ON THE INVERSION OF THE GAUSS TRANSFORMATION, II

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1. Introduction. In an earlier paper **(5)** we studied the inversion theory of the Gauss transformation defined by

$$f(x) = \mathscr{G}(\phi(x)) = \frac{1}{(4\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{1}{4}(x-t)^2} \phi(t)dt. \tag{1.1}$$

Operational methods indicated that

$$\exp(-D^2) f(x) = \phi(x)$$

and we showed that in certain circumstances this equation was true if $\exp(-D^2) f(x)$ was interpreted as the sum of the series

$$\sum_{n=0}^{\infty} (-1)^n f^{(2n)}(x)/n! \tag{1.2}$$

However, another possible interpretation of $\exp(-D^2) f(x)$ arises from the well known formula

$$e^{-x^2} = \lim_{n\to\infty} \left(1 - \frac{x^2}{n}\right)^n,$$

and we shall show here that such an interpretation also leads to an inversion formula for the transformation. This is done in section two.

Pollard **(4)** has developed an L_2 theory for inversion by the series (1.2), and in § 3 we shall develop a similar theory for our inversion.

2. Convergence theory. The two theorems below give sets of conditions for inversion. We first prove a preliminary lemma.

LEMMA. *If*

$$\sum_{n=0}^{\infty} a_n$$

converges to the sum a, *then*

$$\lim_{n\to\infty} \sum_{r=0}^{n} \frac{n!}{(n-r)!\, n^r} a_r = a.$$

Proof. Let $S_n = a_0 + a_1 + \ldots + a_n$. Then

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$$\sum_{r=0}^{n} \frac{n!}{(n-r)!\,n^r} a_r = \sum_{r=0}^{n} \frac{n!r}{(n-r)!\,n^{r+1}} S_r.$$

Hence, if $a_0 = 1$, $a_n = 0$, $n > 0$, we have $S_n = 1$, $n = 0, 1, 2, \ldots$, and

$$\sum_{r=0}^{n} \frac{n!r}{(n-r)!\,n^{r+1}} = 1. \quad \text{Also} \lim_{n\to\infty} \frac{n!r}{(n-r)!\,n^{r+1}} = 0,$$

so that the result follows from **(2**, § 3.1, Theorem 2**)**.

THEOREM 1. *If* $\phi(t) \in L(-\delta, \delta)$, $\delta > 0$, $|t|^\lambda \exp[-\frac{1}{8}(x_0 - t)^2]\,\phi(t) \in L(-\infty, \infty)$ *for some* $\lambda > 3$, *and* $\phi(t)$ *is of bounded variation in a neighbourhood of* $t = x_0$, *then* $f(x)$, *as defined by* (1.1), *exists for all* x, *and*

$$\lim_{n\to\infty} \left(1 - \frac{D^2}{n}\right)^n f(x)\big|_{x=x_0} = \tfrac{1}{2}\{\phi(x_0+) + \phi(x_0-)\}.$$

Proof. By **(5**, Theorem 1**)** $f(x)$ exists for all x and (1.2) converges for $x = x_0$ to $\frac{1}{2}\{\phi(x_0+) + \phi(x_0-)\}$. But then by the lemma, with $a_r = (-1)^r f^{(2r)}(x_0)/r!$,

$$\lim_{n\to\infty} \sum_{r=0}^{n} \frac{n!}{(n-r)!\,n^r} (-1)^r \frac{f^{(2r)}(x_0)}{r!} = \tfrac{1}{2}\{\phi(x_0+) + \phi(x_0-)\}.$$

But this last sum is

$$\sum_{r=0}^{n} \binom{n}{r} \frac{(-1)^r D^{2r}}{n^r} f(x)\bigg|_{x=x_0} = \left(1 - \frac{D^2}{n}\right)^n f(x)\bigg|_{x=x_0}.$$

Hence,

$$\lim_{n\to\infty} \left(1 - \frac{D^2}{n}\right)^n f(x)\bigg|_{x=x_0} = \tfrac{1}{2}\{\phi(x_0+) + \phi(x_0-)\}.$$

THEOREM 2. *If* $\exp[-\frac{1}{8}(x_0 - t)^2]\phi(t) \in L(-\infty, \infty)$, $\phi(t)$ *is of bounded variation in a neighbourhood of* $t = x_0$, *and the series* (1.2) *converges for* $x = x_0$, *then* $f(x)$, *as defined by* (1.1), *exists for all* x, *and*

$$\lim_{n\to\infty} \left(1 - \frac{D^2}{n}\right)^n f(x)\bigg|_{x=x_0} = \tfrac{1}{2}\{\phi(x_0+) + \phi(x_0-)\}.$$

Proof. By **(5**, Theorem 2**)**, the series (1.2) is summable for $x = x_0$ in the Abel sense to $\frac{1}{2}\{\phi(x_0+) + \phi(x_0-)\}$. But, since (1.2) converges for $x = x_0$, and the Abel method is a regular method of summation, (1.2) converges for $x = x_0$ to $\frac{1}{2}\{\phi(x_0+) + \phi(x_0-)\}$. Hence, by the lemma

$$\lim_{n\to\infty} \left(1 - \frac{D^2}{n}\right)^n f(x)\bigg|_{x=x_0} = \lim_{n\to\infty} \sum_{r=0}^{n} \frac{n!}{(n-r)!\,n^r} (-1)^r \frac{f^{(2r)}(x_0)}{r!}$$
$$= \tfrac{1}{2}\{\phi(x_0+) + \phi(x_0-)\}.$$

3. L_2 theory.

THEOREM 3. *If* $\phi \in L_2(-\infty, \infty)$, *then* $f(x)$ *as defined by* (1.1) *exists for all* x, *and*

$$\underset{n\to\infty}{\text{l.i.m.}} \left(1 - \frac{D^2}{n}\right)^n f(x) = \phi(x).$$

Proof. The existence of $f(x)$ is clear. Let Φ be the Fourier transform of ϕ. Then since the Fourier transform of $(4\pi)^{-\frac{1}{2}} \exp[-\frac{1}{4}(x-t)^2]$ is

$$(2\pi)^{-\frac{1}{2}} \exp(iyx - y^2)$$

we have, on applying the Parseval relation (**6**, Theorem 49 and 2.1.2), that

$$f(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-ixy-y^2} \Phi(y) dy.$$

Since for $n = 0, 1, 2, \ldots,$

$$|y^n e^{-ixy-y^2} \Phi(y)| = |y|^n e^{-y^2} |\Phi(y)| \in L(-\infty, \infty),$$

it follows from (**3**, Corollary 39.2), that we may differentiate this integral as often as we like under the integral sign and obtain

$$f^{(2r)}(x) = \frac{(-1)^r}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} y^{2r} e^{-ixy-y^2} \Phi(y) dy$$

and hence

$$\begin{aligned} S_n(x) = \left(1 - \frac{D^2}{n}\right)^n f(x) &= \sum_{r=0}^{n} \binom{n}{r} (-1)^r \frac{f^{(2r)}(x)}{n^r} \\ &= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \left(\sum_{r=0}^{n} \binom{n}{r} \frac{y^{2r}}{n^r}\right) e^{-ixy-y^2} \Phi(y) dy \\ &= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \left(1 + \frac{y^2}{n}\right)^n e^{-ixy-y^2} \Phi(y) dy. \end{aligned}$$

Hence, if σ_n is the Fourier transform of S_n,

$$\sigma_n(y) = \left(1 + \frac{y^2}{n}\right)^n e^{-y^2} \Phi(y).$$

Hence, from (**6**, Theorem 50 and 2.1.3),

$$\begin{aligned} \int_{-\infty}^{\infty} |S_n(x) - \phi(x)|^2 dx &= \int_{-\infty}^{\infty} |\sigma_n(y) - \Phi(y)|^2 dy \\ &= \int_{-\infty}^{\infty} \left(\left(1 + \frac{y^2}{n}\right)^n e^{-y^2} - 1\right)^2 |\Phi(y)|^2 dy. \end{aligned}$$

Now the integrand in this last integral tends to zero a.e. as $n \to \infty$. Also a short calculation shows that

$$\left(\left(1 + \frac{y^2}{n}\right)^n e^{-y^2} - 1\right)^2 \leqslant 1,$$

and thus

$$\left(\left(1+\frac{y^2}{n}\right)^n e^{-y^2} - 1\right)^2 |\Phi(y)|^2 \leqslant |\Phi(y)|^2 \in L_1(-\infty, \infty).$$

Hence, by the theorem of dominated convergence,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |S_n(x) - \phi(x)|^2 \, dx = 0,$$

that is,

$$\underset{n\to\infty}{\text{l.i.m.}} \left(1 - \frac{D^2}{n}\right)^n f(x) = \phi(x).$$

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