcal{H}_1} \geq \|Aw\|^2_{\mathcal{H}(A)},$$

which exactly means that A acts as a contraction from $\mathcal{H}(A^*)$ into $\mathcal{H}(A)$. □

We naturally wonder when the overlapping space is trivial, i.e. $\mathcal{M}(A) \cap \mathcal{H}(A) = \{0\}$. We are now able to fully characterize this situation.

Theorem 16.21 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a Hilbert space contraction. Then the following are equivalent:*

(i) *A is a partial isometry;*
(ii) *$\mathcal{M}(A)$ is a closed subspace of $\mathcal{H}$ and inherits its Hilbert space structure;*
(iii) *$\mathcal{H}(A)$ is a closed subspace of $\mathcal{H}$ and inherits its Hilbert space structure;*
(iv) *$\mathcal{M}(A)$ and $\mathcal{H}(A)$ are orthogonal complements of each other;*

(v) $\mathcal{M}(A) \cap \mathcal{H}(A) = \{0\}$;
(vi) $\mathcal{H}(A^*) \subset \ker A$.

Moreover, under the preceding equivalent conditions, we have the orthogonal decomposition

$$\mathcal{H} = \mathcal{M}(A) + \mathcal{H}(A).$$

Proof The equivalence (i) $\Longleftrightarrow$ (ii) was proved in Corollary 16.9.

The equivalence (i) $\Longleftrightarrow$ (iii) was proved in Lemma 16.14.

(i) $\Longrightarrow$ (iv) If A is a partial isometry, then $\mathcal{M}(A)$ and $\mathcal{H}(A)$ are the range of complementary orthogonal projections AA^* and $I - AA^*$. Hence $\mathcal{M}(A)$ and $\mathcal{H}(A)$ are orthogonal complements of each other.

(iv) $\Longrightarrow$ (v) This is trivial.

(v) $\Longrightarrow$ (vi) This is an immediate consequence of Lemma 16.20.

(vi) $\Longrightarrow$ (i) By assumption $A(I - A^*A)^{1/2} = 0$. If so, then certainly we have $A(I - A^*A) = 0$. Hence, $A = AA^*A$, which implies $(AA^*)^2 = AA^*$. In other words, AA^* is an orthogonal projection. Therefore, by Theorem 7.22, A is a partial isometry.

The orthogonal decomposition $\mathcal{H} = \mathcal{M}(A) + \mathcal{H}(A)$ is a consequence of (iv). □

16.7 The algebraic sum of $\mathcal{M}(A_1)$ and $\mathcal{M}(A_2)$

Given two operators $A_1 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $A_2 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$, we can form the Hilbert spaces $\mathcal{M}(A_1)$ and $\mathcal{M}(A_2)$ inside $\mathcal{H}$. Looking at them as linear submanifolds of $\mathcal{H}$, we can form the algebraic sum

$$\mathcal{M}(A_1) + \mathcal{M}(A_2) = \{w_1 + w_2 : w_1 \in \mathcal{M}(A_1) \text{ and } w_2 \in \mathcal{M}(A_2)\}.$$

We may naturally ask if this sum can be regarded as a new Hilbert space $\mathcal{M}(A)$, for a suitable operator A. The affirmative answer is explained in more detail in the following result.

Theorem 16.22 *Let $A_1 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $A_2 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$, and let $A = [A_1 A_2] \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2,\ \mathcal{H})$ be defined by*

$$A(x_1, x_2) = A_1 x_1 + A_2 x_2 \qquad (x_1 \in \mathcal{H}_1,\ x_2 \in \mathcal{H}_2).$$

Then the following hold.

(i) $\mathcal{M}(A)$ *decomposes as*

$$\mathcal{M}(A) = \mathcal{M}(A_1) + \mathcal{M}(A_2).$$

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(ii) *For each representation*

$$w = w_1 + w_2,$$

where $w_i \in \mathcal{M}(A_i)$, $i = 1, 2$, and $w \in \mathcal{M}(A)$, we have

$$\|w\|^2_{\mathcal{M}(A)} \le \|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)}.$$

(iii) *For each $w \in \mathcal{M}(A)$, there is a unique pair of points $w_1 \in \mathcal{M}(A_1)$ and $w_2 \in \mathcal{M}(A_2)$ such that $w = w_1 + w_2$ and*

$$\|w\|^2_{\mathcal{M}(A)} = \|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)}.$$

Proof (i) By definition, for each $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$, we have

$$A(x_1 \oplus x_2) = A_1 x_1 + A_2 x_2.$$

This immediately implies the set identity $\mathcal{M}(A) = \mathcal{M}(A_1) + \mathcal{M}(A_2)$.

(ii) If $w = w_1 + w_2$ with $w_i \in \mathcal{M}(A_i)$, $i = 1, 2$, then we can write $w_i = A_i x_i$ with $x_i \perp \ker A_i$. Note that a given $w \in \mathcal{H}(A)$ is not necessarily written in a unique way in the form $w = w_1 + w_2$, and in fact it may have infinitely many such representations. Then we can write

$$w = w_1 + w_2 = A_1 x_1 + A_2 x_2 = A(x_1 \oplus x_2).$$

Therefore, by Corollary 16.8(ii) and (16.5),

$$\begin{aligned}\|w\|^2_{\mathcal{M}(A)} &= \|A(x_1 \oplus x_2)\|^2_{\mathcal{M}(A)}\\ &\le \|x_1 \oplus x_2\|^2_{\mathcal{H}_1 \oplus \mathcal{H}_2}\\ &= \|x_1\|^2_{\mathcal{H}_1} + \|x_2\|^2_{\mathcal{H}_2}\\ &= \|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)}.\end{aligned}$$

(iii) Among all possible representations

$$w = w_1 + w_2 = A(x_1 \oplus x_2),$$

if we choose x_1 and x_2 such that $x_1 \oplus x_2 \perp \ker A$, then, in the light of (1.45), we certainly have $x_i \perp \ker A_i$. Hence, in the last paragraph of (ii) equality holds everywhere. Thus, this choice of x_1 and x_2 gives at least a suitable pair w_1 and w_2 for which $\|w\|^2_{\mathcal{M}(A)} = \|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)}$ holds. But, to have this equality, we need $x_1 \oplus x_2 \perp \ker A$ and this choice of $x_1 \oplus x_2$ is unique. Hence, in return, w_1 and w_2 are also unique. □

We now give an explicit example to reveal the contents of the above result. Let $A_1 \in \mathcal{L}(\mathbb{C}^3)$ be defined by

$$A_1(1,0,0) = (0,0,0), \qquad A_1(0,1,0) = (0,1,0), \qquad A_1(0,0,1) = (0,0,1),$$

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and put $A_2 = -A_1$ and $A = A_1 \oplus A_2 \in \mathcal{L}(\mathbb{C}^3 \oplus \mathbb{C}^3 \longrightarrow \mathbb{C}^3)$. Then

$$w = A((\alpha, \beta, \gamma) \oplus (\alpha', \beta', \gamma')) = (0, \beta - \beta', \gamma - \gamma'). \tag{16.20}$$

There are infinitely many ways to write $w = w_1 + w_2$ with $w_i \in \mathcal{M}(A_i)$. For example, the above equality suggests that

$$w_1 = A_1((\alpha, \beta, \gamma)) = A_1((0, \beta, \gamma)) = (0, \beta, \gamma) \tag{16.21}$$

and

$$w_2 = A_2((\alpha', \beta', \gamma')) = A_2((0, \beta', \gamma')) = (0, -\beta', -\gamma'). \tag{16.22}$$

But, we may equally take $w_2 = 0$ and

$$w_1 = A_1((0, \beta - \beta', \gamma - \gamma')) = (0, \beta - \beta', \gamma - \gamma'). \tag{16.23}$$

We naturally seek the unique representation that is promised in Theorem 16.23. To do so, first note that

$$\ker A_1 = \ker A_2 = \{(\alpha, 0, 0) : \alpha \in \mathbb{C}\},$$

and

$$\ker A = \{(\alpha, \beta, \gamma) \oplus (\alpha', \beta, \gamma) : \alpha, \alpha', \beta, \gamma \in \mathbb{C}\},$$

which imply that

$$(\ker A)^\perp = \{(0, \beta, \gamma) \oplus (0, -\beta, -\gamma) : \beta, \gamma \in \mathbb{C}\}. \tag{16.24}$$

Observe that

$$\ker A_1 \oplus \ker A_2 \subsetneqq \ker A.$$

This proper inclusion has some important consequences.

According to (16.24), the *good* representation for $w = (0, \beta - \beta', \gamma - \gamma')$ is

$$w = A\left(\left(0, \frac{\beta - \beta'}{2}, \frac{\gamma - \gamma'}{2}\right) \oplus \left(0, -\frac{\beta - \beta'}{2}, -\frac{\gamma - \gamma'}{2}\right)\right).$$

Note that

$$\left(0, \frac{\beta - \beta'}{2}, \frac{\gamma - \gamma'}{2}\right) \oplus \left(0, -\frac{\beta - \beta'}{2}, -\frac{\gamma - \gamma'}{2}\right) \perp \ker A.$$

For this unique choice, we have the unique decomposition $w = w_1 + w_2$, where

$$w_1 = A_1\left(0, \frac{\beta - \beta'}{2}, \frac{\gamma - \gamma'}{2}\right) = \left(0, \frac{\beta - \beta'}{2}, \frac{\gamma - \gamma'}{2}\right)$$

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and

$$w_2 = A_2\left(0, -\frac{\beta-\beta'}{2}, -\frac{\gamma-\gamma'}{2}\right) = \left(0, \frac{\beta-\beta'}{2}, \frac{\gamma-\gamma'}{2}\right).$$

Since, moreover,

$$\left(0, \frac{\beta-\beta'}{2}, \frac{\gamma-\gamma'}{2}\right) \perp \ker A_1$$

and

$$\left(0, -\frac{\beta-\beta'}{2}, -\frac{\gamma-\gamma'}{2}\right) \perp \ker A_2,$$

we deduce that

$$\begin{aligned}\|w\|^2_{\mathcal{M}(A)} &= \left\|\left(0, \frac{\beta-\beta'}{2}, \frac{\gamma-\gamma'}{2}\right) \oplus \left(0, -\frac{\beta-\beta'}{2}, -\frac{\gamma-\gamma'}{2}\right)\right\|^2 \\ &= \frac{|\beta-\beta'|^2+|\gamma-\gamma'|^2}{2},\end{aligned}$$

$$\|w_1\|^2_{\mathcal{M}(A_1)} = \left\|\left(0, \frac{\beta-\beta'}{2}, \frac{\gamma-\gamma'}{2}\right)\right\|^2 = \frac{|\beta-\beta'|^2+|\gamma-\gamma'|^2}{4}$$

and

$$\|w_2\|^2_{\mathcal{M}(A_2)} = \left\|\left(0, -\frac{\beta-\beta'}{2}, -\frac{\gamma-\gamma'}{2}\right)\right\|^2 = \frac{|\beta-\beta'|^2+|\gamma-\gamma'|^2}{4}.$$

Hence, it is no wonder that, for the *good* representation, we have the norm identity

$$\|w\|^2_{\mathcal{M}(A)} = \|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)}.$$

For any other representation $w = w_1 + w_2$, with $w_i \in \mathcal{M}(A_i)$, we would certainly have

$$\|w\|^2_{\mathcal{M}(A)} < \|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)}.$$

For example, in the representation (16.21) and (16.22), we have

$$\|w_1\|^2_{\mathcal{M}(A_1)} = \|(0, \beta, \gamma)\|^2 = |\beta|^2 + |\gamma|^2$$

and

$$\|w_2\|^2_{\mathcal{M}(A_2)} = \|(0, \beta', \gamma')\|^2 = |\beta'|^2 + |\gamma'|^2.$$

In this case, the inequality

$$|\beta|^2 + |\gamma|^2 + |\beta'|^2 + |\gamma'|^2 > \frac{|\beta-\beta'|^2+|\gamma-\gamma'|^2}{2}$$

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is equivalent to

$$\|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)} > \|w\|^2_{\mathcal{M}(A)}.$$

In a similar manner, the representation (16.23) implies $\|w_2\|_{\mathcal{M}(A_2)} = 0$ and

$$\|w_1\|^2_{\mathcal{M}(A_1)} = \|(0, \beta - \beta', \gamma - \gamma')\|^2 = |\beta - \beta'|^2 + |\gamma - \gamma'|^2.$$

Hence, again, we are faced with the trivial inequality

$$\begin{aligned}\|w_1\|^2_{\mathcal{M}(A_1)} + \|w_2\|^2_{\mathcal{M}(A_2)} &= |\beta - \beta'|^2 + |\gamma - \gamma'|^2 \\ &> \frac{|\beta - \beta'|^2 + |\gamma - \gamma'|^2}{2} = \|w\|^2_{\mathcal{M}(A)}.\end{aligned}$$

A slightly different version of the above algebraic decomposition will be studied in Theorem 16.23.

16.8 A decomposition of $\mathcal{H}(A)$

If an operator decomposes as $A = A_2A_1$, we naturally ask about the relation between $\mathcal{H}(A)$, on the one hand, and $\mathcal{H}(A_1)$ and $\mathcal{H}(A_2)$, on the other. In this section we address this important question.

Theorem 16.23 *Let $A_1 \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_1)$ and $A_2 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be contractions, and let $A = A_2A_1$. Then the following hold.*

(i) *$\mathcal{H}(A)$ decomposes as*

$$\mathcal{H}(A) = A_2\mathcal{H}(A_1) + \mathcal{H}(A_2).$$

(ii) *For any representation*

$$w = A_2w_1 + w_2,$$

where $w_i \in \mathcal{H}(A_i)$, $i = 1, 2$, and $w \in \mathcal{H}(A)$, we have

$$\|w\|^2_{\mathcal{H}(A)} \le \|w_1\|^2_{\mathcal{H}(A_1)} + \|w_2\|^2_{\mathcal{H}(A_2)}.$$

(iii) *For each $w \in \mathcal{H}(A)$ there is a unique pair of points $w_1 \in \mathcal{H}(A_1)$ and $w_2 \in \mathcal{H}(A_2)$ such that $w = A_2w_1 + w_2$ and*

$$\|w\|^2_{\mathcal{H}(A)} = \|w_1\|^2_{\mathcal{H}(A_1)} + \|w_2\|^2_{\mathcal{H}(A_2)}.$$

(iv) *$\mathcal{H}(A_2)$ is contractively contained in $\mathcal{H}(A)$.*

(v) *The operator A_2 acts as a contraction from $\mathcal{H}(A_1)$ into $\mathcal{H}(A)$.*

Proof The proof has the same spirit as the proof of Theorem 16.22. In fact, we can even appeal to this result, and we give a shorter proof below. However, here we provide a complete and independent proof.

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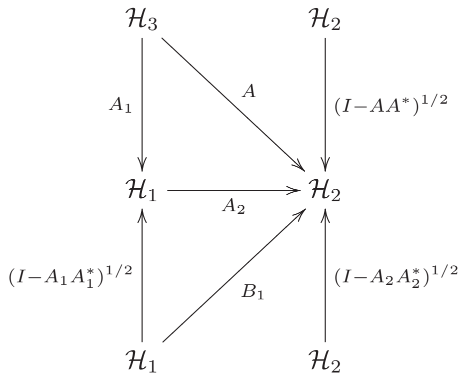
(i) Consider the operators $B_1 = A_2(I - A_1A_1^*)^{1/2} \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $B_2 = (I - A_2A_2^*)^{1/2} \in \mathcal{L}(\mathcal{H}_2)$:



(16.25)

Then, by (1.43), we can write

$$\begin{aligned} I - AA^* &= I - (A_2A_1)(A_2A_1)^* \\ &= A_2(I - A_1A_1^*)A_2^* + (I - A_2A_2^*) \\ &= B_1B_1^* + B_2B_2^* \\ &= BB^*, \end{aligned} \tag{16.26}$$

where $B = [B_1\ B_2] \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2,\ \mathcal{H}_2)$. Therefore, by Corollary 16.8(ii), and that by definition $\mathcal{H}(A) = \mathcal{M}((I - AA^*)^{1/2})$, we have

$$\mathcal{H}(A) \eqcirc \mathcal{M}(B).$$

Hence, at least, we have the set identities

$$\begin{aligned} \mathcal{H}(A) &= \mathcal{M}(B) \\ &= \mathcal{M}(B_1) + \mathcal{M}(B_2) \\ &= A_2\mathcal{M}((I - A_1A_1^*)^{1/2}) + \mathcal{M}((I - A_2A_2^*)^{1/2}) \\ &= A_2\mathcal{H}(A_1) + \mathcal{H}(A_2). \end{aligned}$$

(ii) If $w = A_2w_1 + w_2$ with $w_i \in \mathcal{H}(A_i)$, $i = 1, 2$, then we can write $w_i = (I - A_iA_i^*)^{1/2}x_i$ with $x_i \perp \ker(I - A_iA_i^*)$. Then we have

$$\begin{aligned} w &= A_2w_1 + w_2 \\ &= A_2(I - A_1A_1^*)^{1/2}x_1 + (I - A_2A_2^*)^{1/2}x_2 \\ &= B_1x_1 + B_2x_2 \\ &= B(x_1 \oplus x_2). \end{aligned}$$

Therefore, by Corollary 16.8(ii) and (16.5),

$$\begin{aligned} \|w\|^2_{\mathcal{H}(A)} &= \|w\|^2_{\mathcal{M}(B)} \\ &= \|B(x_1 \oplus x_2)\|^2_{\mathcal{M}(B)} \end{aligned}$$

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$$\begin{aligned} &\leq \|x_1 \oplus x_2\|^2_{\mathcal{H}_1 \oplus \mathcal{H}_2} \\ &= \|x_1\|^2_{\mathcal{H}_1} + \|x_2\|^2_{\mathcal{H}_2} \\ &= \|w_1\|^2_{\mathcal{H}(A_1)} + \|w_2\|^2_{\mathcal{H}(A_2)}. \end{aligned}$$

(iii) Among all possible representations

$$w = A_2 w_1 + w_2 = B(x_1 \oplus x_2),$$

if we choose x_1 and x_2 such that $x_1 \oplus x_2 \perp \ker B$, then, in the light of (1.45), we certainly have $x_i \perp \ker(I - A_i A_i^*)$. Hence, in the last paragraph of (ii) equality holds everywhere. Thus, this choice of x_1 and x_2 gives at least a suitable pair w_1 and w_2 for which $\|w\|^2_{\mathcal{H}(A)} = \|w_1\|^2_{\mathcal{H}(A_1)} + \|w_2\|^2_{\mathcal{H}(A_2)}$ holds. But, to have this equality, we need $x_1 \oplus x_2 \perp \ker B$, and this choice of $x_1 \oplus x_2$ is unique. Hence, in return, w_1 and w_2 are unique too.

(iv) By (i), $\mathcal{H}(A_2) \subset \mathcal{H}(A)$. For each $w_2 \in \mathcal{H}(A_2)$, consider the representation $w = A_2 0 + w_2$. Hence, by (ii),

$$\|w_2\|_{\mathcal{H}(A)} = \|w\|_{\mathcal{H}(A)} \leq \|w_2\|_{\mathcal{H}(A_2)}.$$

This means that $\mathcal{H}(A_2)$ is contractively contained in $\mathcal{H}(A)$.

(v) By (i), $A_2\mathcal{H}(A_1) \subset \mathcal{H}(A)$. For each $w_1 \in \mathcal{H}(A_1)$, consider the representation $w = A_2 w_1 + 0$. Hence, by (ii),

$$\|A_2 w_1\|_{\mathcal{H}(A)} = \|w\|_{\mathcal{H}(A)} \leq \|w_1\|_{\mathcal{H}(A_1)}.$$

This means that A_2 acts as a contraction from $\mathcal{H}(A_1)$ into $\mathcal{H}(A)$. □

In part (iii) of the preceding theorem, the existence of a unique pair of w_1 and w_2 was established. However, we did not offer a procedure or formula to find them. We are able to do this in the following special case. In Corollary 16.16, we saw that $\mathcal{M}(I - AA^*)$ is a dense submanifold of $\mathcal{H}(A)$. Let $w \in \mathcal{M}(I - AA^*)$. Hence, there is $y \in H_2$ such that

$$w = (I - AA^*)y. \tag{16.27}$$

Let

$$x_1 = B_1^* y \quad \text{and} \quad x_2 = B_2^* y,$$

where B_1 and B_2 are as in the proof of Theorem 16.23. Then, by Theorem 1.30,

$$x_1 \oplus x_2 = B_1^* y \oplus B_2^* y = B^* y \in \mathcal{R}(B^*) \subset (\ker B)^\perp.$$

Moreover, by (1.43) and (16.26),

$$\begin{aligned} B(x_1 \oplus x_2) = B_1 x_1 + B_2 x_2 &= (B_1 B_1^* + B_2 B_2^*) y \\ &= BB^* y = (I - AA^*) y = w. \end{aligned}$$

Therefore, the unique pair for an element of the form $w = (I - AA^*)y$ is given by

$$w_1 = (I - A_1 A_1^*)^{1/2} x_1 = (I - A_1 A_1^*) A_2^* y \tag{16.28}$$

and

$$w_2 = (I - A_2 A_2^*)^{1/2} x_2 = (I - A_2 A_2^*) y. \tag{16.29}$$

The decomposition $\mathcal{H}(A) = A_2 \mathcal{H}(A_1) + \mathcal{H}(A_2)$ is an algebraic direct sum of $A_2 \mathcal{H}(A_1)$ and $\mathcal{H}(A_2)$ provided that

$$A_2 \mathcal{H}(A_1) \cap \mathcal{H}(A_2) = \{0\}.$$

However, the relations

$$\mathcal{H}(A) = \mathcal{M}(B), \quad \mathcal{M}(B_1) = A_2 \mathcal{H}(A_1), \quad \mathcal{M}(B_2) = \mathcal{H}(A_2)$$

and (1.47) show that the decomposition $\mathcal{H}(A) = A_2 \mathcal{H}(A_1) + \mathcal{H}(A_2)$ is an algebraic direct sum of $A_2 \mathcal{H}(A_1)$ and $\mathcal{H}(A_2)$ if and only if

$$\ker B = \ker B_1 \oplus \ker B_2.$$

Assuming the decomposition is an algebraic direct sum, if

$$w = A_2 w_1 + w_2 = A_2 w_1' + w_2'$$

then we must have $A_2 w_1 = A_2 w_1'$ and $w_2 = w_2'$. Hence, the choice of w_2 in the representation $w = A_2 w_1 + w_2$ is unique. However, there is still some freedom for w_1.

Corollary 16.24 *Let $A_1 \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_1)$ and $A_2 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be contractions, and let $A = A_2 A_1$. Suppose that*

$$A_2 \mathcal{H}(A_1) \cap \mathcal{H}(A_2) = \{0\}.$$

Then the following hold.

(i) *$\mathcal{H}(A_2)$ is contained isometrically in $\mathcal{H}(A)$.*
(ii) *The operator A_2 acts as a partial isometry from $\mathcal{H}(A_1)$ into $\mathcal{H}(A)$.*
(iii) *Relative to the Hilbert space structure of $\mathcal{H}(A)$, the subspaces $A_2 \mathcal{H}(A_1)$ and $\mathcal{H}(A_2)$ are complementary orthogonal subspaces of $\mathcal{H}(A)$. In other words, the decomposition $\mathcal{H}(A) = A_2 \mathcal{H}(A_1) + \mathcal{H}(A_2)$ is in fact an orthogonal direct sum.*

Proof We use the notation in the proof of Theorem 16.23. The assumption $A_2 \mathcal{H}(A_1) \cap \mathcal{H}(A_2) = \{0\}$ means that the decomposition

$$\mathcal{H}(A) = A_2 \mathcal{H}(A_1) + \mathcal{H}(A_2)$$

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is an algebraic direct sum. Based on this notation, this is equivalent to saying that

$$\mathcal{R}(B) = \mathcal{R}(B_1) + \mathcal{R}(B_2)$$

is an algebraic direct sum. Therefore, by (1.47),

$$P_{(\ker B)^\perp}(x_1 \oplus x_2) = P_{(\ker B_1)^\perp} x_1 \oplus P_{(\ker B_2)^\perp} x_2 \tag{16.30}$$

for all $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$. The identity is the main ingredient in our proof.

(i) Let $w_2 \in \mathcal{H}(A_2)$. Hence $w_2 = B_2 x_2 = (I - A_2 A_2^*)^{1/2} x_2$ with $x_2 \in \mathcal{H}_2$. Therefore, by (16.30),

$$\begin{aligned}
\|w_2\|_{\mathcal{H}(A)} &= \|w_2\|_{\mathcal{M}(B)} \\
&= \|B_2 x_2\|_{\mathcal{M}(B)} \\
&= \|B(0 \oplus x_2)\|_{\mathcal{M}(B)} \\
&= \|P_{(\ker B)^\perp}(0 \oplus x_2)\|_{\mathcal{H}_1 \oplus \mathcal{H}_2} \\
&= \|0 \oplus P_{(\ker B_2)^\perp} x_2\|_{\mathcal{H}_1 \oplus \mathcal{H}_2} \\
&= \|P_{(\ker B_2)^\perp} x_2\|_{\mathcal{H}_2} \\
&= \|B_2 x_2\|_{\mathcal{M}(B_2)} \\
&= \|w_2\|_{\mathcal{H}(A_2)}.
\end{aligned}$$

Hence, $\mathcal{H}(A_2)$ is contained isometrically in $\mathcal{H}(A)$.

(ii) Let us temporarily use the notation

$$\begin{array}{rccc}
\mathbf{A}_2 : & \mathcal{H}(A_1) & \longrightarrow & \mathcal{H}(A) \\
& w_1 & \longmapsto & A_2 w_1
\end{array}$$

for the restriction of A_2 to $\mathcal{H}(A_1)$. By Theorem 16.23(iv), this operator is well defined and $\ker \mathbf{A}_2 = (\ker A_2) \cap \mathcal{H}(A_1)$. To show that $\mathbf{A}_2$ is a partial isometry, we need to verify that, if $w_1 \in \mathcal{H}(A_1)$, with $w_1 \perp \ker \mathbf{A}_2$, with respect to the inner product of $\mathcal{H}(A_1)$, then $\|A_2 w_1\|_{\mathcal{H}(A)} = \|w_1\|_{\mathcal{H}(A_1)}$.

Fix $w_1 \in \mathcal{H}(A_1)$ with $w_1 \perp \ker \mathbf{A}_2$ in $\mathcal{H}(A_1)$. Then there exists $x_1 \in \mathcal{H}_1$, $x_1 \perp \ker(I - A_1 A_1^*)$, such that $w_1 = (I - A_1 A_1^*)^{1/2} x_1$. But, x_1 also satisfies $x_1 \perp \ker B_1$. To verify this fact, let $x \in \ker B_1$, which means that $A_2 (I - A_1 A_1^*)^{1/2} x = 0$. This identity implies that $(I - A_1 A_1^*)^{1/2} x \in \ker A_2 \cap \mathcal{H}(A_1) = \ker \mathbf{A}_2$. Hence, we have

$$\langle x_1, x \rangle_{\mathcal{H}_1} = \langle w_1,\ (I - A_1 A_1^*)^{1/2} x \rangle_{\mathcal{H}(A_1)} = 0.$$

Since this is true for every $x \in \ker B_1$, we obtain that $x_1 \perp \ker B_1$. Therefore, by (16.30),

$$\begin{aligned}
\|A_2 w_1\|_{\mathcal{H}(A)} &= \|A_2 (I - A_1 A_1^*)^{1/2} x_1\|_{\mathcal{H}(A)} \\
&= \|B_1 x_1\|_{\mathcal{M}(B)}
\end{aligned}$$

$$\begin{aligned} &= \|B(x_1 \oplus 0)\|_{\mathcal{M}(B)} \\ &= \|P_{(\ker B_1)^\perp} x_1\|_{\mathcal{H}_1} \\ &= \|x_1\|_{\mathcal{H}_1} \\ &= \|w_1\|_{\mathcal{H}(A_1)}, \end{aligned}$$

which certifies that $\mathbf{A}_2$ is a partial isometry, or, equivalently, A_2 acts as a partial isometry from $\mathcal{H}(A_1)$ into $\mathcal{H}(A)$.

(iii) By parts (i) and (ii), with respect to the structure of $\mathcal{H}(A)$, the sets $\mathcal{H}(A_2)$ and $A_2\mathcal{H}(A_1)$ are closed subspaces of $\mathcal{H}(A)$. Now let $w_i \in \mathcal{H}(A_i)$, $i = 1, 2$. Hence, $w_i = (I - AA^*)^{1/2} x_i$ with $x_i \in \mathcal{H}_i$. Therefore, by (16.30),

$$\begin{aligned} \langle A_2 w_1, w_2 \rangle_{\mathcal{H}(A)} &= \langle A_2 w_1, w_2 \rangle_{\mathcal{M}(B)} \\ &= \langle B_1 x_1, B_2 x_2 \rangle_{\mathcal{M}(B)} \\ &= \langle B(x_1 \oplus 0),\ B(0 \oplus x_2) \rangle_{\mathcal{M}(B)} \\ &= \langle P_{(\ker B)^\perp}(x_1 \oplus 0),\ P_{(\ker B)^\perp}(0 \oplus x_2) \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} \\ &= \langle P_{(\ker B_1)^\perp} x_1 \oplus 0,\ 0 \oplus P_{(\ker B_2)^\perp} x_2 \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} \\ &= \langle P_{(\ker B_1)^\perp} x_1,\ 0 \rangle_{\mathcal{H}_1} + \langle 0,\ P_{(\ker B_2)^\perp} x_2 \rangle_{\mathcal{H}_2} = 0. \end{aligned}$$

Hence, $A_2\,\mathcal{H}(A_1)$ and $\mathcal{H}(A_2)$ are complementary orthogonal subspaces of $\mathcal{H}(A)$. □

To apply Corollary 16.24, we certainly need to verify the condition

$$A_2\mathcal{H}(A_1) \cap \mathcal{H}(A_2) = \{0\}.$$

This is not an easy task. However, in some special cases, it clearly holds. For example, by Lemma 16.20, we have

$$A_2\mathcal{H}(A_1) \cap \mathcal{H}(A_2) \subset \mathcal{M}(A_2) \cap \mathcal{H}(A_2) = A_2\mathcal{H}(A_2^*).$$

Hence, whenever A_2 satisfies $A_2^* A_2 = I$, then we have $\mathcal{H}(A_2^*) = \{0\}$ and thus we conclude that

$$A_2^* A_2 = I \implies A_2\mathcal{H}(A_1) \cap \mathcal{H}(A_2) = \{0\}. \tag{16.31}$$

If $\mathcal{M}$ is a closed subspace of the Hilbert space $\mathcal{H}$, then we have $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. We look at this identity from a different point of view. By Lemma 16.4, we have $\mathcal{M}(P_\mathcal{M}) = \mathcal{M}$ and $\mathcal{H}(P_\mathcal{M}) = \mathcal{M}^\perp$, and thus we can write

$$\mathcal{H} = \mathcal{M}(P_\mathcal{M}) \oplus \mathcal{H}(P_\mathcal{M}).$$

In the following, we generalize this observation.

Theorem 16.25 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ be a Hilbert space contraction. Then*

$$\mathcal{H} = \mathcal{M}(A) + \mathcal{H}(A).$$

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For each decomposition $w = w_1 + w_2$*, with* $w \in \mathcal{H}$*,* $w_1 \in \mathcal{M}(A)$ *and* $w_2 \in \mathcal{H}(A)$*, we have*

$$\|w\|_{\mathcal{H}}^2 \leq \|w_1\|_{\mathcal{M}(A)}^2 + \|w_2\|_{\mathcal{H}(A)}^2.$$

Moreover,

$$\|w\|_{\mathcal{H}}^2 = \|w_1\|_{\mathcal{M}(A)}^2 + \|w_2\|_{\mathcal{H}(A)}^2$$

if and only if

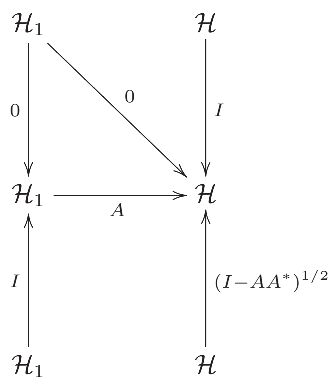
$$w_1 = AA^*w \quad \textit{and} \quad w_2 = (I - AA^*)w.$$

Proof In this proof we write T instead of A. This is because we want to apply Theorem 16.23 and use the notation there, but the operator A that appears in that theorem is not the same as the one introduced in the present theorem.

Consider the decomposition

$$0 = T0,$$

where on the left-hand side we have $0 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and on the right-hand side 0 represents the zero operator in $\mathcal{L}(\mathcal{H}_1)$. Hence, we have the decomposition $A = A_2 A_1$ with $A = 0 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$, $A_2 = T$ and $A_1 = 0 \in \mathcal{L}(\mathcal{H}_1)$. Hence, $\mathcal{H}(A) = \mathcal{H}$ and $\mathcal{H}(A_1) = \mathcal{H}_1$:



(16.32)

The diagram (16.32) is a simplified version of diagram (16.25).

Thus the decomposition $\mathcal{H}(A) = A_2\mathcal{H}(A_1) + \mathcal{H}(A_2)$ obtained in Theorem 16.23(i) can be written as

$$\mathcal{H} = T\mathcal{H}_1 + \mathcal{H}(T) = \mathcal{M}(T) + \mathcal{H}(T),$$

and if $z = Tz_1 + z_2$ with $z_1 \in \mathcal{H}_1$ and $z_2 \in \mathcal{H}(T)$, then, by Theorem 16.23(ii),

$$\|z\|_{\mathcal{H}}^2 \leq \|z_1\|_{\mathcal{H}_1}^2 + \|z_2\|_{\mathcal{H}(T)}^2.$$

In particular, if we take $z_1 \perp \ker T$, we obtain

$$\|z\|_{\mathcal{H}}^2 \leq \|Tz_1\|_{\mathcal{M}(T)}^2 + \|z_2\|_{\mathcal{H}(T)}^2.$$

Finally, by (16.27), (16.28) and (16.29), the unique pair z_1 and z_2 for which

$$\|z\|_{\mathcal{H}}^2 = \|z_1\|_{\mathcal{H}_1}^2 + \|z_2\|_{\mathcal{H}(T)}^2$$

holds is given by

$$z_1 = (I - A_1 A_1^*) A_2^* z = T^* z$$

and

$$z_2 = (I - A_2 A_2^*) z = (I - TT^*) z.$$

But

$$z_1 \in \mathcal{R}(T^*) \subset (\ker T)^{\perp}$$

implies that $\|z_1\|_{\mathcal{H}_1} = \|Tz_1\|_{\mathcal{M}(T)}$. To be consistent with the notation of the theorem, just take $w = z$, $w_1 = Tz_1$ and $w_2 = z_2$. □

We are now able to better understand Theorem 16.21. Generally speaking, an algebraic direct sum is not necessarily an orthogonal direct sum. However, the decomposition $\mathcal{H} = \mathcal{M}(A) + \mathcal{H}(A)$ is a special case. If this is an algebraic direct sum, then it means that $\mathcal{M}(A) \cap \mathcal{H}(A) = \{0\}$. Therefore, Theorem 16.21 ensures that it is in fact an orthogonal direct sum.

16.9 The geometric definition of $\mathcal{H}(A)$

De Branges and Rovnyak had a geometric point of view and gave a different definition of the complementary space $\mathcal{H}(A)$. In this section we treat their definition and show that it is equivalent to that given in Section 16.4. The latter definition is due to Sarason and opened a new world for these spaces. As the motivation for their definition, let us make an observation. According to Theorem 16.25, for each $z \in \mathcal{M}(A)$ and $w \in \mathcal{H}(A)$, we have

$$\|w + z\|_{\mathcal{H}_2}^2 \le \|w\|_{\mathcal{H}(A)}^2 + \|z\|_{\mathcal{M}(A)}^2.$$

Writing this inequality as

$$\|w + z\|_{\mathcal{H}_2}^2 - \|z\|_{\mathcal{M}(A)}^2 \le \|w\|_{\mathcal{H}(A)}^2$$

immediately implies that

$$\sup_{z \in \mathcal{M}(A)} \left(\|w + z\|_{\mathcal{H}_2}^2 - \|z\|_{\mathcal{M}(A)}^2 \right) \le \|w\|_{\mathcal{H}(A)}^2. \tag{16.33}$$

Moreover, the supremum is attained.

Suppose that $\mathcal{H}$ is a Hilbert space and let $\mathcal{M}$ be a Hilbert space contractively contained in $\mathcal{H}$. As we explained at the end of Section 16.1, we have $\mathcal{M} = \mathcal{M}(i)$, where i is the inclusion map $i = i_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$. Then de Branges and

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Rovnyak defined the complementary space of $\mathcal{M}$ as the set of all $w \in \mathcal{H}$ such that

$$\sup_{z\in\mathcal{M}} (\|w+z\|_{\mathcal{H}}^2 - \|z\|_{\mathcal{M}}^2) < \infty. \tag{16.34}$$

By (16.33), surely each element of $\mathcal{H}(i)$ satisfies this property. But, in fact, this is a characterization property.

Lemma 16.26 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a Hilbert space contraction, and let $w \in \mathcal{H}_2$ be such that*

$$\sup_{z\in\mathcal{M}(A)} (\|w+z\|_{\mathcal{H}_2}^2 - \|z\|_{\mathcal{M}(A)}^2) < \infty. \tag{16.35}$$

Then $w \in \mathcal{H}(A)$ and

$$\|w\|_{\mathcal{H}(A)}^2 \le \sup_{z\in\mathcal{M}(A)} (\|w+z\|_{\mathcal{H}_2}^2 - \|z\|_{\mathcal{M}(A)}^2).$$

Proof If we succeed in showing that $A^*w \in \mathcal{H}(A^*)$, then, by Theorem 16.18, w would be in $\mathcal{H}(A)$. Hence, we proceed to show that $A^*w \in \mathcal{H}(A^*)$. Let

$$c = \sup_{x\in\mathcal{H}_1} (\|w+Ax\|_{\mathcal{H}_2}^2 - \|x\|_{\mathcal{H}_1}^2). \tag{16.36}$$

Since $\|Ax\|_{\mathcal{M}(A)} \le \|x\|_{\mathcal{H}_1}$ and the equality holds whenever $x \perp \ker A$, then

$$c = \sup_{z\in\mathcal{M}(A)} (\|w+z\|_{\mathcal{H}_2}^2 - \|z\|_{\mathcal{M}(A)}^2) < \infty.$$

Let $\gamma \in \mathbb{T}$ be such that $\langle w, Ax\rangle_{\mathcal{H}_2} = \gamma|\langle w, Ax\rangle_{\mathcal{H}_2}|$. Replace x by $t\gamma x$, where $t \in \mathbb{R}$, in (16.36). Hence,

$$\|w+t\gamma Ax\|_{\mathcal{H}_2}^2 - \|t\gamma x\|_{\mathcal{H}_1}^2 \le c \qquad (x \in \mathcal{H}_1).$$

This is equivalent to

$$t^2\|(I-A^*A)^{1/2}x\|_{\mathcal{H}_1}^2 - 2t|\langle w, Ax\rangle_{\mathcal{H}_2}| + c - \|w\|_{\mathcal{H}_2}^2 \ge 0.$$

Thus

$$|\langle w, Ax\rangle_{\mathcal{H}_2}| \le C\,\|(I-A^*A)^{1/2}x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1),$$

where $C = (c - \|w\|_{\mathcal{H}_2}^2)^{1/2}$. Since, by Corollary 16.16,

$$\|(I-A^*A)^{1/2}x\|_{\mathcal{H}_1} = \|(I-A^*A)x\|_{\mathcal{H}(A^*)},$$

we can write

$$|\langle A^*w, x\rangle_{\mathcal{H}_1}| \le C\,\|(I-A^*A)x\|_{\mathcal{H}(A^*)} \qquad (x \in \mathcal{H}_1).$$

By the same corollary, $\mathcal{R}(I-A^*A)$ is a dense submanifold of $\mathcal{H}(A^*)$. Hence, the last inequality says that the map

$$\begin{array}{ccc} \mathcal{R}(I-A^*A) & \longrightarrow & \mathbb{C} \\ (I-A^*A)x & \longmapsto & \langle x, A^*w\rangle_{\mathcal{H}_1} \end{array}$$

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gives a bounded linear functional on $\mathcal{R}(I - A^*A)$, whose norm is less than or equal to C. We can extend it by continuity to a bounded linear functional on $\mathcal{H}(A^*)$. Hence, by Riesz's theorem, there is an element $y \in \mathcal{H}(A^*)$, $\|y\|_{\mathcal{H}(A^*)} \le C$, such that

$$\langle x, A^*w\rangle_{\mathcal{H}_1} = \langle (I - A^*A)x,\ y\rangle_{\mathcal{H}(A^*)} \qquad (x \in \mathcal{H}_1).$$

Since, by Corollary 16.16, $\langle (I - A^*A)x,\ y\rangle_{\mathcal{H}(A^*)} = \langle x, y\rangle_{\mathcal{H}_1}$, we thus have

$$\langle x, A^*w\rangle_{\mathcal{H}_1} = \langle x, y\rangle_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

Therefore, $A^*w = y \in \mathcal{H}(A^*)$ and $\|A^*w\|_{\mathcal{H}(A^*)} \le C$. Theorem 16.18 now implies that $w \in \mathcal{H}(A)$ and

$$\begin{aligned}\|w\|^2_{\mathcal{H}(A)} &= \|w\|^2_{\mathcal{H}} + \|A^*w\|^2_{\mathcal{H}(A^*)}\\ &\le \|w\|^2_{\mathcal{H}} + C^2 = c\\ &= \sup_{z\in\mathcal{M}(A)} (\|w + z\|^2_{\mathcal{H}_2} - \|z\|^2_{\mathcal{M}(A)}).\end{aligned}$$

This completes the proof. □

Combining Lemma 16.26 and (16.33), we obtain the following result. It shows that the definition of de Branges and Rovnyak is equivalent to the definition of Sarason.

Corollary 16.27 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a Hilbert space contraction and let $w \in \mathcal{H}_2$. Then the following are equivalent:*

(i) $w \in \mathcal{H}(A)$;
(ii) $\sup_{z\in\mathcal{M}(A)}(\|w + z\|^2_{\mathcal{H}_2} - \|z\|^2_{\mathcal{M}(A)}) < +\infty$.

Moreover, for such an element, we have

$$\|w\|^2_{\mathcal{H}(A)} = \sup_{z\in\mathcal{M}(A)} (\|w + z\|^2_{\mathcal{H}_2} - \|z\|^2_{\mathcal{M}(A)}).$$

Given a Hilbert space $\mathcal{H}$ and $\mathcal{M}$, such that $\mathcal{M}$ is contractively contained in $\mathcal{H}$, we denote by $\mathcal{M}'$ the complementary space of $\mathcal{M}$ defined by (16.34) and we put

$$\|w\|^2_{\mathcal{M}'} = \sup_{z\in\mathcal{M}} (\|w + z\|^2_{\mathcal{H}} - \|z\|^2_{\mathcal{M}}) \qquad (w \in \mathcal{M}'). \tag{16.37}$$

If $i_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$ is the inclusion map, then Corollary 16.27 says that

$$\mathcal{M}' = \mathcal{H}(i_{\mathcal{M}}).$$

In particular, we see that $\mathcal{M}'$ is a Hilbert space that is also contractively contained in $\mathcal{H}$. Remember also that, if $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then Theorem 16.21 gives that $\mathcal{M}'$ coincides with the orthogonal complement of $\mathcal{M}$ in $\mathcal{H}$.

We state the decomposition $\mathcal{H} = \mathcal{M}(A) + \mathcal{H}(A)$ in a slightly different language below. This version will be needed in our discussion of $\mathcal{H}(b)$ spaces.

Corollary 16.28 *Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{M}$ be a Hilbert space that is contractively contained in $\mathcal{H}$ and let $\mathcal{N}$ be the complementary space of $\mathcal{M}$ in $\mathcal{H}$. Denote by $i_{\mathcal{M}}$ (respectively $i_{\mathcal{N}}$) the canonical injection of $\mathcal{M}$ (respectively $\mathcal{N}$) into $\mathcal{H}$. Then, for each $x \in \mathcal{H}$, we have*

$$x = i_{\mathcal{M}}^* x + i_{\mathcal{N}}^* x$$

and

$$\|x\|_{\mathcal{H}}^2 = \|i_{\mathcal{M}}^* x\|_{\mathcal{M}}^2 + \|i_N^* x\|_{\mathcal{N}}^2.$$

Moreover, if $x = x_1 + x_2$, with $x_1 \in \mathcal{M}$ and $x_2 \in \mathcal{N}$, which satisfy

$$\|x\|_{\mathcal{H}}^2 = \|x_1\|_{\mathcal{M}}^2 + \|x_2\|_{\mathcal{N}}^2,$$

then we necessarily have $x_1 = i_{\mathcal{M}}^* x$ and $x_2 = i_{\mathcal{N}}^* x$.

Proof We know that $\mathcal{M} \doteq \mathcal{M}(i_{\mathcal{M}})$ and $\mathcal{N} \doteq \mathcal{M}(i_{\mathcal{N}})$. Moreover, according to Corollary 16.27, we have $\mathcal{N} \doteq \mathcal{H}(i_{\mathcal{M}}) \doteq \mathcal{M}((I - i_{\mathcal{M}} i_{\mathcal{M}}^*)^{1/2})$. Thus Corollary 16.12 implies that

$$i_{\mathcal{N}} i_{\mathcal{N}}^* = I - i_{\mathcal{M}} i_{\mathcal{M}}^*.$$

Therefore $I = i_{\mathcal{N}} i_{\mathcal{N}}^* + i_{\mathcal{M}} i_{\mathcal{M}}^*$, which gives

$$x = i_{\mathcal{M}}^* x + i_{\mathcal{N}}^* x \qquad (x \in H).$$

Furthermore, an application of Theorem 16.25 to $A = i_{\mathcal{M}}$ gives

$$\begin{aligned}\|x\|_{\mathcal{H}}^2 &= \|i_{\mathcal{M}} i_{\mathcal{M}}^* x\|_{\mathcal{M}(i_{\mathcal{M}})}^2 + \|(I - i_{\mathcal{M}} i_{\mathcal{M}}^*) x\|_{\mathcal{H}(i_{\mathcal{M}})}^2 \\ &= \|i_{\mathcal{M}}^* x\|_{\mathcal{M}}^2 + \|i_{\mathcal{N}}^* x\|_{\mathcal{N}}^2.\end{aligned}$$

The second point of Corollary 16.28 follows also immediately from Theorem 16.25. □

This corollary explains why the notion of complementary space can be seen as a generalization of the orthogonal complement, and the map i_M^* (respectively i_N^*) can be seen as a generalization of the orthogonal projection onto M (respectively onto N). We end this section with a result about subspaces that are invariant under the shift operators.

Let $\mathcal{M}$ be a Hilbert space contractively contained in another Hilbert space $\mathcal{H}$, and let $i_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$ be the inclusion map. Then, $\mathcal{M} \doteq \mathcal{M}(i_{\mathcal{M}})$ and the complementary space of $\mathcal{M}$ is $\mathcal{N} = \mathcal{H}(i_{\mathcal{M}})$. In particular, the space $\mathcal{N}$ is itself contractively contained in $\mathcal{H}$. It is natural to wonder what is the complementary space of $\mathcal{N}$. In other words, what is $\mathcal{M}''$?

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Using the relation $I - i_{\mathcal{M}} i_{\mathcal{M}}^* = i_{\mathcal{N}} i_{\mathcal{N}}^*$ and Corollary 16.8, we get that

$$\mathcal{N} \doteq \mathcal{H}(i_{\mathcal{M}}) \doteq \mathcal{M}((I - i_{\mathcal{M}} i_{\mathcal{M}}^*)^{1/2}) \doteq \mathcal{M}(i_{\mathcal{N}}).$$

Hence, the complementary space of $\mathcal{N}$ is $\mathcal{H}(i_{\mathcal{N}})$. Using once more the relation $I - i_{\mathcal{M}} i_{\mathcal{M}}^* = i_{\mathcal{N}} i_{\mathcal{N}}^*$ and Corollary 16.8, we get that

$$\mathcal{H}(i_{\mathcal{N}}) \doteq \mathcal{M}(i_{\mathcal{N}}) \doteq \mathcal{M},$$

which gives that the complementary space of $\mathcal{N}$ is $\mathcal{M}$. In other words, we have proved that, if we start with a space $\mathcal{M}$, contractively contained in $\mathcal{H}$, then

$$\mathcal{M}'' \doteq \mathcal{M}.$$

Theorem 16.29 *Let $\mathcal{M}$ be a Hilbert space contained in H^2. Then the following assertions are equivalent.*

(i) *$\mathcal{M}$ is invariant under the backward shift S^*, i.e. $S^*\mathcal{M} \subset \mathcal{M}$, and*

$$\|S^* f\|_{\mathcal{M}}^2 \le \|f\|_{\mathcal{M}}^2 - |f(0)|^2 \qquad (f \in \mathcal{M}). \tag{16.38}$$

(ii) *The following hold:*

(a) *$\mathcal{M}$ is contractively contained in H^2;*
(b) *$S\mathcal{M}' \subset \mathcal{M}'$ and*

$$\|Sg\|_{\mathcal{M}'} \le \|g\|_{\mathcal{M}'} \qquad (g \in \mathcal{M}').$$

Proof (i) $\Longrightarrow$ (ii) Assume first that $\mathcal{M}$ is invariant under the backward shift S^* and satisfies (16.38). Let us prove first that condition (a) holds. Write $T = S^*|\mathcal{M}$. By induction, from (16.38) we get

$$\|T^n f\|_{\mathcal{M}}^2 \le \|f\|_{\mathcal{M}}^2 - \sum_{k=0}^{n-1} |T^k f(0)|^2 \qquad (n \ge 1,\ f \in \mathcal{M}).$$

Hence,

$$\sum_{k=0}^{n-1} |\hat{f}(k)|^2 \le \|f\|_{\mathcal{M}}^2 \qquad (n \ge 1,\ f \in \mathcal{M}). \tag{16.39}$$

Note that $T^k f(0) = (S^{*k} f)(0) = \hat{f}(k)$. But, for each $f \in H^2$, we have

$$\|f\|_2^2 = \sum_{k=0}^{\infty} |\hat{f}(k)|^2.$$

Therefore, letting $n \longrightarrow \infty$ in (16.39) gives

$$\|f\|_2 \le \|f\|_{\mathcal{M}} \qquad (f \in \mathcal{M}).$$

This inequality means precisely that $\mathcal{M}$ is contractively contained in H^2 and thus its complementary space $\mathcal{M}'$ is well defined.

Let us now prove that (b) holds. Let $f \in \mathcal{M}$, and let $g \in H^2$. Then using (16.38), we have

$$\begin{aligned}\|Sg + f\|_2^2 - \|f\|_{\mathcal{M}}^2 &\le \|Sg + f\|_2^2 - \|S^*f\|_{\mathcal{M}}^2 - |f(0)|^2 \\ &= \|Sg + SS^*f\|_2^2 - \|S^*f\|_{\mathcal{M}}^2,\end{aligned}$$

because

$$\|Sg + f\|_2^2 = \|Sg + SS^*f - f(0)\|_2^2 = \|Sg + SS^*f\|_2^2 + |f(0)|^2.$$

Therefore, we have

$$\|Sg + f\|_2^2 - \|f\|_{\mathcal{M}}^2 \le \|g + S^*f\|_2^2 - \|S^*f\|_{\mathcal{M}}^2, \tag{16.40}$$

for every $g \in H^2$ and every $f \in \mathcal{M}$. If furthermore we assume that $g \in \mathcal{M}'$, then we get

$$\|Sg + f\|_2^2 - \|f\|_{\mathcal{M}}^2 \le \sup_{f \in \mathcal{M}} (\|g + S^*f\|_2^2 - \|S^*f\|_{\mathcal{M}}^2) \le \|g\|_{\mathcal{M}'}^2.$$

By definition, this means that $Sg \in \mathcal{M}'$ and $\|Sg\|_{\mathcal{M}'} \le \|g\|_{\mathcal{M}'}$.

(ii) $\Longrightarrow$ (i) Assume that $\mathcal{M}$ is contractively contained in H^2 and the shift S acts as a contraction on $\mathcal{M}'$. Let $f \in \mathcal{M}$ and $g \in \mathcal{M}'$. We have

$$\|S^*f + g\|_2^2 - \|g\|_{\mathcal{M}'}^2 = \|SS^*f + Sg\|_2^2 - \|g\|_{\mathcal{M}'}^2,$$

and using the fact that $SS^*f + Sg \perp 1$, we also have

$$\|SS^*f + Sg\|_2^2 = \|SS^*f + Sg + f(0)\|_2^2 - |f(0)|^2 = \|f + Sg\|_2^2 - |f(0)|^2.$$

Hence,

$$\|S^*f + g\|_2^2 - \|g\|_{\mathcal{M}'}^2 = \|f + Sg\|_2^2 - |f(0)|^2 - \|g\|_{\mathcal{M}'}^2.$$

Since S acts as a contraction on $\mathcal{M}'$, we deduce that

$$\|S^*f + g\|_2^2 - \|g\|_{\mathcal{M}'}^2 \le \|f + Sg\|_2^2 - \|Sg\|_{\mathcal{M}'}^2 - |f(0)|^2.$$

Using the definition of the norm of the complementary space (see (16.37)), and the fact that $\mathcal{M}'' = \mathcal{M}$, we get

$$\|S^*f + g\|_2^2 - \|g\|_{\mathcal{M}'}^2 \le \|f\|_{\mathcal{M}}^2 - |f(0)|^2.$$

Now taking the supremum over all $g \in \mathcal{M}'$, we deduce that $S^*f \in \mathcal{M}$ and

$$\|S^*f\|_{\mathcal{M}}^2 \le \|f\|_{\mathcal{M}}^2 - |f(0)|^2. \qquad \square$$

Theorem 16.29 says that, if $\mathcal{H}$ is a Hilbert space contractively contained in H^2 and such that $S\mathcal{H} \hookrightarrow \mathcal{H}$, then $S^*\mathcal{H}' \hookrightarrow \mathcal{H}'$, but the converse is not true. Let us provide a simple example. Let $\mathcal{M} = \mathrm{Span}(1, z)$ endowed with a new scalar product such that

$$\|1\|_{\mathcal{M}} = \sqrt{2}, \quad \|z\|_{\mathcal{M}} = \sqrt{3} \quad \text{and} \quad 1 < \langle 1, z\rangle_{\mathcal{M}} < \sqrt{2}.$$

Then, we can easily check that $S^*\mathcal{M} \hookrightarrow \mathcal{M} \hookrightarrow H^2$. Putting $f(z) = 1 - z$, we see that $\|S^*f\|_{\mathcal{M}}^2 = \|1\|_{\mathcal{M}}^2 = 2$ and

$$\|f\|_{\mathcal{M}}^2 - |f(0)|^2 = 1 + \|z\|_{\mathcal{M}}^2 - 2\Re\langle 1, z\rangle_{\mathcal{M}} - 1 = 3 - 2\Re\langle 1, z\rangle_{\mathcal{M}}.$$

In particular,

$$\|f\|_{\mathcal{M}}^2 - |f(0)|^2 < \|S^*f\|_{\mathcal{M}}^2,$$

and Theorem 16.29 implies that $S\mathcal{M}'$ cannot be contained contractively in $\mathcal{M}'$.

The inequality (16.38) might look a bit strange. But note that

$$\|S^*f\|_2^2 = \|f\|_2^2 - |f(0)|^2 \qquad (f \in H^2),$$

and if $\mathcal{H}$ is a Hilbert space that is isometrically contained in H^2, then the inequality holds in the general case. In Section 18.8, we will show that the space $\mathcal{H}(b)$ also satisfies (16.38), and in Sections 23.5 and 25.4, we will study this inequality more precisely depending on whether b is an extreme or nonextreme point of the closed unit ball of H^∞.

Exercise

Exercise 16.9.1 Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, $B \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_2)$, $C \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_1)$ be contractions such that $B = AC$. Show that, for any $f \in \mathcal{H}(A)$, we have

$$\|f\|_{\mathcal{H}(A)}^2 = \sup_{g \in \mathcal{H}(C)} (\|f + Ag\|_{\mathcal{H}(B)}^2 - \|g\|_{\mathcal{H}(C)}^2).$$

Hint: Apply Theorem 16.23 to get

$$\sup_{g \in \mathcal{H}(C)} (\|f + Ag\|_{\mathcal{H}(B)}^2 - \|g\|_{\mathcal{H}(C)}^2) \le \|f\|_{\mathcal{H}(A)}^2.$$

For the other direction, let $\varepsilon > 0$. Apply Corollary 16.27 to show that there exists $h \in \mathcal{H}_1$ such that

$$\|f + Ah\|_{\mathcal{H}_2}^2 - \|h\|_{\mathcal{H}_1}^2 \ge \|f\|_{\mathcal{H}(A)}^2 - \varepsilon.$$

Then, apply Theorem 16.25 with $h = Ch_1 + h_2$, $h_1 = C^*h$, $h_2 = (I - CC^*)h$, which gives that

$$\|h\|_{\mathcal{H}_1}^2 = \|h_1\|_{\mathcal{H}_3}^2 + \|h_2\|_{\mathcal{H}(C)}^2.$$

Hence

$$\begin{aligned}
\|f + Ah_2\|_{\mathcal{H}(B)}^2 - \|h_2\|_{\mathcal{H}(C)}^2 &= \|f + Ah - Bh_1\|_{\mathcal{H}(B)}^2 - \|h_2\|_{\mathcal{H}(C)}^2 \\
&\ge \|f + Ah\|_{\mathcal{H}_2} - \|h_1\|_{\mathcal{H}_3}^2 - \|h_2\|_{\mathcal{H}(C)}^2 \\
&= \|f + Ah\|_{\mathcal{H}_2}^2 - \|h\|_{\mathcal{H}_1}^2 \\
&\ge \|f\|_{\mathcal{H}(A)}^2 - \varepsilon.
\end{aligned}$$

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16.10 The Julia operator $\mathfrak{J}(A)$ and $\mathcal{H}(A)$

As we have seen in Theorem 16.25, the complementary space $\mathcal{H}(A)$ generalizes in some sense the notion of orthogonality. There is a more direct way in which complementarity is related to orthogonality. In Section 7.3, we introduced the Julia operator $J(A)$ associated with a contraction $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$. Let us recall that $J(A) \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}_1)$ is defined by

$$J(A) = \begin{bmatrix} D_{A^*} & A \\ -A^* & D_A \end{bmatrix},$$

where $D_A = (I - A^*A)^{1/2}$ and $D_{A^*} = (I - AA^*)^{1/2}$. We also recall that $\mathcal{D}_A = \mathrm{Clos}_{\mathcal{H}}(D_A\mathcal{H}_1)$ and $\mathcal{D}_{A^*} = \mathrm{Clos}_{\mathcal{H}_1}(D_{A^*}\mathcal{H})$. According to Theorem 7.18, $J(A)$ is a unitary operator on $\mathcal{H} \oplus \mathcal{H}_1$.

We now define a related operator, which is also called the *Julia operator*. The operator

$$\mathfrak{J}(A) : \mathcal{H}_1 \oplus \mathcal{H} \longrightarrow \mathcal{H} \oplus \mathcal{H}_1$$

is defined by

$$\mathfrak{J}(A) = \begin{bmatrix} A & D_{A^*} \\ D_A & -A^* \end{bmatrix}.$$

There is a simple relation between these two operators. The connection is via the unitary operator $U : \mathcal{H} \oplus \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \oplus \mathcal{H}$ defined by $U(x \oplus y) = y \oplus x$, where $x \oplus y \in \mathcal{H} \oplus \mathcal{H}_1$. If we identify this operator with its matrix, then we have

$$U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} : \mathcal{H} \oplus \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \oplus \mathcal{H}.$$

Hence

$$J(A)U = \begin{bmatrix} D_{A^*} & A \\ -A^* & D_A \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} A & D_{A^*} \\ D_A & -A^* \end{bmatrix},$$

which means that

$$\mathfrak{J}(A) = J(A)U.$$

In particular, we deduce from Theorem 7.18 that $\mathfrak{J}(A)$ is a unitary operator from $\mathcal{H}_1 \oplus \mathcal{H}$ onto $\mathcal{H} \oplus \mathcal{H}_1$.

Theorem 16.30 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ be a Hilbert space contraction and assume that A is one-to-one. Write*

$$\mathcal{X}_1 = \mathfrak{J}(A)(\mathcal{H}_1 \oplus \{0\}), \qquad \mathcal{X}_2 = \mathfrak{J}(A)(\{0\} \oplus \mathcal{D}_{A^*}),$$

and let P_1 be the orthogonal projection of $\mathcal{H} \oplus \mathcal{D}_A$ onto its first coordinate $\mathcal{H}$. Then the following hold.

(i) $\mathfrak{J}(A)(\mathcal{H}_1 \oplus \mathcal{D}_{A^*}) = \mathcal{H} \oplus \mathcal{D}_A$.
(ii) $\mathcal{H} \oplus \mathcal{D}_A = \mathcal{X}_1 \oplus \mathcal{X}_2$.
(iii) $P_{1|\mathcal{X}_1}$ *is unitary from* $\mathcal{X}_1$ *onto* $\mathcal{M}(A)$.
(iv) $P_{1|\mathcal{X}_2}$ *is unitary from* $\mathcal{X}_2$ *onto* $\mathcal{H}(A)$.

Proof (i) Let $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{D}_{A^*}$. Then

$$\mathfrak{J}(A)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax + D_{A^*}y \\ D_A x - A^* y \end{bmatrix}.$$

It is clear that $Ax + D_{A^*}y \in \mathcal{H}$ and $D_A x \in \mathcal{D}_A$. The fact that $A^* y \in \mathcal{D}_A$ for $y \in \mathcal{D}_{A^*}$ follows from (7.12). Hence

$$\mathfrak{J}(A)(\mathcal{H}_1 \oplus \mathcal{D}_{A^*}) \subset \mathcal{H} \oplus \mathcal{D}_A. \tag{16.41}$$

Note that

$$\mathfrak{J}(A^*) = \begin{bmatrix} A^* & D_A \\ D_{A^*} & -A \end{bmatrix} = \mathfrak{J}(A)^*, \tag{16.42}$$

and thus if we apply (16.41) to A^*, we get

$$\mathfrak{J}(A)^*(\mathcal{H} \oplus \mathcal{D}_A) \subset \mathcal{H}_1 \oplus \mathcal{D}_{A^*}.$$

Since $\mathfrak{J}(A)\mathfrak{J}(A)^* = I$, we obtain

$$\mathcal{H} \oplus \mathcal{D}_A \subset \mathfrak{J}(A)(\mathcal{H}_1 \oplus \mathcal{D}_{A^*}),$$

which gives (i).

(ii) This part follows immediately from the fact that $\mathfrak{J}(A)$ is a unitary operator from $\mathcal{H}_1 \oplus \mathcal{D}_{A^*}$ onto $\mathcal{H} \oplus \mathcal{D}_A$ and that

$$\mathcal{H}_1 \oplus \mathcal{D}_{A^*} = (\mathcal{H}_1 \oplus \{0\}) \oplus (\{0\} \oplus \mathcal{D}_{A^*}).$$

(iii) We have

$$P_1\mathcal{X}_1 = P_1(\{Ax \oplus D_A x : x \in \mathcal{H}_1\}) = \{Ax : x \in \mathcal{H}_1\} = \mathcal{M}(A).$$

Moreover, if $x_1 \in \mathcal{X}_1$, then $x_1 = Ax \oplus D_A x$ for some $x \in \mathcal{H}_1$. Then, since A is one-to-one, we have

$$\|P_1 x_1\|_{\mathcal{M}(A)} = \|Ax\|_{\mathcal{M}(A)} = \|x\|_{\mathcal{H}_1}.$$

On the other hand, with (7.33), we also have

$$\|x_1\|^2_{\mathcal{H}\oplus\mathcal{H}_1} = \|Ax\|^2_{\mathcal{H}} + \|D_A x\|^2_{\mathcal{H}_1} = \|x\|^2_{\mathcal{H}_1},$$

which gives the assertion (iii).

(iv) We have

$$P_1\mathcal{X}_2 = P_1(\{D_{A^*}y \oplus (-A^*y) : y \in \mathcal{D}_{A^*}\}) = \{D_{A^*}y : y \in \mathcal{D}_{A^*}\} = \mathcal{H}(A).$$

Now if $x_2 \in \mathcal{X}_2$, then $x_2 = D_{A^*}y \oplus (-A^*y)$ for some $y \in \mathcal{D}_{A^*}$, and we have

$$\|P_1x_2\|_{\mathcal{H}(A)} = \|D_{A^*}y\|_{\mathcal{H}(A)} = \|y\|_{\mathcal{H}}.$$

On the other hand, once more using (7.33), we also have

$$\|x\|^2_{\mathcal{H}\oplus\mathcal{H}_1} = \|D_{A^*}y\|^2_{\mathcal{H}} + \|A^*y\|_{\mathcal{H}_1} = \|y\|^2_{\mathcal{H}},$$

which gives the result. □

Theorem 16.30 says that the orthogonal decomposition of $\mathcal{H} \oplus \mathcal{D}_A$ as $\mathcal{X}_1 \oplus \mathcal{X}_2$ is mapped by projecting onto the first coordinate into the complementary decomposition $\mathcal{H} = \mathcal{M}(A) + \mathcal{H}(A)$ (which is not in general, as we have already seen, a direct sum). So the rather exotic definition of complementary spaces is in fact the projection of a more familiar geometric structure.

Notes on Chapter 16

The main part of this chapter is taken from [166]. The notion of complementary space, which is the heart of our study, was introduced in the context of square-summable power series by de Branges and Rovnyak in their book [65] and their paper [64].

Section 16.1

The notion of a Hilbert space boundedly contained into another is crucial in the theory developed by de Branges and Rovnyak. Theorem 16.3 is taken from [139]. See also [19].

Section 16.2

Theorem 16.7 is known as Douglas's criterion and can be found in [67]. This theorem as well as Corollaries 16.8 and 16.9 can also be found in [166, chap. I]. Exercise 16.2.2 is taken from [19, corollary 3.3], but the method presented here is slightly different. Exercise 16.2.3 comes from [139, lemma 5.7].

Section 16.3

The description of linear functionals on $\mathcal{M}(A)$ is taken from [166, sec. I.3].

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Section 16.4

The notions of contractive containment and complementary spaces were crucial in de Branges's proof of the Bieberbach conjecture. See [63, 141, 142].

The terminology and notion of complementary space are due to de Branges. In the context of square-summable power series, it was introduced by de Branges and Rovnyak in [65]. Nevertheless, the definition of complementary space $\mathcal{H}(A)$ used in this book is due to Sarason and appears in [160]. See Section 16.9 for the original definition of de Branges and Rovnyak. In [64] de Branges and Rovnyak used "overlapping space" in a different way. In this text, we use this term in the sense introduced by Lotto and Sarason [123].

The presentation of this section is taken from [166, chap. I]. A special version of Theorem 16.17 appears in [160] without proof.

Section 16.5

Theorem 16.18 on the relation between $\mathcal{H}(A)$ and $\mathcal{H}(A^*)$ is due to Lotto and Sarason [123, lemma 2.1].

Section 16.6

The description of the overlapping space $\mathcal{H}(A) \cap \mathcal{M}(A)$ is due to Lotto and Sarason [123, lemma 2.1].

Section 16.8

The decomposition of $\mathcal{H}(A)$ given by Theorem 16.23 is due to de Branges and Rovnyak [65, problem 52]. See also [64, appdx, theorem 4]. Theorem 16.25 is also due to de Branges and Rovnyak [65, theorem 8 and problem 36]. The presentation used in this text comes from [166].

Section 16.9

The geometric definition of the complementary space $\mathcal{H}(A)$ given in this section is due to de Branges and Rovnyak. The definition we choose to introduce the complementary space emphasizes the role of the contraction A and it will be successful (as we will see later) in the context of the Toeplitz operator on H^2.

A vector-valued version of Theorem 16.29 appears in Nikolskii and Vasyunin [139, theorem 7.4]. See also Ando [19, theorem 4.3]. The example at the end of the section showing that (16.38) is important in Theorem 16.29 comes from [139, theorem 7.4]. Exercise 16.9.1 is due to de Branges and

Rovnyak [65, theorem 11], who proved the formula in the context of square-summable power series.

Section 16.10

The connection between the Julia operator $\mathfrak{J}(A)$ and $\mathcal{H}(A)$ is taken from Timotin [187]. Nevertheless, in the particular case where $A = T_b$, it is implicitly present in the paper of Nikolskii and Vasyunin [139] when they studied the connection between the de Branges–Rovnyak and the Sz.-Nagy–Foiaş models.

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